Optimal Error Estimate of Fourier Spectral Method for the Kawahara Equation

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Received February 13, 2017; Accepted (revised) June 20, 2017

Abstract. An optimal error estimate in $L^2$–norm for Fourier spectral method is presented for the Kawahara equation with periodic boundary conditions. A numerical example is provided to confirm the theoretical analysis. The method and proving skills are also applicable to the periodic boundary problems for some nonlinear dispersive wave equations provided that the dispersive operator is bounded and antisymmetric and commutes with differentiation.

AMS subject classifications: 65M70, 76B15

Key words: Fourier spectral method, Kawahara equation, error estimate.

1 Introduction

We will analyze Fourier spectral method for the Kawahara equation with periodic boundary conditions:

$$
\begin{cases}
\partial_t U + \partial_x F(U) + \partial_x^3 U - \partial_x^5 U = 0, & x \in \mathbb{R}, t \in (0,T], \\
U(x+2\pi,t) = U(x,t), & x \in \mathbb{R}, t \in (0,T], \\
U(x,0) = U_0(x), & x \in \mathbb{R},
\end{cases}
$$

(1.1)

where $U_0$ is $2\pi$-periodic in space and $F(U) = \alpha U + U^2/2$, $\alpha$ is a non-negative real constant. The Kawahara equation, also known as fifth-order Korteweg-de Vries equation, arises in the study of several physical phenomena, such as water waves and plasma physics [1–4]. The Fourier spectral methods for the initial- and periodic boundary-value problems of the Kawahara equation have been studied together with time-stepping methods, e.g., the mixture of integrating factor with fourth-order Runge-Kutta method [5], leapfrog
method [6] and the mixture of exponential time differencing with fourth-order Runge-Kutta method [7]. For non-periodic boundary-value problems of the equation, a fully discrete Crank-Nicolson leapfrog dual-Petrov-Galerkin scheme was used in [8, 9].

The semi-discrete Fourier spectral method for (1.1) is to find $u_N(t) \in V_N$ such that for any $v \in V_N$ and $t \in (0, T]$,

$$
\begin{aligned}
&\left( \partial_t u_N(t) + \partial_x P_N F(u_N(t)) + \partial_x^3 u_N(t) - \partial_x^5 u_N(t), v \right) = 0, \\
&\left( u_N(0), v \right) = \left( P_N U_0, v \right).
\end{aligned}
$$

(1.2)

Here $\langle \cdot, \cdot \rangle$ is the inner product $L^2(I)$, $I = (-\pi, \pi)$, $P_N : L^2(I) \rightarrow V_N$ is the Fourier orthogonal projection operator, i.e.,

$$
\langle P_N u - u, v \rangle = 0, \quad v \in V_N,
$$

and the approximation space $V_N$ of the real trigonometric polynomials of degree $N$ is defined by

$$
V_N = \left\{ u(x) = \sum_{k=-N}^{N} a_k e^{ikx} : \overline{a_k} = a_{-k}, \quad -N \leq k \leq N \right\}.
$$

This semidiscrete scheme was considered in [6] and an optimal error estimate was claimed. The estimate in [6] was based on the optimal estimate of the projection (2.6). However, an optimal estimate of the numerical solution cannot be obtained even though the error estimate for the projection is optimal, see Remark 2.1. Here we present a new projection, see (2.4) and obtain optimal error estimates for both our projection and the numerical solution.

We will focus on optimal error estimate in $L^2$-norm of the semi-discrete Fourier spectral method for (1.1). A fully discrete scheme can be obtained when we discretize in time the semi-discrete scheme using the second-order leapfrog-Crank-Nicolson method [10, 11] and optimal error estimate can also be obtained in space. We will not elaborate on the estimate for the fully discrete scheme since the estimate can be done similarly as in [12].

In Section 2, we give some lemmas and theorems needed in the error estimate. In Section 3, we analyze the stability and convergence of the semi-discrete Fourier spectral method. In Section 4, we present a numerical example for the Kawahara equation showing the accuracy of the fully discrete Fourier spectral method in space and time.

## 2 Preliminaries

In this section, we give some lemmas and theorems needed in the error estimate. Throughout this article, $C$ denotes a generic positive constant, independent of $N$.

Let $I = (-\pi, \pi)$. The inner product and norm of $L^2(I)$ are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. For any non-negative real number $r$, we denote the usual Sobolev space by $H^r(I)$. The subspace of $H^r(I)$ consisting of all periodic functions of period $2\pi$ is denoted
by $H^r_p(I)$. The norm and semi-norm of $H^r_p(I)$ are denoted by $\| \cdot \|_r$ and $| \cdot |_r$, respectively. These norms are defined by

$$\| u \|_r = \left\{ \sum_{k=-\infty}^{\infty} (1+|k|^2)^r |a_k|^2 \right\}^{1/2}, \quad |u|_r = \left\{ \sum_{k=-\infty}^{\infty} |k|^{2r} |a_k|^2 \right\}^{1/2},$$

where

$$u(x) = \sum_{k=-\infty}^{\infty} a_k e^{ikx}, \quad a_k = \frac{1}{2\pi} \int_I u(x) e^{-ikx} dx.$$

Let $N$ be a positive integer.

