A Class of Periodic Solutions of One-Dimensional Landau-Lifshitz Equations

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Received January 10, 2017; Accepted (revised) June 22, 2017

Abstract. We study one-dimensional Landau-Lifshitz Equations and give the sufficient and necessary conditions for the existence of a class of periodic solutions.

AMS subject classifications: 35Q40, 35J20, 35B44

Key words: Landau-Lifshitz, periodic solution, blow up.

1 Introduction

Here we consider the following Schrödinger type flow

$$\frac{\partial u}{\partial t} = u \times (\Delta u + \nabla h \nabla u + \lambda u_3 e_3),$$

(1.1)

where $u(t,x) = (u_1, u_2, u_3) : \mathbb{R} \times \mathbb{R}^n \to S^2 \subseteq \mathbb{R}^3$, $h \in C^\infty(\mathbb{R}^n)$, $\lambda \in \mathbb{R}$ and $e_3 = (0, 0, 1)$ denotes the north pole. In fact, above equation is just Hamilton system with respect to the following functional

$$E_h(u) = \frac{1}{2} \int_{\mathbb{R}^n} |\nabla u|^2 e^h dx + \frac{\lambda}{2} \int_{\mathbb{R}^n} (1 - u_3^2) e^h dx.$$  

(1.2)

Let us recall some results on the Schrödinger type flow (1.1). Despite a great deal of mathematical efforts, some basic mathematical issues such as the global well-posedness

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and global-in-time asymptotics for (1.1) are still unclear. Therefore, some authors (see [5, 9, 10]) focused on finding some soliton solutions to (1.1). As $h$ is a constant function, equation (1.1) is just Landau-Lifshitz equation with easy-axis anisotropic, for simplicity, denoted by LLEE. In [5], Gustafson and Shatah studied the time-periodic solitary wave solutions (also called vortex solutions) to LLEE in two spatial dimensions with $\lambda > 0$ (see also [6]), while Lin and Wei ([9]) constructed traveling wave solutions with $\lambda < 0$. However, Kollar ([8]) showed the nonexistence of vortex solutions to the same problem with $\lambda = 0$ (See also [10]). Then limited work has been done in seeking for soliton solutions to (1.1) for $n = 1$ or $n \geq 3$.

The main purpose of this paper is to obtain some soliton solutions to equation (1.1) on one-dimensional space, i.e. $n = 1$. Specifically, we look for a solution of the following form

$$u(t,x) = (\sin(\alpha(x)\cos(\omega t)),\sin(\alpha(x)\sin(\omega t)),cos(\alpha(x))), \quad (1.3)$$

where $\alpha \in C^\infty(\mathbb{R})$ and $\omega \in \mathbb{R}$ is the angular velocity. After a simple calculation, the equation (1.1) reduces to an ordinary differential equation (ODE) of $\alpha$:

$$\alpha''(x) + h'(x)\alpha'(x) = g(\alpha(x)), \quad (1.4)$$

where

$$g(\cdot) = \lambda \sin(\cdot) \cos(\cdot) + \omega \sin(\cdot). \quad (1.5)$$

In view of physical background, we are more concerned with such solutions with finite energy, i.e. $|E_h(u)| < \infty$, which is equivalent to the following boundary condition of $\alpha(x)$

$$\lim_{x \to \infty} \alpha(x) = k\pi \quad \text{and} \quad \lim_{x \to -\infty} \alpha(x) = (k + 2l)\pi, \quad (1.6)$$

where $k, l \in \mathbb{Z}$.

For simplicity, we only consider the case $k = 1$ and $l = -1$, i.e.

$$\lim_{x \to \infty} \alpha(x) = \pi \quad \text{and} \quad \lim_{x \to -\infty} \alpha(x) = -\pi. \quad (1.6)$$

In order to find solutions to problem (1.4)-(1.6), Let us consider the following boundary value problem

$$\text{(BVP)} \begin{cases} \alpha''(x) + h'(x)\alpha'(x) = g(\alpha(x)), & 0 \leq \alpha(x) \leq \pi, \quad x \in (0,\infty), \\ \alpha(0) = 0, \quad \alpha(\infty) = \pi, \end{cases}$$

where $h \in C^\infty(\mathbb{R})$ is an even function. It’s easy to verify that if $\alpha(x)$ is a solution to (BVP), then

$$\pi(x) = \begin{cases} \alpha(x), & x \in [0,\infty), \\ -\alpha(-x), & x \in (-\infty,0) \end{cases} \quad (1.7)$$
is a solution to (1.4)-(1.6).

Now we can state our main theorem.

**Theorem 1.1.** Suppose \( h(x) \in C_0^\infty(\mathbb{R}) \) is an even function. Then we have

- If \( \lambda = 0 \), the (BVP) admits a solution if and only if \( \omega < \lambda = 0 \);
- If \( \lambda < 0 \), the (BVP) admits a solution if and only if \( \omega \leq \lambda \);
- If \( \lambda > 0 \), the (BVP) admits a solution with \( \omega < 0 \). Moreover, if \( h'(x) \leq 0, \forall x \in [0,\infty) \), then the (BVP) admits a solution if and only if \( \omega < 0 \).

