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L^p-REGULARITY FOR A CLASS OF PSEUDODIFFERENTIAL OPERATORS IN $\mathbb{R}^n$

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Abstract We study here a class of pseudodifferential operators with weighted symbols of Shubin type. First, we develop the basic elements of the pseudodifferential calculus for these operators, proving in particular a result of L^p-boundedness. Then we derive regularity results in the frame of suitably defined functional spaces of Sobolev type.
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1. Introduction

In this paper we shall consider a class of pseudodifferential operators, including the parametrices of the linear partial differential operators of Schrödinger type $P(x, D) = -\Delta + V(x)$ where the potential $V(x) = \sum_{\alpha\in\mathcal{R}} a_\alpha x^\alpha$ is a multi-quasi-elliptic polynomial with constant coefficients $a_\alpha \in \mathbb{C}$, with $\mathcal{R}$ complete polyhedron of $\mathbb{R}^n$; cf. [1-8].

We will summarize later the notion of complete polyhedron of $\mathbb{R}^n$ and multi-quasi-ellipticity. However, to be more clear, we give here two explicit examples of such $V(x)$ in two variables (x_1, x_2):

$$V_1(x_1, x_2) = x_1^h + x_2^k, \tag{1}$$

with k, h even positive integers;

$$V_2(x_1, x_2) = x_1^{h_1} + x_1^{k_1} x_2^{k_2} + x_2^{h_2}, \tag{2}$$

with h_i, k_i even positive integers, $h_i > k_i > 0$ $(i = 1, 2)$ and $\frac{k_1}{h_1} + \frac{k_2}{h_2} > 1$.

By arguing in the framework of the pseudodifferential operators with symbol in the weighted Shubin classes (see [9-12]) it can be shown that $u \in L^2(\mathbb{R}^n)$, $\partial^2_{x_j} u \in L^2(\mathbb{R}^n)$ and $x^\alpha u \in L^2(\mathbb{R}^n)$ for any $j = 1, ..., n$ and $\alpha \in \mathcal{R}$, provided that $P(x, D)u \in L^2(\mathbb{R}^n)$, cf. the preceding examples.

In this paper, the above L^2-regularity result is extended to any $1 < p < \infty$ in the context of a calculus of L^p-bounded weighted pseudodifferential operators.

As is well-known, pseudodifferential operators with symbol in the Shubin class $\Gamma^0_{\rho,\Lambda}$, cf. [9, 10], with weight function $\Lambda(x,\xi)$ and $\rho = \rho(\Lambda) \in (0,1]$, are not in general L^p-bounded for $p \neq 2$ (see [13] for instance).

Here we deal with suitable subclasses $M\Gamma^m_{\rho,\Lambda}$ of the Shubin classes which are obtained by assuming an additional Lizorkin-Marcinkiewicz type condition on the decay of the derivatives of the symbol. By means of the Lizorkin-Marcinkiewicz theorem about Fourier multipliers ([14]) the related pseudodifferential operators are L^p-continuous when the order m is zero; cf. [15, 16], for the related results in a local context.

In Sections 2 and 3, the class of the weight functions and the related weighted symbol classes are introduced. The basic elements of the symbolic calculus in this framework are developed and some properties of the weighted-elliptic symbols are also considered.

In Sections 4 and 5 various kind of pseudodifferential operators with symbol in $M\Gamma^m_{\rho,\Lambda}$ and their algebra properties are investigated, while Section 6 is devoted to the construction of the parametrix of an elliptic operator; since here we follow closely the lines of the standard symbolic calculus, cf. [9], the proofs are usually omitted or outlined only.

The main L^p-continuity result for operators of order zero is proved in detail in Section 7 and in Section 8 the action of operators with arbitrary order on a scale of suitably defined weighted Sobolev spaces is derived.

In Section 9 we consider the same operators acting on the functional spaces defined as the images, under the inverse Fourier transform, of the Sobolev spaces in Section 8. The symmetric behavior of symbols in $M\Gamma^m_{\rho,\Lambda}$ with respect to the variables x, ξ leads to the continuity of the related pseudodifferential operators on such spaces.

In Section 10, the regularity results for weighted-elliptic pseudodifferential operators are stated in the frame of all the previous functional spaces; as an example they are applied in particular to the Schrödinger type operators mentioned at the beginning.

We finally observe that, as far as we know, our results are new also in the homogeneous classes of [11, 12], corresponding to the weight $\Lambda(x,\xi) = \sqrt{|x|^2 + |\xi|^2 + 1}$ and having as basic example the harmonic oscillator $-\Delta + |x|^2$; however in this case L^p-regularity of the solutions can be deduced easily from known results, cf. [17].

2. Weight Functions

Definition 2.1 *We say that a positive function $\Lambda(z) \in C^\infty(\mathbb{R}^N)$ is a weight function if the following conditions are fulfilled:*

1. *there exist positive constants $\mu_0 \le \mu_1$ and C such that*

$$C^{-1}(1+|z|)^{\mu_0} \le \Lambda(z) \le C(1+|z|)^{\mu_1}, \tag{3}$$

for all $z \in \mathbb{R}^N$;

2. *for every $t \in \mathbb{R}^N$ there exists $C = C_t > 0$ such that*

$$\Lambda(tz) \le C\Lambda(z), \tag{4}$$

for all $z \in \mathbb{R}^N$ *(where* $tz := (t_1 z_1, ..., t_N z_N)$*);*

3. *there exists* $\mu > 0$ *such that for any* $\alpha \in \mathbb{Z}_+^N$ *and* $\gamma \in \mathbb{Z}_+^N$ *with* $\gamma_j \in \{0,1\}$ *(*$j = 1, ..., N$*)*
$$|z^\gamma \partial^{\alpha+\gamma} \Lambda(z)| \le C_{\alpha,\gamma} \Lambda(z)^{1-\frac{|\alpha|}{\mu}}, \tag{5}$$
for all $z \in \mathbb{R}^N$ *with a suitable* $C_{\alpha,\gamma} > 0$.

For similar definitions of weight function, cf. [9, 10, 18, 19], the peculiarity is here the invariance of the estimate (5) under the application of the operator $z^\gamma \partial^\gamma$ to the weight.

From now on we set for short $\mathbb{K} := \{\gamma \in \mathbb{Z}_+^N : \gamma_j \in \{0,1\} \quad j = 1, ..., N\}$.

Using Faà di Bruno formula, we can easily derive the following result.

Proposition 2.1 *Let* $\Lambda(z)$ *be a weight function. Then for any* $m \in \mathbb{R}$ *and all* $\alpha \in \mathbb{Z}_+^N$, $\gamma \in \mathbb{K}$ *there exists* $C_{\alpha,\gamma} > 0$ *such that*
$$|z^\gamma \partial^{\alpha+\gamma} (\Lambda(z)^m)| \le C_{\alpha,\gamma} \Lambda(z)^{m-\frac{|\alpha|}{\mu}}, \qquad z \in \mathbb{R}^N. \tag{6}$$
As a consequence of the above proposition, any power $\Lambda(z)^m$, with $m \in \mathbb{R}_+$, of a weight function is also a weight function.

An example of weight function according to Definition 2.1 is given by the so-called *quasi-elliptic polynomials*; namely, we take the square root:
$$P_m(z) := \sqrt{1 + \sum_{j=1}^{N} z_j^{2m_j}}, \tag{7}$$
where $m = (m_1, ..., m_N) \in \mathbb{N}^N$ and $\min_{1 \le j \le N} m_j \ge 1$.

In fact the assumptions *1*, *2*, *3* are easily verified; in particular $\langle z \rangle := \sqrt{1 + |z|^2}$ is a weight function in the sense of Definition 2.1.

A more general class of weight functions is given by the *multi-quasi-elliptic polynomials* of Volevich-Gindikin ([8]) defined by
$$\Lambda_{\mathcal{P}}(z) := \sqrt{\sum_{\alpha \in V(\mathcal{P})} z^{2\alpha}}, \tag{8}$$
where $\mathcal{P}$ is a *complete polyhedron* of $\mathbb{R}^N$.