**Lemma 2.1 ([13, 14]).** If $0 \leq \mu \leq r$ and $u \in H^r_p(I)$, then

$$\| P_N u - u \|_\mu \leq CN^{\mu-r} |u|_r, \quad (2.1)$$

$$|u|_\mu \leq C |u|_r^{\frac{\theta}{r}} |u|_1^{1-\frac{\theta}{r}}, \quad \theta = \frac{\mu}{r}. \quad (2.2)$$

If $u \in H^1_p(I)$, then

$$\| u \|_{L^\infty(I)} \leq C |u|_1^{1/2} |u|_1^{1/2}. \quad (2.3)$$

**Lemma 2.2 ([12]).** Suppose

(i) the functions $E(t)$ and $\rho(t)$ are non-negative and continuous on $[0,T]$, $\rho(t)$ is increasing on $[0,T]$, and $M, C$ are positive constants;

(ii) for any $t \in (0,T]$, if $\max_{0 \leq s \leq t} E(s) \leq M$, then $E(t) \leq \rho(t) + C \int_0^t E(s) ds$;

(iii) $E(0) \leq 0$ and $\rho(T)e^{CT} \leq M$.

Then $E(t) \leq \rho(t)e^{CT}$ for any $t \in (0,T]$.

**Theorem 2.1.** Suppose $C_0$ is a constant from (2.2), $U \in C^1(I)$ and $F(U) = aU + U^2/2$, $a \geq 0$ is a real constant. There exist constants $K \geq \frac{1}{2} \| \partial_x U \|_\infty + \frac{3}{2} N^{-1} C_0^2 \| F'(U) \|_\infty^2$ and $N_0$ depending on $U$, such that for $N \geq N_0$, $P^*_N: H^3_p(I) \to V_N$ is the projection defined by

$$\eta = u - P^*_N u,$$

$$(K\eta + F'(U) \partial_x \eta, v) + (\partial_x \eta, \partial_x^2 v - \partial_t^4 v) = 0, \quad v \in V_N. \quad (2.4)$$

Moreover, let $r \geq 0$, $0 \leq l \leq \min\{5, r\}$. If $u \in H^r_p(I)$, then

$$\| \eta \|_l \leq CN^{l-r} |u|_r. \quad (2.5)$$
Remark 2.1. For the Kawahara equation \((F(U) = U^2/2)\), a different projection was considered in [6]. The projection \(W \in V_N\) is defined by

\[
\begin{align*}
(\partial_t W + U \partial_x W + \partial^3_x W - \partial^5_x W, v) &= 0, \quad t \in (0,T], \, v \in V_N, \\
W(x,0) &= P_N U_0.
\end{align*}
\]  

(2.6)

Even though the optimal estimate for this projection is proved (see Lemma 3.3 in [6]), the error estimate for the numerical solution is not optimal. Let us present below the key steps for the proof of Lemma 3.3 in [6]. Let \(\eta = P_N U - W\), then \(\eta\) satisfies the error equation

\[
(\partial_t \eta + 3^2_x \eta - 5^2_x \eta, v) = -(U \partial_x U - U \partial_x W, v).
\]

It is observed in [6] that

\[
U \partial_x U - U \partial_x W = (U - P_N U) \partial_x U + (P_N U - W) \partial_x U + W \partial_x U - U \partial_x W
\]

and taking \(v = \eta\) in the error equation yields

\[
\frac{1}{2} \frac{d}{dt} \|\eta\|^2 \leq \|\partial_x U\|_{\infty} \|U - P_N U\| \|\eta\| + \|\partial_x U\|_{\infty} \|\eta\|^2 + \|U\|_{\infty} \|\eta\| \|\partial_x W\|.
\]

But the last two terms in the above inequality cannot be bounded by an optimal estimate. Instead, one can write

\[
U \partial_x U - U \partial_x W = U \partial_x (U - P_N U) + U \partial_x \eta
\]

and take \(v = \eta\), then using integration by parts, periodicity and (2.1) yields

\[
\frac{1}{2} \frac{d}{dt} \|\eta\|^2 = - \left( U \partial_x (U - P_N U), \eta \right) + \left( \frac{1}{2} \partial^2_x U, \eta^2 \right)
\]

\[
\leq \|U\|_{\infty} \|\partial_x (U - P_N U)\| \|\eta\| + \frac{1}{2} \|\partial_x U\|_{\infty} \|\eta\|^2
\]

\[
\leq CN^{2-2r} + C \|\eta\|^2.
\]

Since \(\eta(x,0) = 0\), it follows from Gronwall’s lemma that

\[
\max_{0 \leq t \leq T} \|\eta\| \leq CN^{1-r}.
\]

Again, an optimal error estimate cannot be obtained.

Proof of Theorem 2.1. The idea of proof is similar to that in Wahlbin [15]. The proof is based on several steps. The first step is to show the existence and uniqueness. Then the second step is to establish an estimate and then in the following step we refine the estimate using the duality argument.
Putting all the estimates in (2.8), we have
\[ (K\eta + F'(U)\partial_x\eta + \partial_x^3\eta, v - N^{-1}\partial_x v) - (\partial_x^3\eta, \partial_x^2(v - N^{-1}\partial_x v)) = 0, \quad v \in V_N. \] (2.7)

For any \( \phi \in H^3_0(\Omega) \),
\[ \left( K\phi + F'(U)\partial_x\phi + \partial_x^3\phi, \phi - N^{-1}\partial_x\phi \right) - \left( \partial_x^3\phi, \partial_x^2(\phi - N^{-1}\partial_x\phi) \right) \geq K\|\phi\|^2 + N^{-1}(\|\partial_x^3\phi\|^2 + \|\partial_x^3\phi\|^2) - |(\partial_x u, 1/2\phi^2)| - |N^{-1}(F'(U), (\partial_x \phi)^2)|. \] (2.8)