**Remark 1.1.** It’s trivial to see that Theorem 1.1 also holds true by replacing the condition \( h(x) \in C_0^\infty(\mathbb{R}) \) by \( h(x) = h_0(x) + C \), where \( h_0 \in C_0^\infty(\mathbb{R}) \) and \( C \) is a constant.

An immediate consequence of Theorem 1.1 and Remark 1.1 is the following theorem, also obtained by Chen in [2].

**Theorem 1.2.** If \( h \) is a constant function, then (BVP) admits a solution if and only if \( \omega < 0 \) and \( \omega \leq \lambda \). Moreover, the solution to (BVP) is unique.

**Proof.** It is obvious that we only need to prove the uniqueness of the solutions to (BVP). If \( a(x) \) is a solution to (BVP), multiplying the both sides of (1.4) by \( a'(x) \) and integrating from \( x_1 \) to \( x_2 \), we have

\[
(a'(x_2))^2 - (a'(x_1))^2 = 2G(a(x_2)) - 2G(a(x_1)),
\]

where

\[
G(a(x)) = \frac{\lambda}{2} \sin^2(a(x)) - \omega \left( 1 + \cos(a(x)) \right).
\]

Since \( a(x) \) is a solution to (BVP), we can choose a sequence \( x_k \to \infty \) such that

\[
\lim_{k \to \infty} a(x_k) = \pi \quad \text{and} \quad \lim_{k \to \infty} a'(x_k) = 0.
\]

Replacing \( x_2, x_1 \) by \( x_k, 0 \) respectively and letting \( k \to \infty \), we have

\[
(a'(0))^2 = 2G(0) = -4\omega > 0.
\]

Since \( a'(0) \geq 0 \), we obtain \( a'(0) = \sqrt{-4\omega} \). It follows that the solution to (BVP) is unique. \( \square \)

This paper is organized as follows. Some important facts such as the well-posedness of initial value problem to (1.4), the blow-up analysis and the nonexistence of solutions to (BVP) are presented in section 2. In section 3, we establish the theorem of existence of solutions to (BVP) by the variational method. Finally, we will employ shooting target method to prove Theorem 1.1 in section 4.
2 Some preliminaries

In this section, we collect some important facts that will be used later. First, we establish the well-posedness of initial value problem (IVP). Then we show the behavior of the solutions to (IVP) as initial data goes to infinity. Finally, we prove a lemma on the nonexistence of solutions to (BVP).

2.1 Initial value problem

Let us consider the following initial value problem

\[
(\text{IVP}) \begin{cases} 
\alpha''(x) + h'(x)\alpha'(x) = g(\alpha(x)), & x \in (0, \infty), \\
\alpha(0) = 0, & \alpha'(0) = a, \quad a \in \mathbb{R},
\end{cases}
\]

where \( g(\cdot) = \lambda \sin(\cdot) \cos(\cdot) + \omega \sin(\cdot) \) and \( h \in C_0^\infty(\mathbb{R}) \).

Theorem 2.1. The (IVP) admits a unique smooth solution \( \alpha_a(x) \). Moreover, continuous dependence on initial values of the solutions holds, i.e., fix \( a_0 \in \mathbb{R} \) and \( R_0 > 0 \), \( \forall \varepsilon > 0 \) and \( k \in \mathbb{N} \), there exists \( \delta > 0 \) such that

\[
||\alpha_a - \alpha_{a_0}||_{C^k([0,R_0])} \leq \varepsilon, \quad \forall \alpha \in (a_0 - \delta, a_0 + \delta).
\]

Proof. For \( g \in C^\infty(\mathbb{R}) \), the theorem follows from the standard well-posedness theory of second-order ordinary differential equation (see Chapter 1 in [3]). \( \square \)

2.2 The blow-up analysis

In this subsection, we will give a characterization of the solutions to IVP as initial value \( a \to \infty \) by blow-up analysis.

First, suppose \( \alpha(x) \) is a solution to (IVP), denote

\[
\tilde{\alpha}(x) = \alpha\left(\frac{1}{a}x\right), \quad a > 0.
\]

Then, \( \tilde{\alpha}(x) \) satisfies following equation

\[
\tilde{\alpha}''(x) + \frac{1}{a}h'( \frac{1}{a} x ) \tilde{\alpha}'(x) = \frac{1}{a^2} g(\tilde{\alpha}(x)), \quad x \in (0, \infty)
\]

and initial values

\[
\tilde{\alpha}(0) = 0, \quad \tilde{\alpha}'(0) = 1.
\]