Namely $\mathcal{P}$ is the convex hull of a finite set $V(\mathcal{P}) \subset \mathbb{N}^N$ of convex-linear independent points, called *vertices* of $\mathcal{P}$ which are uniquely determined by $\mathcal{P}$ itself. Since $\mathcal{P}$ has non empty interior, it is completely described by
$$\mathcal{P} = \{z \in \mathbb{R}^N; \nu \cdot z \ge 0, \forall \nu \in \mathcal{N}_0(\mathcal{P})\} \cap \{z \in \mathbb{R}^N; \nu \cdot z \le 1, \forall \nu \in \mathcal{N}_1(\mathcal{P})\},$$
where $\mathcal{N}_0(\mathcal{P}) \subset \{\nu \in \mathbb{R}_+^N; |\nu| = 1\}$, $\mathcal{N}_1(\mathcal{P}) \subset \mathbb{R}_+^N$ are finite sets and as usual $\nu \cdot z = \sum_{j=1}^N \nu_j z_j$.

We assume moreover that for $j = 1, \dots, N$:

i) $(0,\dots,0) \in V(\mathcal{P})$, and $V(\mathcal{P}) \neq \{(0,\dots,0)\}$;

ii) $\mathcal{N}_0(\mathcal{P}) = \{e_1,\dots,e_N\}$ with $e_j = (0,\dots,1_{j-th\ entry},\dots 0) \in \mathbb{R}^N_+$;

iii) every $\nu \in \mathcal{N}_1(\mathcal{P})$ has strictly positive components ν_j, $(j = 1,...,N)$.

Let us notice that the boundary of $\mathcal{P}$, $\mathcal{F}(\mathcal{P})$, consists of faces which are the convex hull of the vertices of $\mathcal{P}$ lying on the hyperplane H_ν orthogonal to $\nu \in \mathcal{N}_0(\mathcal{P}) \cup \mathcal{N}_1(\mathcal{P})$, i. e. :

$$\nu \cdot z = 0 \quad \text{if} \quad \nu \in \mathcal{N}_0(\mathcal{P}), \qquad \nu \cdot z = 1 \quad \text{if} \quad \nu \in \mathcal{N}_1(\mathcal{P}).$$

When $\mathcal{P}$ is a complete polyhedron, $\Lambda_{\mathcal{P}}(z)$ satisfies the assumptions *1*, *2* and *3* with $\mu_0 = \min_{\substack{\alpha \in V(\mathcal{P}) \\ \alpha \neq 0}} |\alpha|$, $\mu_1 = \max_{\alpha \in V(\mathcal{P})} |\alpha|$ and $\mu = \max \left\{ \nu_j^{-1}; \nu \in \mathcal{N}_1(\mathcal{P}),\ j = 1,...,N \right\}$ the *formal order* of the multi-quasi-elliptic polynomial, cf. [8].

The following properties of a weight function $\Lambda(z)$ are easy consequence of the assumptions *1-3* in the above definition.

Proposition 2.2 *If $\Lambda(z)$ is a weight function, there exist positive constants C and ϵ such that for all $\xi, \eta \in \mathbb{R}^N$*

1. $|\Lambda(\xi)^{\frac{1}{\mu}} - \Lambda(\eta)^{\frac{1}{\mu}}| \le C|\xi - \eta|$;
2. $C^{-1} \le \frac{\Lambda(\xi)}{\Lambda(\eta)} \le C, \qquad$ *as* $\quad |\xi - \eta| < \epsilon \Lambda(\xi)^{\frac{1}{\mu}}$;
3. $\Lambda(\xi) \le C(1 + |\xi - \eta|)^\mu \Lambda(\eta)$.

Remark 1 From assumption *1* in Definition 2.1 and Proposition 2.2 it follows that $\mu_0 \le \mu_1 \le \mu$.

3. Classes of Symbols

Let $\Lambda(z)$ be a weight function according to Definition 2.1. The following weighted classes of Shubin type related to $\Lambda(z)$ are considered (see [9-12, 19].)

Definition 3.1 *Let $m \in \mathbb{R}$ and $0 < \rho \le \frac{1}{\mu}$. $\Gamma^m_{\rho,\Lambda}$ is then the class of all the functions $a(z) \in C^\infty(\mathbb{R}^N)$ such that for all $\alpha \in \mathbb{Z}^N_+$*

$$|\partial^\alpha a(z)| \le C_\alpha \Lambda(z)^{m - \rho|\alpha|}, \qquad z \in \mathbb{R}^N. \tag{9}$$

We introduce now suitable subclasses of $\Gamma^m_{\rho,\Lambda}$ whose related pseudodifferential operators will be L^p-continuous for all $1 < p < \infty$ when the order m is zero.

Definition 3.2 *For $m \in \mathbb{R}$ and $0 < \rho \le \frac{1}{\mu}$, $M\Gamma^m_{\rho,\Lambda}$ is the class of the functions $a(z) \in C^\infty(\mathbb{R}^N)$ such that*

$$z^\gamma \partial^\gamma a(z) \in \Gamma^m_{\rho,\Lambda}, \tag{10}$$

for any $\gamma \in \mathbb{K}$.

We list some propositions; they are obvious variants of the standard symbolic calculus and proofs are omitted.

Proposition 3.1 *The following statements are equivalent*

1. $a(z) \in M\Gamma^m_{\rho,\Lambda}$;
2. $z^\gamma \partial^{\alpha+\gamma} a(z) \in \Gamma^{m-\rho|\alpha|}_{\rho,\Lambda}$ *for all* $\alpha \in \mathbb{Z}^N_+$ *and* $\gamma \in \mathbb{K}$;
3. *for every* $\alpha \in \mathbb{Z}^N_+$ *and* $\gamma \in \mathbb{K}$
$$|z^\gamma \partial^{\alpha+\gamma} a(z)| \le C_{\alpha,\gamma} \Lambda(z)^{m-\rho|\alpha|}, \qquad z \in \mathbb{R}^N. \tag{11}$$

Proposition 3.2 *For* $\Lambda(z)$ *weight function,* $m, m' \in \mathbb{R}$, $0 < \rho' \le \rho \le \frac{1}{\mu}$ *the following properties hold*

1. *if* $m \le m'$ *then* $M\Gamma^m_{\rho,\Lambda} \subset M\Gamma^{m'}_{\rho',\Lambda}$;
2. *if* $a(z) \in M\Gamma^m_{\rho,\Lambda}$ *and* $b(z) \in M\Gamma^{m'}_{\rho,\Lambda}$ *then* $(ab)(z) \in M\Gamma^{m+m'}_{\rho,\Lambda}$;
3. *if* $a(z) \in M\Gamma^m_{\rho,\Lambda}$ *then* $\partial^\alpha a(z) \in M\Gamma^{m-\rho|\alpha|}_{\rho,\Lambda}$, *for any* $\alpha \in \mathbb{Z}^N_+$;
4. *if* $a(z) \in M\Gamma^m_{\rho,\Lambda}$ *and* $t \in \mathbb{R}^N$, *then* $a(z-t) \in M\Gamma^m_{\rho,\Lambda}$ *and* $a(tz) \in M\Gamma^m_{\rho,\Lambda}$.

Proposition 3.3 *Let* $\Lambda(z)$ *be a weight function,* $m \in \mathbb{R}$ *and* $0 < \rho \le \frac{1}{\mu}$. *Then the following inclusions hold*
$$\Gamma^{m-N_0}_{\rho,\Lambda} \subset M\Gamma^m_{\rho,\Lambda} \subset \Gamma^m_{\rho,\Lambda}, \tag{12}$$
where $N_0 := N(\frac{1}{\mu_0} - \rho)$.

Definition 3.3 *We say that a sequence* $\{a_j\}_{j\in\mathbb{N}}$ *of symbols* $a_j(z) \in M\Gamma^{m_j}_{\rho,\Lambda}$, *such that* $m_j > m_{j+1}$ *and* $\lim_{j\to\infty} m_j = -\infty$ *is an asymptotic expansion for* $a(z) \in M\Gamma^{m_1}_{\rho,\Lambda}$ *and write*
$$a(z) \sim \sum_j a_j(z),$$
if, for any integer $N > 1$, $a(z) - \sum_{j<N} a_j(z) \in M\Gamma^{m_N}_{\rho,\Lambda}$.

Proposition 3.4 *If* $\{a_j\}_{j\in\mathbb{N}}$ *is a sequence of symbols as in Definition 3.3, there exists* $a(z) \in M\Gamma^{m_1}_{\rho,\Lambda}$ *for which* $\{a_j\}_{j\in\mathbb{N}}$ *is an asymptotic expansion. Moreover the symbol* $a(z)$ *is unique modulo* $\Gamma^{-\infty}_{\rho,\Lambda} := \bigcap_{m\in\mathbb{R}} \Gamma^m_{\rho,\Lambda} = \mathcal{S}(\mathbb{R}^N)$.