The last two terms in the above inequality can be bounded as follows.
\[ \left| \left( \partial_x u, 1/2\phi^2 \right) \right| \leq \frac{1}{2}||\partial_x U||\|\phi\|^2. \]

Using (2.2) and Young’s inequality \( ab \leq a^p/p + b^q/q, a, b > 0, p, q > 1, 1/p + 1/q = 1 \) gives
\[ |N^{-1}(F'(U), (\partial_x \phi)^2)| \leq N^{-1}\|F'(U)\|\|\partial_x \phi\|^2 \leq N^{-1}\|F'(U)\|\|\partial_x \phi\|^2 \leq N^{-1}(\frac{2}{3}C_0\|F'(U)\|^{3/2}\|\phi\|^2 + \frac{1}{2}\|\partial_x^3\phi\|^2). \]

Putting all the estimates in (2.8), we have
\[ \left( K - \frac{1}{2}||\partial_x U|| - \frac{2}{3}N^{-1}C_0\|F'(U)\|^{3/2} \right)\|\phi\|^2 + N^{-1}(\|\partial_x^3\phi\|^2 + \frac{2}{3}\|\partial_x^3\phi\|^2) \leq K\phi + F'(U)\partial_x\phi + \partial_x^3\phi, \phi - N^{-1}\partial_x\phi \right) - \left( \partial_x^3\phi, \partial_x^2(\phi - N^{-1}\partial_x\phi) \right). \] (2.9)

Hence, if
\[ K \geq \frac{1}{2}||\partial_x U|| + \frac{2}{3}N^{-1}C_0\|F'(U)\|^{3/2}, \]
then
\[ N^{-1}\|\partial_x^3\phi\|^2 + \|\phi\|^2 \leq C \left( (K\phi + F'(U)\partial_x\phi + \partial_x^3\phi, \phi - N^{-1}\partial_x\phi \right) - \left( \partial_x^3\phi, \partial_x^2(\phi - N^{-1}\partial_x\phi) \right), \] (2.10)

where \( C_0 \) is a constant from (2.2), and
\[ \frac{1}{C} = \min \left( \frac{2}{3}K - \frac{1}{2}||\partial_x U|| - \frac{2}{3}N^{-1}C_0\|F'(U)\|^{3/2} \right). \]
Hence, the form \( \phi, \psi \mapsto (K\phi + F'(U)\partial_x\phi + \partial_x^3\phi, \psi - N^{-1}\partial_x\psi) - (\partial_x^3\phi, \partial_x^5(\psi - N^{-1}\partial_x\psi)) \) is positive definite. By Lax-Milgram Theorem, the solution \( P_N^*u \) of the equation (2.4) exists and is unique.

**Step 2.** Next prove (2.5). By periodicity, we obtain from (2.4) that

\[
(K\eta + F'(U)\partial_x\eta + \partial_x^3\eta - \partial_x^5\eta, v - N^{-5}\partial_x^5\psi) = 0, \quad v \in V_N.
\]

(2.11)

A derivation analogous to (2.9) yields that for any \( \phi \in H^5_p(1) \),

\[
\left( K - \frac{1}{2}\|\partial_x U\|_\infty - N^{-5} \left( 54 \left( \frac{C_0\|F'(U)\|_\infty}{5} \right)^{5/2} + \frac{20736}{3125}C_0^5 \right) \right) \|\phi\|^2 + \frac{1}{3}N^{-5}\|\partial_x^5\phi\|^2 \\
\leq (K\phi + F'(U)\partial_x\phi + \partial_x^3\phi - \partial_x^5\phi, \phi - N^{-5}\partial_x^5\phi).
\]

Hence

\[
N^{-5}\|\partial_x^5\phi\|^2 + \|\phi\|^2 \leq C_1((K\phi + F'(U)\partial_x\phi + \partial_x^3\phi - \partial_x^5\phi, \phi - N^{-5}\partial_x^5\phi)),
\]

(2.12)

where

\[
\frac{1}{C_1} = \min \left( \frac{1}{3}K - \frac{1}{2}\|\partial_x U\|_\infty - \frac{54}{N^5} \left( \frac{C_0\|F'(U)\|_\infty}{5} \right)^{5/2} - \frac{20736C_0^5}{3125N^5} \right).
\]

From (2.12) for \( \phi = \eta \) and (2.11) for \( v = P_N^*u - P_Nu \) we get for \( N \) large enough,

\[
N^{-5}\|\partial_x^5\eta\|^2 + \|\eta\|^2 \leq C_1((K\eta + F'(U)\partial_x\eta + \partial_x^3\eta - \partial_x^5\eta, \eta - N^{-5}\partial_x^5\eta) \\
= C_1(K\eta + F'(U)\partial_x\eta + \partial_x^3\eta - \partial_x^5\eta, u - P_Nu - N^{-5}\partial_x^5(u - P_Nu)).
\]

Using (2.2) and Young’s Inequality, we have

\[
\|\partial_x\eta\| \leq C_0\|\eta\|^{4/5}\|\partial_x^5\eta\|^{1/5} \leq C_0\left( \frac{4}{5}\|\eta\| + \frac{1}{5}\|\partial_x^5\eta\| \right),
\]

and also

\[
\|\partial_x^3\eta\| \leq C_0\|\eta\|^{2/5}\|\partial_x^5\eta\|^{3/5} \leq C_0\left( \frac{2}{3}\|\eta\| + \frac{3}{5}\|\partial_x^5\eta\| \right).
\]