Theorem 2.2. Fix \( R_0 > 0 \) and \( \forall \varepsilon > 0 \), there exists \( a_0 > 0 \) such that for all \( a \in [a_0, \infty) \) the solution \( \tilde{\alpha}(x) \) to (2.1)-(2.2) satisfies

\[
||\tilde{\alpha}(x) - x||_{C^1([0,R_0])} \leq \varepsilon.
\]
Proof. Since \( g(t) = \lambda \sin(t) \cos(t) + \omega \sin(t) \) and \( h \in C_0^\infty(\mathbb{R}) \), there exists a positive constant \( M \) such that
\[
\|g\|_{C(\mathbb{R})} + \|h\|_{C(\mathbb{R})} \leq M. \tag{2.3}
\]
By simple calculation, we can obtain the integral form of VIP,
\[
\tilde{\alpha}(x) = e^{h(0)} \int_0^x e^{-h(s)} dt + \frac{1}{a^2} \int_0^x \int_0^t g(\tilde{\alpha}(s)) e^{h(s) - h(t)} dt ds. \tag{2.4}
\]
Fix \( R_0 > 0 \) and \( \forall \varepsilon > 0 \), for \( h \in C_0^\infty(\mathbb{R}) \), there exists \( \delta > 0 \) such that, for all \( t \in [0, \delta] \),
\[
|e^{-h(t)} - e^{-h(0)}| \leq \frac{1}{2R_0 e^{h(0)}} \varepsilon. \tag{2.5}
\]
From (2.4) and (2.3), we have
\[
\|	ilde{\alpha}(x) - x\|_{C([0, R_0])} \leq \left| e^{h(0)} \int_0^x e^{-h(s)} dt - x \right| + \frac{1}{a^2} \int_0^x \int_0^t g(\tilde{\alpha}(s)) e^{h(s) - h(t)} ds dt
\leq e^{h(0)} \int_0^x e^{-h(s)} dt - e^{-h(0)} dt + \frac{1}{2a^2} Me^{2M} x^2.
\]
Choosing \( a_0 > 0 \) such that
\[
a_0 > \max \left\{ \frac{R_0}{\delta}, \sqrt{\frac{R_0^2}{2} Me^{2M}} \right\},
\]
it immediately follows that, for all \( a \in [a_0, \infty) \),
\[
\|	ilde{\alpha}(x) - x\|_{C([0, R_0])} \leq e^{h(0)} R_0 \frac{1}{2R_0 e^{h(0)}} \varepsilon + \frac{1}{2a^2} Me^{2M} R_0^2 \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]
By a similar argument, we can obtain, up to choosing a bigger \( a_0 \),
\[
\|	ilde{\alpha}'(x) - 1\|_{C([0, R_0])} \leq \varepsilon.
\]
The proof is complete. \( \square \)

### 2.3 Nonexistence of the Solutions

In this section we will establish a lemma about nonexistence of the solutions to (BVP).

**Lemma 2.1.** If there exists \( \delta > 0 \) such that \( g(t) > 0, t \in (\pi - \delta, \pi) \), then (BVP) admits no solutions.

**Proof.** We will prove the lemma by contradiction. Suppose there exists a solution \( \alpha(x) \) to (BVP), then we can find \( R_0 > 0 \) such that
\[
\alpha'(R_0) > 0, \quad \pi - \delta < \alpha(x) < \pi \quad \text{and} \quad h'(x) \equiv 0, \quad x \in [R_0, \infty). \tag{2.6}
\]
Note that for a smooth function $\alpha(x)$ satisfies $\alpha(0) = 0, \alpha(\infty) = \pi$ and $h \in C^\infty_0$, we can find $R'_0 > 0$ such that

$$\alpha(R'_0) = \pi - \delta < \alpha(x) < \pi \quad \text{and} \quad h'(x) \equiv 0, \quad x \in [R'_0, \infty).$$

By choosing $R'_1 \in (R'_0, \infty)$, we have $\alpha(R'_0) < \alpha(R'_1)$. By mean value theorem, there exists a $R_0 \in (R'_0, R'_1)$ such that

$$\alpha'(R_0) = \frac{\alpha(R'_1) - \alpha(R'_0)}{R'_1 - R'_0} > 0.$$

For $\alpha(x)$ satisfies equation (1.4), we have that, for $x \in [R_0, \infty)$,

$$\alpha'(x) = \alpha'(R_0) + \int_{R_0}^{x} g(\alpha(t)) dt \geq \alpha'(R_0) > 0.$$

It implies that

$$\alpha(x) = \alpha(R_0) + \int_{R_0}^{x} \alpha'(t) dt \geq \alpha(R_0) + \alpha'(R_0)(x - R_0) \to \infty \quad \text{as} \quad x \to \infty,$$

which contradicts the fact $\lim_{x \to \infty} \alpha(x) = \pi$. The lemma is proved.

Thus we immediately obtain the following result on nonexistence of solutions to (BVP).

**Proposition 2.1.** The (BVP) admits no solutions with $\omega > \lambda$ or $\omega = \lambda > 0$.

**Proof.** For the case $\omega > \lambda$, note that

$$g'(t) = \lambda \cos 2t + \omega \cos t.$$

Then $g(\pi) = 0, g'(\pi) = \lambda - \omega < 0$ which implies that there exits $\delta > 0$ such that

$$g(t) > 0, \quad t \in (\pi - \delta, \pi).$$

For the case $\omega = \lambda > 0$,

$$g(t) = \lambda \sin t (\cos t + 1) > 0, \quad t \in (0, \pi).$$

From above argument, the Proposition holds true by Lemma 2.1.
3 Existence of solutions to (BVP)

In this section, we will establish theorem of existence of solutions to (BVP) by variational method.