The following notion of *ellipticity* is the same as that in the frame of the Shubin type classes (see [9]).

Definition 3.4 *A symbol* $a(z) \in \Gamma^m_{\rho,\Lambda}$ *is* Λ*-elliptic of order* m *if there are two constants* $C, R > 0$ *such that*
$$|a(z)| \ge C\Lambda(z)^m, \qquad |z| \ge R. \tag{13}$$

We will write $E\Gamma^m_{\rho,\Lambda}$ for the class of the Λ-elliptic symbols of order m and $EM\Gamma^m_{\rho,\Lambda} := E\Gamma^m_{\rho,\Lambda} \cap M\Gamma^m_{\rho,\Lambda}$.

Concerning the elliptic symbols we have the following proposition; since it plays a key role in the following, we shall give the details of the proof.

Proposition 3.5 *Consider $a(z) \in E\Gamma^m_{\rho,\Lambda}$ and let $\psi(z)$ be a function in $C^\infty(\mathbb{R}^N)$ which is identically zero for $|z| \le R'$ and identically one for $|z| \ge R''$, with $0 < R' < R''$ sufficiently large. Then $\frac{\psi(z)}{a(z)} \in E\Gamma^{-m}_{\rho,\Lambda}$.*

If $a(z) \in EM\Gamma^m_{\rho,\Lambda}$, then $\frac{\psi(z)}{a(z)} \in EM\Gamma^{-m}_{\rho,\Lambda}$.

Proof Since $a(z)$ fulfills the estimate (13) with a suitable positive R, taking any $\psi(z)$ satisfying the prescribed assumptions with $R'' > R' > R$, we have $\frac{\psi(z)}{a(z)} \in C^\infty(\mathbb{R}^N)$.

See for example [9] for the proof of the first statement.

For the second one, it suffices to prove that $z^\gamma \partial^\gamma \frac{\psi(z)}{a(z)} \in \Gamma^{-m}_{\rho,\Lambda}$ for any nonzero $\gamma \in \mathbb{K}$.

By use of the Leibnitz rule and Faà di Bruno formula we get for $\gamma \neq 0$,

$$z^\gamma \partial^\gamma \frac{\psi(z)}{a(z)} = \sum_{\nu \le \gamma} \sum_{k=0}^{|\nu|} \sum C_{k,\nu,\nu^1,\dots,\nu^k} \frac{z^{\gamma-\nu}\partial^{\gamma-\nu}\psi(z)}{a(z)^{k+1}} z^{\nu^1}\partial^{\nu^1}a(z)\dots z^{\nu^k}\partial^{\nu^k}a(z), \qquad (14)$$

where, for any $\nu \le \gamma$ and $1 \le k \le |\nu|$, the last sum in the right-hand side is taken all over the k-tuples $(\nu^1, \dots, \nu^k) \in \mathbb{K} \times ... \times \mathbb{K}$ such that $\nu^J > 0$ $(J = 1, \dots, k)$ and $\nu^1 + \cdots + \nu^k = \nu$, while $C_{k,\nu,\nu^1,\dots,\nu^k}$ are suitable non negative constants depending on $\nu \le \gamma$, $\nu^1, \dots, \nu^k$ and k (for $\nu = 0$ the related term on the right-hand side reduces to $\frac{z^\gamma \partial^\gamma \psi(z)}{a(z)}$).

Since $a(z) \in E\Gamma^m_{\rho,\Lambda}$, $\frac{z^{\gamma-\nu}\partial^{\gamma-\nu}\psi(z)}{a(z)^{k+1}} \in \Gamma^{-m(k+1)}_{\rho,\Lambda}$ for any $\nu \le \gamma$, $0 \le k \le |\nu|$ (it belongs in particular to $\Gamma^{-\infty}_{\rho,\Lambda}$ when $\nu < \gamma$); on the other hand, $z^{\nu^J}\partial^{\nu^J}a(z) \in \Gamma^m_{\rho,\Lambda}$ for $J = 1, \dots, k$.

This just shows that $z^\gamma \partial^\gamma \frac{\psi(z)}{a(z)} \in \Gamma^{-m}_{\rho,\Lambda}$ and completes the proof.

Remark 2 As an immediate consequence of Propositions 2.1 and 3.1, we see that any real power of a weight function $\Lambda(z)^m$ is a Λ-elliptic symbol of order m; more precisely $\Lambda(z)^m \in EM\Gamma^m_{\frac{1}{\mu},\Lambda}$ for any $m \in \mathbb{R}$.

4. Pseudodifferential Operators

In this section we set $N = 2n$ and $z = (x, \xi)$.

Firstly, we list some notions and related results which will be useful in the following. We shall associate to each symbol $a(x, \xi) \in \Gamma^m_{\rho,\Lambda}$ the pseudodifferential operator of generalized Weyl type

$$Au(x) := \int e^{i(x-y)\cdot\xi} a((1-\tau)x + \tau y, \xi) u(y) dy đ\xi, \qquad u \in \mathcal{S}, \qquad (15)$$

where $\tau = (\tau_1, ..., \tau_n) \in \mathbb{R}^n$ is fixed, we denote by τy the vector $(\tau_1 y_1, ..., \tau_n y_n)$ and $đ\xi := (2\pi)^{-n} d\xi$.
We can consider from the very beginning more general operators of the form

$$Au(x) := \int e^{i(x-y)\cdot\xi} a(x, y, \xi) u(y) dy đ\xi, \qquad u \in \mathcal{S}, \tag{16}$$

where the function $a(x, y, \xi) \in C^\infty(\mathbb{R}^{3n})$, called *amplitude*, satisfies suitable estimates. More precisely, according to [9, 10, 12], we give the following

Definition 4.1 *Let $\Lambda(x, \xi)$ be a weight function, $0 < \rho \le \frac{1}{\mu}$ and $m \in \mathbb{R}$. We define $\overline{\Gamma}^m_{\rho,\Lambda}$ to be the class of all $a(x, y, \xi) \in C^\infty(\mathbb{R}^{3n})$ such that for any $\alpha, \beta, \gamma \in \mathbb{Z}^n_+$*

$$|\partial^\alpha_\xi \partial^\beta_x \partial^\gamma_y a(x, y, \xi)| \le C_{\alpha,\beta,\gamma} \lambda_{m,m',\alpha,\beta,\gamma}(x, y, \xi), \qquad C_{\alpha,\beta,\gamma} > 0, \tag{17}$$

with

$$\lambda_{m,m',\alpha,\beta,\gamma}(x, y, \xi) := \Lambda(x, \xi)^m \langle x - y\rangle^{m'} \big(1 + \Lambda(x, \xi)\langle x - y\rangle^{-m'}\big)^{-\rho|\alpha+\beta+\gamma|},$$

for a suitable $m' \in \mathbb{R}$ depending on $a(x, y, \xi)$, but independent of α, β, γ.

When $a(x, y, \xi) \in \overline{\Gamma}^m_{\rho,\Lambda}$, the operator A defined in (16) continuously maps the space $\mathcal{S}$ into itself and it extends to a continuous map from $\mathcal{S}'$ into itself.

We set $L^m_{\rho,\Lambda}$ for the class of all the operators defined by (16) for some $a(x, y, \xi) \in \overline{\Gamma}^m_{\rho,\Lambda}$. When $a(x, \xi) \in \Gamma^m_{\rho,\Lambda}$, any pseudodifferential operator defined by (15) is an operator in $L^m_{\rho,\Lambda}$ thanks to the following

Proposition 4.1 *If $a(x, \xi) \in \Gamma^m_{\rho,\Lambda}$ then for every $\tau \in \mathbb{R}^n$ the function*

$$b(x, y, \xi) := a((1 - \tau)x + \tau y, \xi) \tag{18}$$

belongs to $\overline{\Gamma}^m_{\rho,\Lambda}$. In particular $b(x, y, \xi) = a(x, \xi)$ and $b(x, y, \xi) = a(y, \xi)$ are in $\overline{\Gamma}^m_{\rho,\Lambda}$.

In fact, $L^m_{\rho,\Lambda}$ does not enlarge the class of the pseudodifferential operators defined by (15) with $a(x, \xi) \in \Gamma^m_{\rho,\Lambda}$.