Thus

\[
N^{-5}\|\partial_x^5\eta\|^2 + \|\eta\|^2 \\
\leq C_2 \left( K\|\eta\| + \|F'(U)\|_\infty\|\partial_x\eta\| + \|\partial_x^3\eta\| + \|\partial_x^5\eta\| \right) \left( \|u - P_Nu\| + N^{-5}\|\partial_x^5(u - P_Nu)\| \right) \\
\leq C_2N^{-r}|u|r \left( \|\eta\| + \|\partial_x^5\eta\| \right),
\]

(2.13)

where we used (2.1) and

\[
\|u - P_Nu\| + N^{-5}\|\partial_x^5(u - P_Nu)\| \leq CN^{-r}|u|r.
\]
Then we get
\[ \frac{1}{2} N^{-5} (||\partial^5_x \eta|| + ||\eta||)^2 \leq N^{-5} ||\partial^5_x \eta||^2 + ||\eta||^2 \leq C_2 N^{-r} |u|, (||\eta|| + ||\partial^5_x \eta||), \]
which leads to
\[ ||\partial^5_x \eta|| + ||\eta|| \leq 2C_2 N^{5-r} |u|, \tag{2.14a} \]
\[ ||\eta||^2 \leq N^{-5} ||\partial^5_x \eta||^2 + ||\eta||^2 \leq C_2 N^{-r} |u|, (||\eta|| + ||\partial^5_x \eta||) \leq 2C_2^2 N^{5-2r} |u|^2. \tag{2.14b} \]
Hence
\[ ||\partial^5_x \eta|| \leq 2C_2 N^{5-r} |u|, \quad ||\eta|| \leq \sqrt{2} C_2 N^{5/2-r} |u|. \tag{2.15} \]

**Step 3.** To get an optimal bound for ||\eta||, we apply the duality argument. There exists a constant C_3 such that for N large enough, if \( \eta \in L^2(I) \) and \( \phi \in H^2(I) \) satisfies
\[ (\phi - N^{-5} \partial^5_x \phi, K \psi + F'(U) \partial_x \psi + \partial^3_x \psi - \partial^5_x \psi) = (\eta, \psi), \quad \psi \in H^2(I), \]
then
\[ ||\phi||^5 \leq C_3 ||\eta||. \tag{2.17} \]
Taking \( \psi = \phi \) in (2.16), we have
\[ (\phi - N^{-5} \partial^5_x \phi, K \partial^5_x \phi + F'(U) \partial^5_x \phi + \partial^3_x \phi - \partial^5_x \phi) = (\eta, \phi). \tag{2.18} \]
By (2.12) and (2.18), we have
\[ N^{-5} ||\partial^5_x \phi||^2 + ||\phi||^2 \leq C_1 ||\eta|| ||\phi||. \]
Then
\[ ||\phi|| \leq C_1 ||\eta||, \tag{2.19} \]
\[ N^{-5/2} ||\partial^5_x \phi|| \leq C_1 ||\eta||. \tag{2.20} \]
Taking \( \psi = \partial^5_x \phi \) in (2.16), we have
\[ (\phi - N^{-5} \partial^5_x \phi, K \partial^5_x \phi + F'(U) \partial^5_x \phi + \partial^3_x \phi - \partial^10_x \phi) = (\eta, \partial^5_x \phi). \]
By integration by parts, we have
\[ ||\partial^4_x \phi||^2 + (1 - KN^{-5}) ||\partial^5_x \phi||^2 + \frac{1}{2} N^{-5}(\partial_x(U, \partial^6_x \phi)^2) = (\eta, \partial^5_x \phi) + (\partial_x U \phi + F'(U) \partial_x \phi, \partial^5_x \phi). \tag{2.21} \]
Suppose that for some constant M_1 independent of N,
\[ ||F'(U)||_{\infty} + ||\partial_x U||_{\infty} \leq M_1. \tag{2.22} \]
Thus, by (2.19), we have

\[ (\eta, \partial_5^2 \phi) + (\partial_4 \nu \phi + F'(U) \partial_5 \phi, \partial_4^2 \phi) \leq \left( (1 + CM_1) \| \eta \| + M_1 \| \partial_4 \phi \| \right) \| \partial_5^2 \phi \|, \tag{2.23} \]

and for \( N \) large enough

\[ \| \partial_4^2 \phi \|^2 + (1 - KN^{-5}) \| \partial_5^2 \phi \|^2 + \frac{1}{2} N^{-5} (\partial_4 \nu, \partial_5 \phi)^2 \]

\[ \geq (1 - KN^{-5} - \frac{1}{2} N^{-5} M_1) \| \partial_5^2 \phi \|^2. \tag{2.24} \]

By (2.21), (2.23), (2.24) and (2.19) we find that

\[ (1 - KN^{-5} - \frac{1}{2} N^{-5} M_1) \| \partial_5^2 \phi \|^2 \leq (1 + CM_1) \| \eta \| + M_1 \| \partial_4 \phi \| \| \partial_5^2 \phi \|. \]

Then we have

\[ \| \partial_5^2 \phi \| \leq C_4 \| \eta \| + C_5 \| \partial_4 \phi \|. \tag{2.25} \]

Taking \( \psi = \partial_4 \phi \) in (2.16), we have

\[ (\phi - N^{-5} \partial_5^2 \phi, K \partial_4 \phi + F'(U) \partial_5^2 \phi + \partial_4^2 \phi - \partial_5^2 \phi) = (\eta, \partial_4 \phi). \]

By integration by parts, we have

\[ \| \partial_5^2 \phi \|^2 + \| \partial_4^2 \phi \|^2 \]

\[ = (\eta, \partial_4 \phi) + KN^{-5} (\partial_5^2 \phi, \partial_4 \phi) - (\phi, F'(U) \partial_5^2 \phi) + N^{-5} (\partial_5^2 \phi, F'(U) \partial_5^2 \phi). \]