Denote
\[ G(t) = - \int_t^\pi g(s)ds = \frac{\lambda}{2}\sin^2 t - \omega(1 + \cos t). \]  
(3.1)

Then \( G(\pi) = 0. \)

**Theorem 3.1.** Suppose \( G(t) \) satisfies
\[ C^{-1}(\pi - t)^2 \leq G(t) \leq C(\pi - t)^2, \quad t \in [0, \pi], \]
for some positive constant \( C, \) then \( \text{(BVP)} \) admits a solution.

**Proof.** First, define a functional
\[ J_\lambda(\alpha) = \int_0^\infty (\alpha'(x))^2 e^{h(x)}dx + 2 \int_0^\infty G(\alpha(x)) e^{h(x)}dx \]
(3.2)
with respect to variational space
\[ X = \{ \alpha \mid \pi - \alpha \in H^1([0, \infty)) \} \text{ and } \alpha(0) = 0 \leq \alpha(x) \leq \pi, x \in (0, \infty) \}, \]
(3.3)
then it is easy to verify that above definition is well defined. Because, for all \( \alpha \in X, \) we have
\[ J_\lambda(\alpha) = \int_0^\infty ((\pi - \alpha(x))')^2 e^{h(x)}dx + 2 \int_0^\infty G(\alpha(x)) e^{h(x)}dx \]
\[ \leq C \int_0^\infty ((\pi - \alpha(x))')^2 dx + C \int_0^\infty (\pi - \alpha(x))^2 dx < \infty. \]
As \( J_\lambda(\alpha) \geq 0, \forall \alpha \in X, \) we can choose a minimizing sequence \( \alpha_k \subseteq X \) such that
\[ \inf_{\alpha \in X} J_\lambda(\alpha) = \lim_{k \to \infty} J_\lambda(\alpha_k). \]
Since
\[ \int_0^\infty ((\pi - \alpha_k(x))')^2 dx + \int_0^\infty (\pi - \alpha_k(x))^2 dx \]
\[ \leq C \int_0^\infty (\alpha_k'(x))^2 e^{h(x)}dx + C \int_0^\infty G(\alpha_k(x)) e^{h(x)}dx \]
\[ \leq C J_\lambda(\alpha_k) \leq C < \infty, \]
thus \( \{ \pi - \alpha_k \mid k \in \mathbb{N} \} \) is bounded in \( H^1([0,\infty)) \). Then there exists \( \pi - \hat{\alpha} \in H^1([0,\infty)) \) such that

\[
\pi - \alpha_k \rightharpoonup \pi - \hat{\alpha} \quad \text{weakly in} \quad H^1([0,\infty)).
\]

(3.4)

It follows that

\[
\int_0^\infty (\hat{\alpha}')^2 e^h \, dx \leq \lim_{k \to \infty} \int_0^\infty (\alpha_k')^2 e^h \, dx.
\]

(3.5)

By Sobolev embedding theorem, we know that for any fixed \( R > 0 \),

\[
H^1([0,R]) \hookrightarrow C^{1/2}([0,R]).
\]

Up to a subsequence, there holds that

\[
\hat{\alpha}(x) = \lim_{k \to \infty} \alpha_k(x), \quad \forall x \in [0,\infty).
\]

Thus \( \hat{\alpha} \in X \). Moreover, by Fatou’s lemma, we obtain

\[
\int_0^\infty G(\hat{\alpha}) e^h \, dx \leq \lim_{k \to \infty} \int_0^\infty G(\alpha_k) e^h \, dx.
\]

(3.6)

So, combining (3.5) and (3.6), we know

\[
J_h(\hat{\alpha}) = \inf_X J_h.
\]

Now it is time to prove \( \alpha \) satisfies the equation (1.4).

We say \( \varphi \in C_0^\infty(0,\infty) \) is an admissible variational function for \( \hat{\alpha} \), if there exists \( \epsilon > 0 \) such that \( \hat{\alpha} + t \varphi \in X, \forall t \in [0,\epsilon) \).

Since \( \hat{\alpha} \) is the minimal point of \( J_h(\cdot) \), then, for any admissible variational function \( \varphi \), we have

\[
\frac{d}{dt} J_h(\hat{\alpha} + t \varphi) \bigg|_{t=0} \geq 0.
\]

(3.7)

More precisely,

\[
\int_0^{+\infty} \hat{\alpha}' \varphi' e^h \, dx + \int_0^{+\infty} g(\hat{\alpha}) \varphi e^h \, dx \geq 0.
\]

(3.8)

(1). If \( 0 < \hat{\alpha}(x) < \pi, \forall x \in (x_1,x_2) \), then for any \( \varphi \in C_0^\infty(x_1,x_2) \), we have

\[
\int_0^{+\infty} \hat{\alpha}' \varphi' e^h \, dx + \int_0^{+\infty} g(\hat{\alpha}) \varphi e^h \, dx = 0.
\]
Thus \( \hat{a} \) is a weak solution to equation (1.4) on the interval \( (x_1, x_2) \). Applying the standard elliptic estimate, we know \( \hat{a} \in C^\infty(x_1, x_2) \) which satisfies the equation (1.4).