Proposition 4.2 ([9]) *Let $A \in L^m_{\rho,\Lambda}$ with amplitude $a(x, y, \xi) \in \overline{\Gamma}^m_{\rho,\Lambda}$ and $\tau \in \mathbb{R}^n$. Then there exists one and only one symbol $b_\tau(x, \xi) \in \Gamma^m_{\rho,\Lambda}$ such that*

$$Au(x) := \int e^{i(x-y)\cdot\xi} b_\tau((1 - \tau)x + \tau y, \xi) u(y) dy đ\xi. \tag{19}$$

$b_\tau(x, \xi)$ will be called the τ-symbol of A. For $b_\tau(x, \xi)$ the following asymptotic expansion holds

$$b_\tau(x, \xi) \sim \sum_{\beta,\gamma} \frac{(-1)^{|\beta|}}{\beta!\gamma!} \tau^\beta (1 - \tau)^\gamma \big(\partial^{\beta+\gamma}_\xi D^\beta_x D^\gamma_y a\big)(x, x, \xi). \tag{20}$$

If in particular $b_{\tau_1}(x, \xi)$ and $b_{\tau_2}(x, \xi)$ are the τ_1 and the τ_2-symbol of A respectively, then

$$b_{\tau_2}(x, \xi) \sim \sum_\alpha \frac{1}{\alpha!} (\tau_1 - \tau_2)^\alpha \partial^\alpha_\xi D^\alpha_x b_{\tau_1}(x, \xi). \tag{21}$$

According to the notations of the above proposition, $b_0(x,\xi)$ corresponding to $\tau = (0,...,0)$ is called the *left symbol* of A, $b_1(x,\xi)$ corresponding to $\tau = (1,...,1)$ the *right symbol* of A and $b_W(x,\xi) = b_{\frac{1}{2}}(x,\xi)$ corresponding to $\tau = \left(\frac{1}{2},...,\frac{1}{2}\right)$ the *Weyl symbol* of A.

We refer to [9, 10, 12] for details concerning the pseudodifferential calculus in classes $L^m_{\rho,\Lambda}$.

Now we are interested in the operators in $L^m_{\rho,\Lambda}$ with τ-symbol $a(x,\xi) \in M\Gamma^m_{\rho,\Lambda}$.

Proposition 4.2 and the results of Section 3 yield the following

Proposition 4.3 *Let $A \in L^m_{\rho,\Lambda}$ and $\tau \in \mathbb{R}^n$. If the τ-symbol $b_\tau(x,\xi)$ of A belongs to $M\Gamma^m_{\rho,\Lambda}$ then the τ'-symbol $b_{\tau'}(x,\xi)$ also belongs to $M\Gamma^m_{\rho,\Lambda}$, for any τ'.*

Proof In view of Proposition 4.2, for any $\tau' \neq \tau$ the τ'-symbol of A can be expressed in terms of its τ-symbol by means of the asymptotic expansion (21) (for $\tau_2 = \tau'$ and $\tau_1 = \tau$).

By Propositions 3.2 and 3.4, $\partial^\alpha_\xi D^\alpha_x a(x,\xi) \in M\Gamma^{m-2\rho N}_{\rho,\Lambda}$ for $|\alpha| = N$ and then $b_{\tau'}(x,\xi) \in M\Gamma^m_{\rho,\Lambda}$.

We set $ML^m_{\rho,\Lambda}$ for the subclass of $L^m_{\rho,\Lambda}$ consisting of the operators with τ-symbol in $M\Gamma^m_{\rho,\Lambda}$. By the above proposition, $ML^m_{\rho,\Lambda}$ is independent of τ.

Proposition 4.4 *If $A \in ML^m_{\rho,\Lambda}$ and $B \in ML^{m'}_{\rho,\Lambda}$, then*

1. *$AB \in ML^{m+m'}_{\rho,\Lambda}$;*
2. *$A^t \in ML^m_{\rho,\Lambda}$;*
3. *$A^* \in ML^m_{\rho,\Lambda}$;*

where, with standard notations, A^t is the transposed operator defined by $\langle A^t u, v\rangle = \langle u, Av\rangle$, $u,v \in \mathcal{S}$, A^* is the conjugate operator defined by $(A^*u,v) = (u,Av)$, $u,v \in \mathcal{S}$ and $(.,.)$ stands for the inner product in $L^2(\mathbb{R}^n)$.

Proof Let $a_W(x,\xi) \in M\Gamma^m_{\rho,\Lambda}$ and $b_W(x,\xi) \in M\Gamma^{m'}_{\rho,\Lambda}$ the Weyl symbols of A and B respectively. Then for the Weyl symbol $c(x,\xi) \in \Gamma^{m+m'}_{\rho,\Lambda}$ of $C := AB$ we have

$$c_W(x,\xi) \sim \sum_{\alpha,\beta} \frac{(-1)^{|\beta|}}{\alpha!\beta!} 2^{|\alpha+\beta|} \partial^\alpha_\xi D^\beta_x a_W(x,\xi) \partial^\beta_\xi D^\alpha_x b_W(x,\xi). \tag{22}$$

In view of Propositions 3.2 and 3.4, $\partial^\alpha_\xi D^\beta_x a_W(x,\xi)\partial^\beta_\xi D^\alpha_x b_W(x,\xi) \in M\Gamma^{m+m'-2\rho N}_{\rho,\Lambda}$, for $|\alpha+\beta| = N$, and then $c_W(x,\xi) \in M\Gamma^{m+m'}_{\rho,\Lambda}$. This proves the first statement.

For Statements *2* and *3*, it suffices to observe that the Weyl symbols $a^t_W(x,\xi)$ and $a^*_W(x,\xi)$ of A^t and A^* respectively are related to the Weyl symbol $a_W(x,\xi)$ of A by the following formulas

$$a^t_W(x,\xi) = a_W(x,-\xi), \tag{23}$$

$$a^*_W(x,\xi) = \overline{a_W(x,\xi)}. \tag{24}$$

Thus $a^t_W(x,\xi), a^*_W(x,\xi) \in M\Gamma^m_{\rho,\Lambda}$, in view of Proposition 3.2.

5. Anti-Wick Symbols

In this section, we review some known results concerning the *operators with Anti-Wick symbol* (see [12]). First we recall the basic definition.

Definition 5.1 *If $a(x,\xi) \in \Gamma^m_{\rho,\Lambda}$, we define the operator $A = \mathrm{Op}_{\mathrm{AW}}[a]$ with Anti-Wick symbol $a(x,\xi)$ by the following formula*

$$Au(x) := \int a(y,\eta) P_{y,\eta} u(x) dy đ\eta, \qquad u \in \mathcal{S}. \tag{25}$$

Here $P_{y,\eta}$ is the orthogonal projection of $L^2(\mathbb{R}^n)$ on the vector

$$\phi_{y,\eta}(x) := \pi^{-\frac{n}{4}} e^{ix\cdot\eta} e^{-\frac{1}{2}|y-x|^2}, \tag{26}$$

namely $P_{y,\eta}u(x) := (u, \phi_{y,\eta})\phi_{y,\eta}(x), \qquad u \in L^2(\mathbb{R}^n)$.

As is well known, operators with Anti-Wick symbol in $\Gamma^m_{\rho,\Lambda}$ constitute a subclass of the class $L^m_{\rho,\Lambda}$. More precisely, the following holds

Proposition 5.1 ([9, 10, 12])*Any operator $A = \mathrm{Op}_{\mathrm{AW}}[a]$ with Anti-Wick symbol $a(x,\xi) \in \Gamma^m_{\rho,\Lambda}$ belongs to $L^m_{\rho,\Lambda}$ and its Weyl symbol $b_W(x,\xi)$ is given by*

$$b_W(x,\xi) := (2\pi)^{-n}(a * \sigma_{0,0})(x,\xi) := 2^n \int a(y,\eta) e^{-(|x-y|^2+|\xi-\eta|^2)} dy đ\eta, \tag{27}$$

where $\sigma_{0,0}(x,\xi) = 2^n e^{-(|x|^2+|\xi|^2)}$.
Moreover for the Weyl symbol of A we have the following asymptotic expansion

$$b_W(x,\xi) \sim \sum_{\alpha,\beta} \frac{c_{\alpha,\beta}}{\alpha!\beta!} \partial_\xi^\alpha \partial_x^\beta a(x,\xi), \tag{28}$$

where $c_{\alpha,\beta} := 2^n \int \eta^\alpha y^\beta e^{-(|y|^2+|\eta|^2)} dy đ\eta$.