Thus, by (2.2), (2.19) and (2.20), we have

\[ \| \partial_5^2 \phi \|^2 \leq \| \partial_4 \phi \| (\| \eta \| + KN^{-5} \| \partial_5^2 \phi \|) + M_1 (\| \phi \| + N^{-5} \| \partial_5^2 \phi \|) \| \partial_4^2 \phi \| \]

\[ \leq C_0 \| \partial_5^2 \phi \|^{1/2} \| \phi \|^{1/2} (1 + KN^{-5/2} C_1) \| \eta \| + 2M_1 C_1 \| \eta \| \| \partial_5^2 \phi \| \]

\[ \leq C_0 \| \partial_5^2 \phi \|^{1/2} \| \eta \|^{3/2} + 2M_1 C_1 \| \partial_5^2 \phi \| \| \eta \|. \]

Let \( y = \| \partial_5^2 \phi \|^{1/2} \geq 0 \). Then (2.26) becomes

\[ F(y) = (y^2 - 2M_1 C_1 \| \eta \|) y - C_0 \| \eta \|^{3/2} \leq 0. \]

Hence we conclude that

\[ \| \partial_5^2 \phi \| = y^2 \leq C_7 \| \eta \|, \quad C_7 = \left( 2M_1 C_1 + C_0 / \sqrt{2M_1 C_1} \right). \tag{2.27} \]

Then by (2.2), we have

\[ \| \partial_4 \phi \| \leq C_0 \| \partial_5^2 \phi \|^{1/2} \| \phi \|^{1/2} \leq C_0 \sqrt{C_1 C_7} \| \eta \|. \tag{2.28} \]
Hence, by (2.25) and (2.28), we have $\| \partial_x^5 \phi \| \leq C_8 \| \eta \|$. Thus (2.17) holds.

Now we are ready to obtain an optimal estimate. Taking $\psi = \eta$ in (2.16) and using (2.11) for $v = P_N \phi$, we have by (2.2), (2.1), (2.14a) and (2.17)

$$
\| \eta \|^2 = (K \eta + F(U) \partial_x \eta + \partial_x^5 \eta - \partial_x^5 \eta \phi - P_N \phi - N^{-5} \partial_x^5 (\phi - P_N \phi))
\leq C_9 (\| \eta \| + \| \partial_x^5 \eta \|) N^{-5} \| \phi \|_5
\leq C_9 C_3 N^{-r-5} |u|_r N^{-5} \| \eta \|.
$$

Hence

$$
\| \eta \| \leq C_9 C_3 N^{-r-5} |u|_r.
$$

By (2.2), (2.15) and (2.29), we have (2.5). This ends the proof of Theorem 2.1. \qed

**Lemma 2.3.** Let $r \geq 0$, $0 \leq l \leq \min(5, r)$. If $U \in C^l(0, T; C^l(I)), u \in C^l(0, T; H^l_r(I))$ and $P_N u$ is defined by (2.4), then $\eta(t) = u(t) - P_N u(t)$ satisfies

$$
\| \partial_t \eta \|_l \leq C N^{1-l} (\| \partial_t u \|_r + |u|_r).
$$

**Proof.** First, it is clear that $\eta(t)$ satisfies (2.11) at each $t$. Hence, if we differentiate (2.11) with respect to $t$, then we obtain that for any $v \in V_N$,

$$
(K \partial_t \eta + F'(U) \partial_x \eta + \partial_x^5 \partial_t \eta - \partial_x^5 \partial_t \eta - \partial_t \nabla_x \eta + \partial_t U \partial_x \eta, v - N^{-5} \partial_x^5 v) = 0.
$$

Now, from (2.12) for $\phi = \partial_t \eta$ and (2.31) for $v = \partial_t P_N u - P_N \partial_t u$, we get for $N$ large enough,

$$
N^{-5} \| \partial_x^5 \partial_t \eta \|^2 + \| \partial_t \eta \|^2
\leq C_1 (K \partial_t \eta + F'(U) \partial_x \eta + \partial_x^5 \partial_t \eta - \partial_x^5 \partial_t \eta - \partial_t \nabla_x \eta - N^{-5} \partial_x^5 \partial_t \eta)
\leq C_1 (|\partial_t U \partial_x \eta, \partial_t P_N u - P_N \partial_t u - N^{-5} \partial_x^5 (\partial_t P_N u - P_N \partial_t u)|)
+ C_2 N^{1-r} |\partial_t u|_r (\| \partial_t \eta \| + \| \partial_x^5 \partial_t \eta \|),
$$

where the last term follows by a derivation analogous to (2.13). Using (2.1), (2.5) and (2.2), we have

$$
|\partial_t U \partial_x \eta, \partial_t P_N u - P_N \partial_t u - N^{-5} \partial_x^5 (\partial_t P_N u - P_N \partial_t u)|
\leq (|\partial_x \partial_t U \partial_x \eta, \partial_t P_N u - P_N \partial_t u| + |\partial_t U \partial_x \eta, \partial_t P_N u - P_N \partial_t u|)
+ (|\partial_t U \partial_x \eta, N^{-5} \partial_x^5 (\partial_t P_N u - P_N \partial_t u)|)
\leq (\| \partial_x \partial_t U \|_\infty \| \partial_t \eta \| + \| \partial_t U \|_\infty \| \partial_x \partial_t \eta \|) \| \eta \| + N^{-5} \| \partial_t U \|_\infty \| \partial_x \eta \| \| \partial_x^5 \partial_t \eta \|
\leq C_{10} N^{-r} |u|_r (\| \partial_t \eta \| + \| \partial_x^5 \partial_t \eta \|).
Then by a derivation analogous to (2.14a) and (2.15), we have
\[ \| \partial_t^5 \partial_t \eta \| + \| \partial_t \eta \| \leq C_{11} N^{5-r} \left( |\partial_t u|_r + |u|_r \right). \] (2.32)
Hence
\[ \| \partial_t^5 \partial_t \eta \| \leq C_{11} N^{5-r} \left( |\partial_t u|_r + |u|_r \right). \] (2.33)