(2) We claim \( \hat{a}(x) < \pi \) on the interval \((0, \infty)\).

If above assertion were false, define

\[ x^* = \inf \{ x \in (0, +\infty) | \hat{a}(x) = \pi \}. \]

From the definition of \( X \), we know \( 0 < x^* < +\infty \). For \( \hat{a} \) is a minimal point of \( f_\alpha \) on \( X \), we obtain \( \hat{a}(x) \equiv \pi \) on the interval \([x^*, \infty)\). By the definition of \( x^* \), we can choose \( \delta > 0 \) such that

\[ 0 < \hat{a}(x) < \pi, \quad \forall x \in (x^* - \delta, x^*). \]  

Thus by the conclusion (1), we know \( \hat{a} \) satisfies the equation (1.4) on the interval \((x^* - \delta, x^*)\) and

\[ \hat{a}'(x^*) = \lim_{x \to x^*} \hat{a}'(x) > 0. \]

Note that if \( \hat{a}'(x^*) = 0 \), by the uniqueness of ODE, we can deduce that \( \hat{a} \equiv \pi \) on \((x^* - \delta, x^*)\) from the condition \( \hat{a}(x^*) = \pi \), which contradicts (3.9).

Let us choose \( \varphi \in C_0^\infty(x^* - \delta, x^* + \delta) \) such that \( \varphi(x^*) = -1 \) and \( \varphi \leq 0 \). Thus \( \varphi \) is an admissible variational function for \( \hat{a} \). By (3.8), we have

\[
0 \leq \int_0^{+\infty} \hat{a}' \varphi' e^h dx + \int_{-\infty}^{0} g(\hat{a}) \varphi e^h dx = \int_{x^* - \delta}^{x^*} \hat{a}' \varphi' e^h dx + \int_{x^* - \delta}^{x^*} g(\hat{a}) \varphi e^h dx = \hat{a}' \varphi e^h_{|_{x^* - \delta}} - \int_{x^* - \delta}^{x^*} \{ \hat{a}'' + h' \hat{a}' - g(\hat{a}) \} \varphi e^h dx = -\hat{a}'(x^*) e^{h(x^*)} < 0.
\]

Thus there exists a contradiction and our assertion is true.

(3) We claim \( \hat{a}(x) > 0 \) on the interval \((0, \infty)\).

If above assertion were false, define

\[ x_* = \sup \{ x \in (0, +\infty) | \hat{a}(x) = 0 \}, \]

and we have \( 0 < x_* < +\infty \). By the definition of \( x_* \) and the conclusion (2), we know \( 0 < \hat{a}(x) < \pi \) on the interval \((x_*, \infty)\), which implies \( \hat{a} \) satisfies the equation (1.4) on the interval \((x_*, \infty)\). So there holds

\[ \hat{a}'(x_*) = \lim_{x \to x_*} \hat{a}'(x) > 0. \]

We can choose \( \delta > 0 \) such that \( 0 < \hat{a}(x) < \pi \) or \( \hat{a}(x) \equiv 0 \) on the interval \((x_* - \delta, x_*)\). In either case, we have

\[ \hat{a}'(x_* -) = \lim_{x \to x_* -} \hat{a}'(x) \leq 0. \]
Let us choose \( \varphi \in C^\infty_0(x^* - \delta, x^* + \delta) \) such that \( \varphi(x^*) = 1 \) and \( \varphi \geq 0 \). Thus \( \varphi \) is an admissible variational function for \( \hat{\alpha} \). By (3.8), we have
\[
0 \leq \int_0^{+\infty} \hat{\alpha}' \varphi' e^h dx + \int_0^{\infty} g(\hat{\alpha}) \varphi e^h dx
= \int_{x^* - \delta}^{x^* + \delta} \hat{\alpha}' \varphi' e^h dx + \int_{x^* - \delta}^{\infty} g(\hat{\alpha}) \varphi e^h dx
= \hat{\alpha}' \varphi e^h|_{x^* - \delta}^{x^* + \delta} + \hat{\alpha}' \varphi e^h|_{x^*}^{x^* + \delta}
= (\hat{\alpha}'(x^*) - \hat{\alpha}'(x^*)) e^{h(x^*)} < 0.
\]

Thus a contradiction occurs and our assertion is true.

Combining (1), (2) and (3), we obtain that \( 0 < \hat{\alpha}(x) < \pi \) satisfies the equation (1.4) on the interval \((0, \infty)\).