As an easy consequence of the above proposition and the results of Section 3, we obtain

Proposition 5.2 *If $a(x,\xi) \in M\Gamma^m_{\rho,\Lambda}$, then $A = \mathrm{Op}_{\mathrm{AW}}[a] \in ML^m_{\rho,\Lambda}$.*

Proof It suffices to argue on the asymptotic expansion (28) of the Weyl symbol $b_W(x,\xi)$ of A as in the proof of Proposition 4.4.

6. Elliptic Pseudodifferential Operators

We set $EL^m_{\rho,\Lambda}$ for the class of all the pseudodifferential operators in $L^m_{\rho,\Lambda}$ whose τ-symbol belongs to $E\Gamma^m_{\rho,\Lambda}$.

$EL^m_{\rho,\Lambda}$ is independent of τ in view of

Proposition 6.1 ([9])*Let $\tau_1, \tau_2 \in \mathbb{R}^n$ be given. If the τ_1-symbol $b_{\tau_1}(x,\xi)$ of $A \in L^m_{\rho,\Lambda}$ belongs to $E\Gamma^m_{\rho,\Lambda}$, the same holds for the τ_2-symbol $b_{\tau_2}(x,\xi)$.*

By Proposition 4.3 we can also introduce the set $EML^m_{\rho,\Lambda}$ of the pseudodifferential operators with τ-symbol in $EM\Gamma^m_{\rho,\Lambda}$.

Recall that Q is said to be a *parametrix* of P if $PQ = I + R$ and $QP = I + S$, where I is the identity operator and $R, S \in L^{-\infty}_{\rho,\Lambda} := \bigcap_{m\in\mathbb{R}} L^m_{\rho,\Lambda}$. For the parametrix of an operator in $EM\Gamma^m_{\rho,\Lambda}$ the following holds:

Proposition 6.2 *Let $P \in EML^m_{\rho,\Lambda}$ be given. Then there exists an operator $Q \in EML^{-m}_{\rho,\Lambda}$ such that*

$$PQ = I + R, \qquad QP = I + S, \tag{29}$$

where $R, S \in L^{-\infty}_{\rho,\Lambda}$ and I is the identity operator. Moreover if Q' is another operator satisfying (29)*, then $Q - Q' \in L^{-\infty}_{\rho,\Lambda}$.*

Proof Since $EML^m_{\rho,\Lambda} \subset EL^m_{\rho,\Lambda}$, we know already from [9] that there exists $Q \in EL^{-m}_{\rho,\Lambda}$ satisfying the identities (29).

Moreover, if $p(x,\xi) \in EM\Gamma^m_{\rho,\Lambda}$ is the Weyl symbol of P and $\psi(x,\xi)$ a C^∞ function as in Proposition 3.5 (with $z = (x,\xi)$) we may write $Q = RB_1$, where B_1 is the operator with Weyl symbol $b_1(x,\xi) = \frac{\psi(x,\xi)}{p(x,\xi)}$ while R is an operator in $L^0_{\rho,\Lambda}$ whose Weyl symbol $r(x,\xi)$ has the following asymptotic expansion

$$r(x,\xi) \sim \sum_{j\geq 0} (-1)^j r_j(x,\xi), \tag{30}$$

$r_1(x,\xi)$ is the Weyl symbol of the operator $R_1 := B_1 P - I$ and, for any $j \geq 0$, $r_j(x,\xi)$ is the Weyl symbol of the j-th power R_1^j of R_1 (see [9]).

By Proposition 3.5, we know that $b_1(x,\xi) \in M\Gamma^{-m}_{\rho,\Lambda}$.
Since moreover $r_1(x,\xi)$ has the following asymptotic expansion

$$r_1(x,\xi) \sim \sum_{|\alpha+\beta|>0} \frac{(-1)^{|\beta|}}{\alpha!\beta!} 2^{-|\alpha+\beta|} \partial_\xi^\alpha D_x^\beta \frac{\psi(x,\xi)}{p(x,\xi)} \partial_\xi^\beta D_x^\alpha p(x,\xi) \tag{31}$$

(see [9] again) and, for all α,β, $\partial_\xi^\alpha D_x^\beta \frac{\psi(x,\xi)}{p(x,\xi)} \partial_\xi^\beta D_x^\alpha p(x,\xi) \in M\Gamma^{-2\rho|\alpha+\beta|}_{\rho,\Lambda}$ in view of Propositions 3.2 and 3.4, it follows that $r_1(x,\xi) \in M\Gamma^{-2\rho}_{\rho,\Lambda}$ and, by virtue of (30), $r(x,\xi) \in M\Gamma^0_{\rho,\Lambda}$.

So $Q = RB_1 \in ML^{-m}_{\rho,\Lambda}$ is the required parametrix.

The uniqueness of Q modulo $L^{-\infty}_{\rho,\Lambda}$ also follows from the same result in the $L^m_{\rho,\Lambda}$ classes.

7. L^p-Continuity

The following is the main result concerning the L^p behavior of the operators in $ML^m_{\rho,\Lambda}$.

Proposition 7.1 *Any operator $A \in ML^0_{\rho,\Lambda}$ extends to a bounded operator from $L^p(\mathbb{R}^n)$ to itself, for all $1 < p < \infty$.*

A similar L^p-continuity result is given by Taylor [15] in the context of a local theory for classical pseudodifferential operators.

As well as Taylor's theorem, Proposition 7.1 essentially relies on a classical theorem about Fourier multipliers due to Lizorkin and Marcinkiewicz (see [14] and [20] for the proof).

Theorem 7.1 (Lizorkin - Marcinkiewicz [14]) *Let the function $m(\xi)$ be continuous together with its derivatives $\partial_\xi^\gamma m(\xi)$, for any $\gamma \in \mathbb{K}$.*
If there is a constant $B > 0$ such that

$$|\xi^\gamma \partial_\xi^\gamma m(\xi)| \leq B, \qquad \xi \in \mathbb{R}^n, \quad \gamma \in \mathbb{K}, \tag{32}$$

then for every $1 < p < \infty$ we can find a constant $A_p > 0$, depending only on p, B and the dimension n, such that

$$\|m(D)u\|_{L^p} \leq A_p \|u\|_{L^p},$$

for all $u \in \mathcal{S}$.

Proof of Proposition 7.1 We use for A the left-symbol representation

$$A\varphi(x) = a(x,D)\varphi(x) = \int e^{ix\cdot\xi} a(x,\xi)\hat{\varphi}(\xi)đ\xi, \qquad \varphi \in \mathcal{S}, \tag{33}$$

with $a(x,\xi) \in M\Gamma^0_{\rho,\Lambda}$. For $m = (m_1, ..., m_n) \in \mathbb{Z}^n$ let us consider

$$Q_m := \Big\{x \in \mathbb{R}^n : |x_j - m_j| \leq \frac{1}{2},\ j = 1, 2, ..., n\Big\},$$

$$Q_m^{**} := \{x \in \mathbb{R}^n : |x_j - m_j| \leq 1, \quad j = 1, 2, ..., n\}$$

and cut-off functions $\psi_m(x) \in C_0^\infty(\mathbb{R}^n)$ such that $0 \leq \psi_m(x) \leq 1$, $\operatorname{supp}\psi_m \subset Q_m^{**}$ and $\psi_m(x) = 1$ in a neighborhood of a cube Q_m^*, where $Q_m \subset Q_m^* \subset Q_m^{**}$. Setting $\varphi_{1,m} := \psi_m\varphi$ and $\varphi_{2,m} := (1-\psi_m)\varphi$, we have

$$\int |a(x,D)\varphi(x)|^p dx \leq 2^{p-1} \sum_{m\in\mathbb{Z}^n} \{I_{1,m} + I_{2,m}\}, \tag{34}$$

where

$$I_{i,m} := \int_{Q_m} |a(x,D)\varphi_{i,m}(x)|^p dx \qquad i = 1, 2. \tag{35}$$

In fact, since $\mathbb{R}^n = \bigcup_{m\in\mathbb{Z}^n} Q_m$ and the measure of $Q_m \cap Q_n$ vanishes when $n \neq m$, we can write for $\varphi(x) \in \mathcal{S}$:

$$\|a(x,D)\varphi\|_p^p = \sum_{m\in\mathbb{Z}^n} \int_{Q_m} |a(x,D)\varphi(x)|^p dx. \tag{36}$$

Moreover for any $m \in \mathbb{Z}^n$,

$$\begin{aligned}\int_{Q_m} |a(x,D)\varphi(x)|^p dx &= \int_{Q_m} |a(x,D)(\varphi_{1,m} + \varphi_{2,m})(x)|^p dx \\ &\leq 2^{p-1}\{I_{1,m} + I_{2,m}\}.\end{aligned} \tag{37}$$

Therefore (34) is proved. To complete the proof, we shall use the following lemma.