Next, replace \( \eta \) in the right hand side of (2.16) by \( \partial_t \eta \) and find \( \phi \in H^5_p(I) \) such that
\[ (\phi - N^{-5} \partial_t^5 \phi, K\psi + F'(U)\partial_x \psi + \partial_t^3 \psi - \partial_t^3 \phi) = (\partial_t \eta, \psi), \quad \psi \in H^5_p(I). \] (2.34)
From (2.34) for \( \psi = \partial_t \eta \) and (2.31) for \( v = P_N \phi \), we have by (2.2), (2.32) and the analogues of (2.17)
\[
\begin{align*}
\| \partial_t \eta \|^2 &= (K \partial_t \eta + F'(U) \partial_x \partial_t \eta + \partial_t^2 \partial_t \eta - \partial_t^2 \partial_t \eta, \phi - P_N \phi - N^{-5} \partial_t^5 (\phi - P_N \phi)) \\
&\quad + (\partial_t U \partial_x \partial_t \eta, P_N \phi - N^{-5} \partial_t^5 P_N \phi) \\
&\leq C_{12} \left( (\| \partial_t \eta \| + \| \partial_t^2 \partial_t \eta \|) N^{-5} \| \phi \|_5 + \| \eta \| (\| \phi \| + \| \partial_x \phi \|) + N^{-5} \| \partial_x \eta \| \| \partial_t^5 \phi \| \right) \\
&\leq C_{13} N^{-r} \left( |\partial_t u|_r + |u|_r \right) \| \phi \|_5 \\
&\leq C_{13} C_3 N^{-r} \left( |\partial_t u|_r + |u|_r \right) \| \partial_t \eta \|.
\end{align*}
\]
Hence
\[ \| \partial_t \eta \| \leq C_{13} C_3 N^{-r} \left( |\partial_t u|_r + |u|_r \right). \] (2.35)
By (2.2), (2.33) and (2.35), we obtain (2.30). This ends the proof of Lemma 2.3. \( \Box \)

3 The stability and convergence of the semi-discrete scheme

In this section we first prove stability of semi-discrete scheme (1.2), and then obtain the convergence of the scheme from the stability.

Assume that \( u_N \) and the term on the right-hand side in (1.2) have errors \( \tilde{u} \) and \( \tilde{f} \in V_N \), respectively. Then by (1.2), we have the error equation
\[
\begin{cases}
(\partial_t \tilde{u}(t) + \partial_x P_N \tilde{F}(t) + \partial_t^2 \tilde{u}(t) - \partial_t^2 \tilde{u}(t) - \tilde{f}(t), v) = 0, & t \in (0, T], \\
(\tilde{u}(0), v) = (\tilde{u}_0, v),
\end{cases}
\] (3.1)
where
\[ \tilde{F} = F(u_N + \tilde{u}) - F(u_N). \]

For given \( t \in [0, T] \), assume
\[ \max_{0 \leq s \leq t} \| \tilde{u}(s) \| \leq M. \]
By Lemma 2.2, we obtain the following stability result.

By Cauchy-Schwarz inequality, we have

\[
1 \frac{d}{dt} \| \tilde{u}(t) \|^2 = (\tilde{f}(t) - \partial_x \tilde{F}(t), \tilde{u}(t)).
\]

Then by integration in time, we have for any \( t \in [0, T] \)

\[
\| \tilde{u}(t) \|^2 = \| \tilde{u}(0) \|^2 + 2 \int_0^t \left( (\tilde{f}(s), \tilde{u}(s)) - (\partial_x \tilde{F}(s), \tilde{u}(s)) \right) ds.
\]  

(3.2)

By Cauchy-Schwarz inequality, we have

\[
\| (\tilde{f}(s), \tilde{u}(s)) \| \leq \frac{1}{2} \| \tilde{f}(s) \|^2 + \frac{1}{2} \| \tilde{u}(s) \|^2.
\]  

(3.3)

Since \( F''(z) = 1 \) and \( F'(u_N + \tilde{u}) = \int_0^1 F''(u_N + \theta \tilde{u}) d\theta \tilde{u} = \tilde{u} \), we observe that

\[
(\partial_x \tilde{F}, \tilde{u}) = (F'(u_N + \tilde{u}) \partial_x u_N + \tilde{u}) - F'(u_N) \partial_x u_N, \tilde{u})
\]

\[
= \left( (F'(u_N + \tilde{u}) - F'(u_N)) \partial_x u_N, \tilde{u} \right) + \left( F'(u_N + \tilde{u}), \partial_x (\tilde{u}^2 / 2) \right)
\]

\[
= (\partial_x u_N, \tilde{u}^2 / 2).
\]

Thus, by (2.3), the embedding theorem, and (A.1), we have

\[
| (\partial_x \tilde{F}, \tilde{u}) | \leq \frac{1}{2} \| \partial_x u_N \|_{\infty} \| \tilde{u} \|^2
\]  