Finally, we have to check that \( \hat{\alpha} \) satisfies the boundary condition. Since \( \hat{\alpha} \in X \), we know \( \hat{\alpha}(0) = 0 \) and there exists a sequence \( x_k \to \infty \) such that
\[
\lim_{k \to \infty} \hat{\alpha}(x_k) = \pi. \tag{3.10}
\]

For
\[
C^{-1}(\pi - t)^2 \leq G(t) \leq C(\pi - t)^2, \quad t \in [0, \pi],
\]
there exists \( \delta > 0 \) such that
\[
g(t) = G'(t) < 0, \quad t \in [\pi - \delta, \pi), \tag{3.11}
\]
\[
G(s) > G(t), \quad (s, t) \in [0, \pi - \delta) \times (\pi - \delta, \pi). \tag{3.12}
\]

By (3.10), we can choose \( R_0 > 0 \) such that
\[
\hat{\alpha}(R_0) \in (\pi - \delta, \pi) \quad \text{and} \quad h(x) \equiv 0, \quad x \in [R_0, \infty). \tag{3.13}
\]

We claim that
\[
\hat{\alpha}'(x) > 0, \quad x \in [R_0, \infty).
\]

If above assertion were false, for simplicity, we assume \( \hat{\alpha}'(R_0) \leq 0 \). By (1.4), we have
\[
\hat{\alpha}'(x) = \hat{\alpha}'(R_0) + \int_{R_0}^{x} g(\hat{\alpha}(t)) dt.
\]

Combining (3.11) and (3.13), we obtain \( \hat{\alpha}'(x) < 0 \) on the interval \((R_0, R_0 + \epsilon) \) for some small constant \( \epsilon > 0 \). By (3.10), there exists \( R_1 \in (R_0, \infty) \) such that
\[
\hat{\alpha}'(x) < 0, \quad x \in (R_0, R_1) \quad \text{and} \quad \hat{\alpha}'(R_1) = 0. \tag{3.14}
\]
However, by multiplying the both sides of (1.4) by $\alpha'(x)$ and integrating from $R_0$ to $R_1$, we have

$$\left(\hat{\alpha}'(R_1)\right)^2 = \left(\hat{\alpha}'(R_0)\right)^2 + 2G(\hat{\alpha}(R_1)) - 2G(\hat{\alpha}(R_0)).$$

(3.15)

Since $\hat{\alpha}(R_1) < \hat{\alpha}(R_0)$, by (3.11) and (3.12), we have

$$G(\hat{\alpha}(R_1)) > G(\hat{\alpha}(R_0)),$$

which implies $\hat{\alpha}'(R_1) \neq 0$. Thus we obtain a contradiction and the assertion

$$\hat{\alpha}'(x) > 0, \quad x \in [R_0, \infty),$$

holds true.

It follows from (3.10) that

$$\lim_{x \to \infty} \hat{\alpha}(x) = \pi.$$

So $\hat{\alpha}$ is a solution to the (BVP).

**Proposition 3.1.** The (BVP) admits a solution with $\omega < \lambda \leq 0$ or $\omega < 0 < \lambda$.

**Proof.** It suffices to prove $G(t) = \frac{\lambda}{2} \sin^2 t - \omega(1 + \cos t)$ satisfies

$$C^{-1}(\pi - t)^2 \leq G(t) \leq C(\pi - t)^2, \quad t \in [0, \pi],$$

for some constant $C > 0$, under the condition $\omega < \lambda \leq 0$ or $\omega < 0 < \lambda$.

For

$$G(t) = \frac{\lambda}{2} \sin^2 t - \omega(1 + \cos t) = 2\cos^2 \frac{t}{2} \left(\frac{\lambda}{2} \sin^2 \frac{t}{2} - \omega\right),$$

we obtain $G(t) > 0 = G(\pi)$ on the interval $[0, \pi]$ with $\omega < \lambda \leq 0$ or $\omega < 0 < \lambda$.

Define

$$\psi(t) = \frac{G(t)}{(\pi - t)^2},$$

and we have $\psi(t)$ is a continuous function on the interval $[0, \pi]$ and

$$\psi(t) > 0, \quad \forall t \in [0, \pi).$$

For

$$\lim_{t \to \pi^-} \psi(t) = \frac{\lambda - \omega}{2} > 0,$$

it follows that $\psi(t)$ is a continuous positive function on the interval $[0, \pi]$. Then we have

$$m(\pi - t)^2 \leq G(t) \leq M(\pi - t)^2, \quad \forall t \in [0, \pi],$$

where $m = \inf\{\psi(t) | t \in [0, \pi]\} > 0$ and $M = \sup\{\psi(t) | t \in [0, \pi]\} > 0$.  

$\square$
4 The proof of theorem 1.1

In this section, we will employ shooting target method to prove Theorem 1.1.

First, let us denote the solution to the (IVP) by \( \alpha_a \).

**Lemma 4.1.** Suppose \( \alpha_a \) is the solution to the (IVP) and \( g'(0) < 0 \). Then there exists \( a_0 > 0 \) such that for all \( a \in (0, a_0) \) we have \( x_a \in (0, \infty) \) satisfies

\[
0 < \alpha(x) < \pi, \quad x \in (0, x_a) \quad \text{and} \quad \alpha_a(x_a) = 0.
\]

**Proof.** As \( h \in C_0^\infty(\mathbb{R}) \), there exists \( R_0 > 0 \) such that \( h'(x) \equiv 0, \forall x \in [R_0, \infty) \).