Lemma 7.1 *Let $\Lambda(x,\xi)$ be a weight function, $a(x,\xi) \in M\Gamma^0_{\rho,\Lambda}$, $\chi(x) \in C_0^\infty(\mathbb{R}^n)$ and set $a_\chi(x,\xi) := \chi(x)a(x,\xi)$; we can then consider the Fourier transform of $a_\chi(x,\xi)$ with respect to the x variable*

$$\hat{a}_\chi(\eta,\xi) := \int e^{-ix\cdot\eta} a_\chi(x,\xi)dx. \tag{38}$$

For any $N > 0$ there is a positive constant $C_{N,\chi}$ such that

$$|\xi^\gamma \partial_\xi^\gamma \hat{a}_\chi(\eta,\xi)| \leq C_{N,\chi}(1+|\eta|)^{-N}, \tag{39}$$

for all $\eta, \xi \in \mathbb{R}^n$ and $\gamma \in \mathbb{K}$.

Proof Since $a_\chi(x,\xi)$ has compact support with respect to the x variable, differentiation and integration by parts in (38) give

$$(-i\eta)^\beta \partial_\xi^\gamma \hat{a}_\chi(\eta,\xi) = (-1)^{|\beta|} \int e^{-ix\cdot\eta} \partial_x^\beta \partial_\xi^\gamma a_\chi(x,\xi)dx, \tag{40}$$

where β is an arbitrary multi-index. Then by Leibnitz formula we get

$$|(-i\eta)^\beta \xi^\gamma \partial_\xi^\gamma \hat{a}_\chi(\eta,\xi)| \leq \sum_{\nu\leq\beta} \binom{\beta}{\nu} \int |\partial_x^{\beta-\nu}\chi(x)||\xi^\gamma \partial_x^\nu \partial_\xi^\gamma a(x,\xi)|dx. \tag{41}$$

Since $|\xi^\gamma \partial_x^\nu \partial_\xi^\gamma a(x,\xi)| \leq C_{\gamma,\nu}\Lambda(x,\xi)^{-\rho|\nu|} \leq C'_{\gamma,\nu}$ (cf. (11)), we obtain

$$|(-i\eta)^\beta \xi^\gamma \partial_\xi^\gamma \hat{a}_\chi(\eta,\xi)| \leq C_{\beta,\chi}, \tag{42}$$

for every $\eta, \xi \in \mathbb{R}^n$ and $\gamma \in \mathbb{K}$, where the positive constant on the right-hand side is given by

$$C_{\beta,\chi} := \max_{\gamma\in\mathbb{K}} \sum_{\nu\leq\beta} \binom{\beta}{\nu} C'_{\gamma,\nu} \int |\partial_x^{\beta-\nu}\chi(x)|dx. \tag{43}$$

Thus by summing on $|\beta| \leq N$ we obtain the proof.

End of the proof of Proposition 7.1 In order to estimate $I_{1,m}$ in (34), (35), let us consider a cut-off function $\eta(x) \in C_0^\infty(\mathbb{R}^n)$ such that $\eta(x) = 1$ for $x \in Q_0$ and define $\eta_m(x) := \eta(x-m)$ for any $m \in \mathbb{Z}^n$.

By setting now $a_m(x,\xi) := \eta_m(x)a(x,\xi) \in M\Gamma^0_{\rho,\Lambda}$ we get

$$I_{1,m} = \int_{Q_m} |a_m(x,D)\varphi_{1,m}(x)|^p dx \leq \int_{\mathbb{R}^n} |a_m(x,D)\varphi_{1,m}(x)|^p dx. \tag{44}$$

If we set now $\chi(x) = \eta_m(x)$ in Lemma 7.1, we can show that for any $N > 0$ there exists a positive constant C_N such that

$$|\xi^\gamma \partial_\xi^\gamma \hat{a}_m(\eta,\xi)| \leq C_N(1+|\eta|)^{-N} \qquad \eta,\xi \in \mathbb{R}^n, \quad \gamma \in \mathbb{K}. \tag{45}$$

Notice that, from to the definition of the functions $\eta_m(x)$ and (43), the constant C_N in (45) depends on $\eta(x)$ but not on $m \in \mathbb{Z}^n$. Then by Theorem 7.1 for every $N > 0$ there exists $M_N > 0$, which does not depend on m, such that

$$\|\hat{a}_m(\eta, D)u\|_{L^p} \le M_N(1+|\eta|)^{-N}\|u\|_{L^p}, \tag{46}$$

for every $u \in \mathcal{S}$, $\eta \in \mathbb{R}^n$ and $m \in \mathbb{Z}^n$.

On the other hand, from the Fubini's theorem, we have

$$a_m(x, D)u(x) = \int e^{ix\cdot\xi} a_m(x,\xi)\hat{u}(\xi)đ\xi = \int e^{ix\cdot\eta}\hat{a}_m(\eta, D)u(x)đ\eta. \tag{47}$$

So using (46) and the Minkovski's inequality in integral form, we have

$$\begin{aligned} I_{1,m} &\le \int\left|\int e^{ix\cdot\eta}\hat{a}_m(\eta,D)\varphi_{1,m}(x)đ\eta\right|^p dx \le \left\{\int \|\hat{a}_m(\eta,D)\varphi_{1,m}\|_{L^p}đ\eta\right\}^p \\ &\le M_N^p\|\varphi_{1,m}\|_{L^p}^p\left\{\int(1+|\eta|)^{-N}đ\eta\right\}^p = M_N^pC_N^p\|\varphi_{1,m}\|_{L^p}^p, \end{aligned} \tag{48}$$

where, for suitably large N, $C_N := \int(1+|\eta|)^{-N}đ\eta < \infty$.

To estimate the second term $I_{2,m}$, we may apply the same argument used by Wong [17] to prove L^p-continuity of classical pseudodifferential operators. Namely following step by step the second part of the proof of Theorem 10.7 in [17], we get the following estimate

$$I_{2,m} \le H_{N,p}\int_{\mathbb{R}^n\setminus Q_m^*}\frac{|\varphi_{2,m}(z)|^p}{(\mu+|m-z|)^{\frac{Np}{2}}}dz, \tag{49}$$

where $\mu = \frac{\sqrt{n}}{2}+1$ and $H_{N,p}$ is a positive constant only depending on N, p and the dimension n.

From the definition of $\varphi_{1,m}$ and Q_m^{**}, $m \in \mathbb{Z}^n$, we obtain

$$\sum_{m\in\mathbb{Z}^n}\int_{Q_m^{**}}|\varphi_{1,m}(x)|^p dx \le C_n\|\varphi\|_{L^p}^p, \tag{50}$$

where the constant $C_n > 0$ depends only on the dimension n. Moreover

$$\begin{aligned} \sum_{m\in\mathbb{Z}^n}\int_{\mathbb{R}^n\setminus Q_m^*}\frac{|\varphi_{2,m}(z)|^p}{(\mu+|m-z|)^{\frac{Np}{2}}}dz &\le \sum_{m\in\mathbb{Z}^n}\sum_{l\ne m}\int_{Q_l}\frac{|\varphi_{2,m}(z)|^p}{(\mu+|m-z|)^{\frac{Np}{2}}}dz \\ &\le \sum_{m\in\mathbb{Z}^n}\sum_{l\in\mathbb{Z}^n}\frac{1}{(1+|m-l|)^{\frac{Np}{2}}}\int_{Q_l}|\varphi(z)|^p dz \\ &\le \|\varphi\|_{L^p}^p\sum_{m\in\mathbb{Z}^n}\frac{1}{(1+|m|)^{\frac{Np}{2}}}, \end{aligned} \tag{51}$$

and $\sum_{m\in\mathbb{Z}^n}\frac{1}{(1+|m|)^{\frac{Np}{2}}} < \infty$ for suitably large N.