(3.4)

\[
\leq \frac{1}{2} \left( \| \partial_x (u_N - P_N^\infty U) \|_{\infty} + \| \partial_x (P_N^\infty U - U) \|_{\infty} + \| \partial_x U \|_{\infty} \right) \| \tilde{u} \|^2
\]

\[
\leq C \left( \| \partial_x (u_N - P_N^\infty U) \|_{1/2} + \| \partial_x (u_N - P_N^\infty U) \|_{1/2} + \| P_N^\infty U - U \|_2 + \| U \|_2 \right) \| \tilde{u} \|^2
\]

\[
\leq C \left( \| \partial_x (U - P_N^\infty U) \|_{1/2} + \| \partial_x (U - P_N^\infty U) \|_{1/2} + \| P_N^\infty U - U \|_2 + \| U \|_2 \right) \| \tilde{u} \|^2
\]

\[
\leq C \left( N^{3-r} \| U \|_r + \| U \|_2 \right) \| \tilde{u} \|^2.
\]

where if \( r \geq 3 \), then \( \| u_N \|_{C(0, T, C^1(\Omega))} \) is bounded.

Putting (3.3) and (3.4) into (3.2) and defining

\[
E(t) = \| \tilde{u}(t) \|^2, \quad \rho(t) = \| \tilde{u}(0) \|^2 + \int_0^t \| \tilde{f}(s) \|^2 ds,
\]

we have

\[
E(t) \leq \rho(t) + C \int_0^t E(s) ds, \quad 0 < t \leq T.
\]

By Lemma 2.2, we obtain the following stability result.
Theorem 3.1. There exist constants $C$ depending on $K$ in Theorem 2.1 and $\|u_N\|_{C(0,T;C^1(I))}$, and $M$, such that if

$$\rho(t) \leq Me^{-CT},$$

then

$$E(t) \leq \rho(t)e^{CT}, \quad 0 < t \leq T.$$  

Next, consider the convergence of scheme (1.2). Let $w = P_N^* U$, and $e = u_N - w$. Then $u_N - U = e - (U - w)$, and we note that Theorem 2.1 implies that $u_N - U$ can be estimated by estimating $e$. From (1.1), (1.2) and (2.4), we have for any $v \in V_N$

$$\left\{ \begin{array}{ll}
(\partial_t e(t) + \partial_t P_N \tilde{F}(t) + \partial^2_t e(t) - \delta^2_t e(t) - f(t), v) = 0, & 0 < t \leq T, \\
(e(0), v) = (u_N(0) - w(0), v),
\end{array} \right.$$  

where $K$ is the constant of Theorem 2.1,

$$\tilde{F}(t) = F(u_N(t)) - F(w(t)),$$

$$f(t) = K(w(t) - U(t)) + (F'(U(t)) - F'(w(t))) \partial_x w(t) + \partial_t(U(t) - w(t)).$$

We first estimate $f$. Consider $0 \leq s \leq t$. By (2.5), we have

$$\|K(w(s) - U(s))\| \leq CN^{-r}|U(s)|.$$  

Since $F''(z) = 1$ and $F'(U) - F'(w) = \int_0^1 F''(w + \theta(U - w)) d\theta(U - w) = U - w$, we have by the embedding theorem, and (2.5),

$$\|\partial_x w\| \leq \|\partial_x w\| \leq (\|\partial_x (w - U)\| + \|\partial_x U\|) \|\partial_x w\| + \|\partial_t(U - w)\|$$

$$\leq C(\|w - U\|_2 + |U|_2 N^{-r}|U|_r)$$

$$\leq C(N^{2-r} |U|_r + |U|_2 N^{-r} |U|_r),$$

where if $r \geq 2$, then $\|w\|_{C(0,T;C^1(I))}$ is bounded. By (2.30), we have

$$\|\partial_t (U(s) - w(s))\| \leq CN^{-r} |\partial_t U(s)|_r.$$  

Hence

$$\|f(s)\| \leq CN^{-r} (|\partial_t U(s)|_r + |U(s)|_r).$$

For the initial error, we have from (2.1) and (2.5) that

$$\|e(0)\| = \|P_N U(0) - U_0 + U_0 - w(0)\| \leq CN^{-r} |U(0)|_r.$$  

Thus, if apply Theorem 3.1 to (3.5) and use the triangle inequality and (2.5), then we arrive at the following convergence result.
Remark 3.1. The method and proving skills in this paper also can be applied to the periodic boundary problems for some nonlinear dispersive wave equations

Theorem 3.2. Assume that $r \geq 5$, $U \in C^1(0,T;H^r_p(1))$. Then there exist constants $C$ depending on $K$ in Theorem 2.1 and $\|U\|_{C^1(0,T;H^r_p(1))}$, and $M$, such that

$$\|u_N(t) - U(t)\| \leq CN^{-r}, \quad 0 \leq t \leq T.$$ 

provided that the dispersive operator $D$ is bounded $(C_1 \|u\| \leq \|D u\| \leq C_2 \|u\|, s > 1)$ and antisymmetric $(\langle Du, u \rangle = -\langle u, Du \rangle)$ and commutes with differentiation. The projection $P_N^* : H^r_p(1) \to V_N$ is defined by

$$(K(u - P_N^* u) + F(U)\partial_x(u - P_N^* u) + D(u - P_N^* u), v) = 0, \quad v \in V_N.$$ 

4 A numerical example

In this section we first discretize in time the semi-discrete Fourier spectral method (1.2) using the second-order leapfrog-Crank-Nicolson method [8–11], and then give an example for Kawahara equation showing the accuracy of the resulting fully discrete Fourier spectral method in space and time.