Thus we can rewrite the equation (1.4) as following form

\[
\alpha''(x) + C(x)\alpha(x) = 0, \quad x \in (R_0, R_0 + L), \tag{4.1}
\]

where

\[
C(x) = -\frac{g(\alpha(x))}{\alpha(x)}
\]

is a continuous function.

Since \( g'(0) < 0 \), there exists \( \delta \in (0, \pi) \) such that

\[
\frac{g(t)}{t} \leq \frac{1}{2}g'(0), \quad \forall t \in [-\delta, \delta]. \tag{4.2}
\]

On the other hand, by the dependence of solutions on initial values, we know that there exists \( a_0 > 0 \) such that for any \( a \in (0, a_0) \),

\[
|\alpha_a(x)| \leq \delta < \pi, \quad x \in (0, R_0 + L), \tag{4.3}
\]

where \( L > 0 \) will be determined later. Combining (4.2) and (4.3), we can obtain that \( \forall a \in (0, a_0) \)

\[
C(x) = -\frac{g(\alpha_a(x))}{\alpha_a(x)} \geq -\frac{1}{2}g'(0) > 0, \quad x \in (R_0, R_0 + L).
\]

Comparing Eq. (4.1) with the following equation

\[
u'' + \ell^2 u = 0
\]

with \( \ell = \sqrt{-\frac{1}{2}g'(0)} \), whose solutions have period \( \frac{2\pi}{\ell} \) and infinite zero points in \( [R_0, +\infty) \),

by Sturm-Liouville theorem we conclude that \( \alpha_a(r) \) has at least a zero point in the interval \( [R_0, R_0 + L] \) with \( L > \frac{4\pi}{\ell} \).

Now, define

\[
x_a = \inf\{x | \alpha_a(x) = 0, x \in (0, \infty)\},
\]

it’s easy to verify \( x_a \in (0, \infty) \) for \( a \in (0, a_0) \). Thus, the lemma is proved. \( \square \)
Definition 4.1. \( \alpha_a \) is called a solution of Type (I) to the (IVP) if there exists \( x_a \in (0,\infty) \) such that
\[
\alpha_a(x_a) = \pi, \quad \alpha_a'(x) > 0, \quad \forall x \in [0,x_a].
\]

Now we are in the position to prove Theorem 1.1.

**Proof of Theorem 1.1.** We need to consider three cases in order to prove the theorem.

**Case 1:** \( \lambda = 0 \).

By Proposition 2.1 and Proposition 3.1, we only need to prove the nonexistence of the solutions to the (BVP) with \( \omega = 0 \).

Since \( \lambda = \omega = 0 \), equation (1.4) reduces to the following equation
\[
\alpha''(x) + h'(x)\alpha'(x) = 0, \quad x \in (0,\infty).
\] (4.4)

For \( h \in C_0^\infty(\mathbb{R}) \), there exists \( R_0 > 0 \) such that \( h \equiv 0, \quad x \in [R_0,\infty) \). Thus we obtain the explicit solution to equation (4.4) as follows
\[
\alpha(x) = C(x-R_0) + \alpha(R_0), \quad x \in [R_0,\infty),
\]
where \( C \) is a constant. It is obvious that \( \alpha \) is not a bound function on \( [0,\infty) \) with \( C \neq 0 \), while \( \alpha \) is a constant function on \( [R_0,\infty) \) with \( C = 0 \). Thus there exist no solutions to the (BVP) with \( \omega = 0 \).

**Case 2:** \( \lambda < 0 \).

By Proposition 2.1 and Proposition 3.1, it suffices to prove the existence of the solutions to the (BVP) with \( \omega = \lambda < 0 \). In the following, we will apply shooting target method to find solutions to the (BVP) with \( \omega = \lambda < 0 \).

First, let us define a set
\[
A = \{ a \in (0,\infty) | \alpha_a \text{ is a solution of Type (I) to the (IVP)} \}.
\]

By Theorem 2.1, we know the set \( A \) is an open set. By Theorem 2.2, we obtain the set \( A \) is not empty.

Since \( g'(0) = \lambda + \omega < 0 \) with \( \omega = \lambda < 0 \), by Lemma 4.1, we have
\[
\underline{a} = \inf \{ a \in A \} > 0.
\]

Next, we would like to prove that \( \alpha_{\underline{a}} \) is a solution to the (BVP).