By (34) and the estimates from (48) to (51) we then get:

$$\|a(x,D)\varphi\|_{L^p}^p \leq \left\{ C''_{N,p,n} + H'_{N,p} \sum_{m\in\mathbb{Z}^n} \frac{1}{(1+|m|)^{\frac{Np}{2}}} \right\} \|\varphi\|_{L^p}^p, \tag{52}$$

which ends the proof.

Remark 3 Notice that in order to apply the Lizorkin Marcinkiewicz theorem it would be enough to consider pseudodifferential operators in $L^0_{\rho,\Lambda}$ with τ-symbol satisfying the following assumption

$$\xi^\gamma \partial_\xi^\gamma a(x,\xi) \in \Gamma^0_{\rho,\Lambda}, \qquad \gamma \in \mathbb{Z}^n_+, \quad \gamma_j \in \{0,1\}\ (j=1,...,n), \tag{53}$$

instead of (10) involving the whole variables $z=(x,\xi)$.

Actually the condition (53) is sufficient to assure the L^p-continuity of the related pseudodifferential operators.

The symmetry of the role played by the x and the ξ variables in the classes $\Gamma^m_{\rho,\Lambda}$ suggested here the more restrictive condition (10). Moreover the same condition will allow us to study the behavior of the related operators on functional spaces defined by means of the Fourier transform (cf. the next Propositions 9.1 and 9.3).

8. Weighted Sobolev Spaces

Definition 8.1 *Let $\Lambda(x,\xi)$ be a weight function. For $s\in\mathbb{R}$ and $1<p<\infty$, $H^{s,p}_\Lambda$ is the space of all the tempered distributions $u\in\mathcal{S}'$ such that $\Lambda^s(x,D)u \in L^p(\mathbb{R}^n)$. Here $\Lambda^s(x,D)$ is the pseudodifferential operator with left symbol $\Lambda(x,\xi)^s \in M\Gamma^s_{\frac{1}{\mu},\Lambda}$, that is*

$$\Lambda^s(x,D)u(x) := \int e^{ix\cdot\xi}\Lambda(x,\xi)^s \hat{u}(\xi) đ\xi, \qquad u\in\mathcal{S}.$$

Since $\Lambda(x,\xi)^s$ is Λ-elliptic of order s (see Remark 2), in view of Proposition 6.2, there exists an operator $\overline{\Lambda}_{-s}(x,D) \in ML^{-s}_{\rho,\Lambda}$ such that

$$\overline{\Lambda}_{-s}(x,D)\Lambda^s(x,D) = I + R_s, \tag{54}$$

where $R_s \in L^{-\infty}_{\rho,\Lambda}$. When $\Lambda(x,\xi)=\sqrt{1+|x|^2+|\xi|^2}$ and $p=2$, the spaces $H^{s,p}_\Lambda$ coincide with the spaces $\tilde{\mathrm{B}}^s$ studied in Helffer [11]. For the case of a more general weight function $\Lambda(x,\xi)$ see also [9, 10] and [12]. We impose a Banach topology on $H^{s,p}_\Lambda$ by setting for any $u\in H^{s,p}_\Lambda$

$$\|u\|_{s,p,\Lambda} := \|\Lambda^s(x,D)u\|_{L^p} + \|R_s u\|_{L^p}, \tag{55}$$

where R_s is defined by (54).

Proposition 8.1 *$H^{s,p}_\Lambda$ is a Banach space with respect to the norm $\|.\|_{s,p,\Lambda}$ defined by* (55).

Proof In the following we write for short $\Lambda^s := \Lambda^s(x, D)$ and $\overline{\Lambda}_{-s} := \overline{\Lambda}_{-s}(x, D)$.

Of course, $\|.\|_{s,p,\Lambda}$ is a semi-norm in $H^{s,p}_\Lambda$.

Let us suppose $\|u\|_{s,p,\Lambda} = 0$. By (55) it follows $\Lambda^s u = 0$ and $R_s u = 0$; then $u = \overline{\Lambda}_{-s}(\Lambda^s u) - R_s u = 0$, in view of (54).

To prove the completeness, let $\{u_\nu\}_{\nu\in\mathbb{N}}$ be a Cauchy sequence in $H^{s,p}_\Lambda$ with respect to $\|.\|_{s,p,\Lambda}$. Then it follows that $\{\Lambda^s u_\nu\}_{\nu\in\mathbb{N}}$ and $\{R_s u_\nu\}_{\nu\in\mathbb{N}}$ are Cauchy sequences in $L^p(\mathbb{R}^n)$; let v and w be their limits in $L^p(\mathbb{R}^n)$, respectively.

We prove now that $\{u_\nu\}_{\nu\in\mathbb{N}}$ converges to $u := \overline{\Lambda}_{-s}v - w$ in $H^{s,p}_\Lambda$. Setting $v_\nu := \Lambda^s u_\nu$ and $w_\nu := R_s u_\nu$, we get $\Lambda^s u_\nu = \Lambda^s\overline{\Lambda}_{-s}v_\nu - \Lambda^s w_\nu$.

Since $v_\nu \to v$ in $L^p(\mathbb{R}^n)$ and $\Lambda^s\overline{\Lambda}_{-s} \in ML^0_{\frac{1}{\mu},\Lambda}$, $\Lambda^s\overline{\Lambda}_{-s}v_\nu \to \Lambda^s\overline{\Lambda}_{-s}v$ in $L^p(\mathbb{R}^n)$ in view of Proposition 7.1.

On the other hand, since R_s is a regularizing operator, $u_\nu \to \overline{\Lambda}_{-s}v - w$ in $\mathcal{S}'$ yields $w_\nu = R_s u_\nu \to R_s(\overline{\Lambda}_{-s}v - w)$ in $\mathcal{S}$. Thus, from the uniqueness of the limit, $w_\nu \to w$ in $\mathcal{S}$ and then $\Lambda^s w_\nu \to \Lambda^s w$ in $\mathcal{S}$.

This shows that $\Lambda^s u \in L^p(\mathbb{R}^n)$. Moreover $\Lambda^s u_\nu = \Lambda^s\overline{\Lambda}_{-s}v_\nu - \Lambda^s w_\nu \to \Lambda^s\overline{\Lambda}_{-s}v - \Lambda^s w = \Lambda^s u$ in $L^p(\mathbb{R}^n)$ and the proof is concluded.

In fact the previous Sobolev spaces can be better described by means of any pseudodifferential operator with positive Anti-Wick elliptic symbol. Firstly we recall the following result.

Lemma 8.1 ([9]) *Let $a(x,\xi) \in \Gamma^m_{\rho,\Lambda}$ be Λ-elliptic of order m and assume that $a(x,\xi) > 0$, for any $x, \xi \in \mathbb{R}^n$. Then the operator $A = \mathrm{Op_{AW}}[a] \in L^m_{\rho,\Lambda}$ with Anti-Wick symbol a is an automorphism of $\mathcal{S}$ extending to an automorphism of $\mathcal{S}'$. Moreover $A^{-1} \in L^{-m}_{\rho,\Lambda}$ is Λ-elliptic of order $-m$.*

Proposition 8.2 *Let $a(x,\xi) \in M\Gamma^s_{\rho,\Lambda}$ be an arbitrary Λ-elliptic symbol of order s such that $a(x,\xi) > 0$, for any $x, \xi \in \mathbb{R}^n$. Then for any $1 < p < \infty$,*

$$H^{s,p}_\Lambda = A^{-1}(L^p(\mathbb{R}^n)) = \{u \in \mathcal{S}' : \ Au \in L^p(\mathbb{R}^n)\},$$

where $A = \mathrm{Op_{AW}}[a]$ is the operator with Anti-Wick symbol a. Moreover a norm on $H^{s,p}_\Lambda$ equivalent to (55) *is given by $\|Au\|_{L^p}$.*

Proof Firstly suppose that $Au \in L^p(\mathbb{R}^n)$ and write $\Lambda^s u = \Lambda^s A^{-1}(Au)$. Then $\Lambda^s u \in L^p(\mathbb{R}^n)$ since $\Lambda^s A^{-1} \in ML^0_{\rho,\Lambda}$.