Let $\tau$ be time-step and $t_k = k\tau$ ($k = 0, 1, \cdots, m$; $T = m\tau$). For simplicity, denote $w(x, t_k)$ by $w_k$ and set

$$w^k_1 = \frac{1}{2\tau}(w^{k+1} - w^{k-1}), \quad \tilde{w}^k = \frac{1}{2}(w^{k+1} - w^{k-1}).$$

The fully discrete Fourier spectral method for (1.1) is to find $u^k_N \in V_N$ such that for any $v \in V_N$ and $1 \leq k \leq m - 1$,

$$\begin{cases} 
(u^k_N + \partial_x P_N F(u^k_N) + \partial_x^2 a^k_N - \partial_x^2 a_{jN}^k, v) = 0, \\
(u^1_N, v) = (P_N[U_0 + \tau \partial_t U(0)], v), \\
(u^0_N, v) = (P_N U_0, v).
\end{cases} \tag{4.1}$$

Let

$$u^k_N = \sum_{|l| \leq N} a^k_l e^{i lx}, \quad P_N F(u^k_N) = \sum_{|l| \leq N} b^k_l e^{i lx}.$$ 

Then by using the orthogonality of $e^{i lx}$, (4.1) can be solved as

$$a^{k+1}_l = \frac{(1 + i\tau l^3(1 + l^2))a^k_l - i2\tau lb^k_l}{1 - i\tau l^3(1 + l^2)}, \quad |l| \leq N.$$
Now we use the fully discrete Fourier spectral method (4.1) to solve the following Kawa-
hara equation (see [4])
\[
\partial_t U + \partial_x (U + \frac{1}{2} U^2) + \partial_x^3 U - \partial_x^5 U = 0,
\]
the soliton solution of which is
\[
U(x,t) = \frac{105}{169} \text{sech}^4 \left( \frac{1}{2\sqrt{13}} \left( x - \frac{205}{169} t \right) \right).
\]
Take \( x \in [-150,150] \). In Table 1 we list the numerical errors at \( t = 10 \) with various \( N \) and \( \tau \). The results show the second order convergence in time and the spectral accuracy in space of the Fourier spectral method (4.1).

**Table 1**: Errors at \( t=10 \) of the Fourier spectral method.

<table>
<thead>
<tr>
<th>( \tau )</th>
<th>( 2N )</th>
<th>( L^2)-error</th>
<th>( L^\infty)-error</th>
<th>( L^2)-order</th>
<th>( L^\infty)-order</th>
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<td>8.4269e-04</td>
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<tr>
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<td>8.4205e-06</td>
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<td>( \tau^{2.00} )</td>
<td></td>
</tr>
<tr>
<td>1e-3</td>
<td>2.0188e-07</td>
<td>8.4222e-08</td>
<td>( \tau^{2.00} )</td>
<td>( \tau^{2.00} )</td>
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</tr>
<tr>
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<td>1.8861e-02</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1e-3</td>
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<td>4.5556e-05</td>
<td>( N^{-7.92} )</td>
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<td>( N^{-10.17} )</td>
<td>( N^{-9.08} )</td>
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</tr>
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</table>

**Acknowledgments**

The author thanks the referees for their valuable comments and helpful suggestions. The work was supported by the Scientific Research Foundation of Guangxi University (No. XBZ120542), the Guangxi Natural Science Foundation (No. 2013GXNSFBA019021), NSFC(Grant No. 11362002).

**Appendix A**

**Proof of an error estimate in** \( H^1 \) — **norm of the solution** \( P_N^* u \) **of the equation** (2.4) **for any** \( u \in H^3_p(I) \).

Here we derive an estimate in \( H^1 \) — norm of \( \eta = u - P_N^* u \). From (2.10) for \( \phi = \eta \) and (2.7) for \( v = P_N^* u - P_N u \) we get for \( N \) large enough,
\[
N^{-1} \| \partial_x^3 \eta \|^2 + \| \eta \|^2 \\
\leq C \left( (K\eta + F'(U)\partial_x \eta + \partial_x^3 \eta, \eta - N^{-1} \partial_x \eta) - (\partial_x^3 \eta, \partial_x^2 (\eta - N^{-1} \partial_x \eta)) \right) \\
= C \left( K\eta + F'(U)\partial_x \eta + \partial_x^3 \eta, u - P_N u - N^{-1} \partial_x (u - P_N u) \right) \\
- C (\partial_x^3 \eta, \partial_x^2 (u - P_N u - N^{-1} \partial_x (u - P_N u))).
\]
A derivation analogous to (2.13) yields that
\[ N^{-1} \| \partial^3_x \eta \|^2 + \| \eta \|^2 \leq CN^{2-r} |u|_r (\| \eta \| + \| \partial^3_x \eta \|). \]

Then we get
\[ \frac{1}{2} N^{-1} (\| \partial^3_x \eta \| + \| \eta \|)^2 \leq N^{-1} \| \partial^3_x \eta \|^2 + \| \eta \|^2 \leq CN^{2-r} |u|_r (\| \eta \| + \| \partial^3_x \eta \|), \]
which leads to
\[ \| \partial^3_x \eta \| \leq CN^{3-r} |u|_r, \quad \| \eta \| \leq CN^{5/2-r} |u|_r. \]

Hence,
\[ \| \partial^2_x \eta \| \leq CN^{3-r} |u|_r, \quad \| \partial_x \eta \| \leq CN^{3-r} |u|_r. \quad (A.1) \]

This ends the proof of the error estimate.

References