For any \( a \in A \), there exists \( x_a \in (0,\infty) \) such that
\[
\alpha_a(x_a) = \pi, \quad \alpha_a'(x) > 0, \quad \forall x \in [0,x_a].
\]

Let us choose a sequence \( a_k \in A \) such that \( \lim_{k \to \infty} a_k = \underline{a} \). Up to a subsequence, we know there exists \( x^* \in (0,\infty) \) such that
\[
\lim_{k \to \infty} x_{a_k} = x^*.
\]
If \( x^* \in (0, \infty) \), by Theorem 2.1, we obtain
\[
\alpha_\pm(x^*) = \pi, \quad \alpha_\pm'(x) \geq 0, \quad x \in [0, x^*].
\]
Since \( \alpha_\pm(x) \) satisfies equation (1.4), we have
\[
e^h(x) \alpha_\pm'(x) = e^h(x^*) \alpha_\pm'(x^*) - \int_x^{x^*} g(\alpha_\pm(t))e^h(t)dt, \quad x \in [0, x^*].
\]
(4.5)
Since \( \pi \) is a constant solution to equation (1.4), we have
\[
\alpha_\pm'(x^*) > 0.
\]
Moreover, \( g(t) = \lambda \sin t \cos t + \omega \sin t < 0 \) with \( \lambda = \omega < 0 \), by (4.5), we obtain
\[
\alpha_\pm'(x) > 0, \quad x \in [0, \pi].
\]
This is to say that \( \alpha_\pm \in A \) which contradicts the fact that \( A \) is an open set. Thus we obtain
\[
x^* = \infty,
\]
and
\[
\alpha_\pm'(x) \geq 0, \quad x \in (0, \infty).
\]
(4.6)
It immediately follows that \( \lim_{x \to \infty} \alpha_\pm(x) \) exists, denoted by \( l \). Since \( g > 0 \), thus
\[
l \in (0, \pi].
\]
If \( l \in (0, \pi) \), then \( g(l) < 0 \) with \( \lambda = \omega < 0 \). By equation (1.4), we have
\[
\alpha_\pm'(x) = \alpha_\pm'(R_0) + \int_{R_0}^{x} g(\alpha_\pm(t))dt, \quad x \in [R_0, \infty)
\]
\[
\leq \alpha_\pm'(R_0) + \frac{1}{2}g(l)(x - R_0) \to -\infty \quad \text{as} \quad x \to \infty,
\]
(4.7)
where \( R_0 > 0 \) is large enough such that
\[
h(x) \equiv 0 \quad \text{and} \quad g(\alpha_\pm(t)) \leq \frac{1}{2}g(l) < 0, \quad x \in [R_0, \infty).
\]
The conclusion in (4.7) contradicts the fact in (4.6). Thus \( l = \pi \), i.e.
\[
\lim_{x \to \infty} \alpha_\pm(x) = \pi.
\]
So, \( \alpha_\pm \) is a solution to the (BVP).

**Case 3:** \( \lambda > 0 \).
By Propositions 2.1 and 3.1, we only need to show that the (BVP) admits no solutions under the condition \(0 \leq \omega < \lambda\) and \(h'(x) \leq 0, x \in [0, \infty)\). By (1.4), we have, for any \(0 \leq s < t < \infty\)

\[
e^{2h(t)}(\alpha'(t))^2 - e^{2h(s)}(\alpha'(s))^2 = \int_s^t g(\alpha(x))\alpha'(x)e^{2h(x)}dx
\]

\[
= 2\left[G(\alpha(t))e^{2h(t)} - G(\alpha(s))e^{2h(s)}\right] - 4\int_s^t G(\alpha(x))e^{h(x)}h'(x)dx,
\]

where \(G(\cdot) = \frac{\lambda}{2}\sin^2(\cdot) - \omega(1 + \cos(\cdot))\).

If there exists a solution \(\alpha(x)\) to the (BVP), then we can choose a sequence \(x_k\) such that

\[
\lim_{k \to \infty} \alpha'(x_k) = 0 \quad \text{and} \quad \lim_{k \to \infty} \alpha(x_k) = \pi.
\]

Replacing \(s,t\) by \(0, x_k\) in (4.8), respectively, and letting \(k \to \infty\), then we have

\[
e^{2h(0)}(\alpha'(0))^2 = 2G(0)e^{2h(0)} + 4\int_0^\infty G(\alpha(x))e^{2h(x)}h'(x)dx
\]

\[
= 2G(0)e^{2h(0)} + 4\int_0^{R_0} G(\alpha(x))e^{2h(x)}h'(x)dx,
\]

where \(R_0 > 0\) satisfies \(h(x) \equiv 0, x \in [R_0, \infty)\).

When \(0 \leq \omega < \lambda\), we have

\[
G(t) \geq G(0) = -2\omega, \quad t \in [0, \pi].
\]

By (4.9), we have, for \(h'(x) \leq 0, x \in [0, \infty)\),

\[
0 < e^{2h(0)}(\alpha'(0))^2 = 2G(0)e^{2h(0)} + 4\int_0^{R_0} G(\alpha(x))e^{2h(x)}h'(x)dx
\]

\[
\leq 2G(0)e^{2h(0)} + 4G(0)\int_0^{R_0} e^{2h(x)}h'(x)dx
\]

\[
= 2G(0) = -4\omega \leq 0.
\]

So there exists a contradiction and the (BVP) admits no solutions under the condition \(0 \leq \omega < \lambda\) and \(h'(x) \leq 0, x \in [0, \infty)\).

Based on above arguments, the proof of Theorem 1.1 is completed. \(\Box\)

**Acknowledgments**

The research of Ruiqi Jiang was supported by a grant from the Fundamental Research Funds for the Central Universities; The research of Youde Wang was supported by NSFC (Grant No. 11471316).
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