Conversely let us take $u \in \mathcal{S}'$ such that $\Lambda^s u \in L^p(\mathbb{R}^n)$. By (54) we may write

$$Au = A\overline{\Lambda}_{-s}(\Lambda^s u) + AR_s\overline{\Lambda}_{-s}(\Lambda^s u) + AR_s(R_s u). \tag{56}$$

So using Proposition 7.1, we conclude that $Au \in L^p(\mathbb{R}^n)$ and there is a positive constant C independent of u such that

$$\|Au\|_{L^p} \leq C(\|\Lambda^s u\|_{L^p} + \|R_s u\|_{L^p}), \tag{57}$$

since $A\overline{\Lambda}_{-s} \in ML^0_{\rho,\Lambda}$ and $AR_s\overline{\Lambda}_{-s}, AR_s \in L^{-\infty}_{\rho,\Lambda}$.

On the other hand, since A is an automorphism of $\mathcal{S}'$, $\|A.\|_{L^p}$ is a norm on $H^{s,p}_\Lambda$ and it is easy to show that $H^{s,p}_\Lambda$ is complete with respect to it; thus the equivalence with the norm (55) follows from the open mapping theorem.

Remark 4 It is evident from the preceding arguments that the Sobolev spaces $H^{s,p}_\Lambda$ can be described by means of any elliptic pseudodifferential operator of order s. Namely for any $T \in EML^s_{\rho,\Lambda}$

$$H^{s,p}_\Lambda = \{u \in \mathcal{S}' : Tu \in L^p(\mathbb{R}^n)\}. \tag{58}$$

If moreover $Q \in EML^{-s}_{\rho,\Lambda}$ is a parametrix of T and $QT = I + R$, with $R \in L^{-\infty}_{\rho,\Lambda}$, then $\|Tu\|_{L^p} + \|Ru\|_{L^p}$ is a norm in $H^{s,p}_\Lambda$ equivalent to (55).

Remark 5 Notice also that for any weight function $\Lambda(x,\xi)$, $t < s$ and $1 < p < \infty$ the following inclusions hold

$$\mathcal{S} \subset H^{s,p}_\Lambda \subset H^{t,p}_\Lambda \subset \mathcal{S}',$$

with continuous embedding.

Using the above description of the spaces $H^{s,p}_\Lambda$ we can plainly deduce the action of pseudodifferential operators in classes $ML^m_{\rho,\Lambda}$. Namely the following holds

Proposition 8.3 *If $B \in ML^m_{\rho,\Lambda}$, then*

$$B : H^{s+m,p}_\Lambda \to H^{s,p}_\Lambda,$$

continuously, for all $s \in \mathbb{R}$ and $1 < p < \infty$.

Proof In the following we set $G_t = \mathrm{Op}_{\mathrm{AW}}[\Lambda^t]$ for the operator with Anti-Wick symbol $\Lambda(x,\xi)^t \in M\Gamma^t_{\frac{1}{\mu},\Lambda}$; since $\Lambda(x,\xi)^t$ is positive and Λ-elliptic of order t, in view of Proposition 8.2 we have $H^{t,p}_\Lambda = G_t^{-1}(L^p(\mathbb{R}^n))$ for any $1 < p < \infty$.

Since $Q := G_s A G^{-1}_{s+m} \in ML^0_{\rho,\Lambda}$ and, for any $u \in H^{s+m,p}_\Lambda$, $G_{s+m}u \in L^p(\mathbb{R}^n)$, $G_s A u = Q(G_{s+m}u) \in L^p(\mathbb{R}^n)$; moreover there is a positive constant C such that

$$\|G_s A u\|_{L^p} = \|Q(G_{s+m}u)\|_{L^p} \leq C\|G_{s+m}u\|_{L^p}.$$

This shows the continuity of B and concludes the proof.

9. B_p-Continuity

First we give some preliminary definitions.

Definition 9.1 *For any $1 \leq p \leq \infty$, we define $B_p(\mathbb{R}^n)$ (B_p for short) as the space of all the tempered distributions $u \in \mathcal{S}'$ such that $\hat{u}(\xi) \in L^p(\mathbb{R}^n)$. Here $\hat{u}(\xi)$ means the Fourier transform of u.*

Of course, B_p is a Banach space, with respect to the norm $\|u\|_{B_p} := \|\hat{u}\|_{L^p}$.

Notice that by the Plancherel's theorem $B_2 \equiv L^2(\mathbb{R}^n)$.

The spaces B_p are the starting point for the definition of a more general type of function spaces $B_{p,k}$, where $1 \leq p \leq \infty$ and $k = k(\xi)$ is a *weight function* in a suitable sense; namely Hörmander ([21]) sets $u \in B_{p,k}$ if $\hat{u}(\xi)k(\xi) \in L^p(\mathbb{R}^n)$. We shall introduce here similar function spaces, adapted to our class of operators.

Let $f(x,\xi)$ be any complex valued function in $\mathbb{R}^{2n}$; we set

$$f^\#(x,\xi) := f(\xi,x). \tag{59}$$

The following lemma can be easily derived.

Lemma 9.1 *If $\Lambda(x,\xi)$ is a weight function, then $\Lambda^{\#}(x,\xi)$ is also a weight function. If $a(x,\xi) \in \Gamma^m_{\rho,\Lambda}$, then $a^{\#}(x,\xi) \in \Gamma^m_{\rho,\Lambda^{\#}}$; moreover $a^{\#}(x,\xi) \in M\Gamma^m_{\rho,\Lambda^{\#}}$ if in particular $a(x,\xi) \in M\Gamma^m_{\rho,\Lambda}$.*

Remark 6 When the weight function $\Lambda(x,\xi)$ is the multi-quasi-elliptic polynomial

$$\Lambda_{\mathcal{P}}(x,\xi) = \sqrt{\sum_{(\alpha,\beta)\in V(\mathcal{P})} x^{2\alpha}\xi^{2\beta}}$$

corresponding to a complete polyhedron $\mathcal{P} \subset \mathbb{R}^n_x \times \mathbb{R}^n_\xi$, then $\Lambda^{\#}(x,\xi)$ is the multi-quasi-elliptic polynomial related to the polyhedron $\mathcal{P}^{\#}$ symmetric of $\mathcal{P}$ with respect to the diagonal $\{x=\xi\}$, obtained by interchanging the role of the x and ξ variables.

The following lemma provides an expression of an operator with given left symbol $a(x,\xi)$ in terms of another operator with left symbol $a^{\#}(x,\xi)$ by means of the Fourier transform.

Lemma 9.2 ([9]) *Take $a(x,\xi) \in \Gamma^m_{\rho,\Lambda}$ and let $A_a \in L^m_{\rho,\Lambda}$ be the pseudodifferential operator with left symbol $a(x,\xi)$. Then for the transposed operator A^t_a we have*

$$A^t_a = \mathcal{F} A_{a^{\#}} \mathcal{F}^{-1}, \tag{60}$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ are the Fourier transform and the inverse Fourier transform respectively and $A_{a^{\#}}$ is the pseudodifferential operator with left symbol $a^{\#}(x,\xi) \in \Gamma^m_{\rho,\Lambda^{\#}}$.

Proof By the Fubini's theorem we get for any $u, v \in \mathcal{S}$

$$\begin{aligned}\langle A^t_a v, u\rangle = \langle v, A_a u\rangle &= \int v(x)\Big\{\int e^{ix\cdot\xi} a(x,\xi)\hat{u}(\xi) đ\xi\Big\}dx \\ &= \int\int e^{ix\cdot\xi} a(x,\xi)\hat{u}(\xi)v(x)dx đ\xi = \int\Big\{\int e^{ix\cdot\xi} a(x,\xi)v(x)đx\Big\}\hat{u}(\xi)d\xi.\end{aligned} \tag{61}$$

Let us change now the role of the variables x and ξ and set $v=\hat{w}$. Then

$$\int\Big\{\int e^{ix\cdot\xi} a(\xi,x)\hat{w}(\xi)đ\xi\Big\}\hat{u}(x)dx = \int (A_{a^{\#}}w)(x)\Big(\int e^{-ix\cdot y}u(y)dy\Big)dx \tag{62}$$

and applying again the Fubini's theorem we get

$$\begin{aligned}\int (A_{a^{\#}}w)(x)\Big(\int e^{-ix\cdot y}u(y)dy\Big)dx &= \int\int e^{-ix\cdot y}(A_{a^{\#}}w)(x)u(y)dydx \\ &= \int\Big(\int e^{-ix\cdot y}(A_{a^{\#}}w)(x)dx\Big)u(y)dy \\ &= \langle \mathcal{F}A_{a^{\#}}\mathcal{F}^{-1}v, u\rangle.\end{aligned} \tag{63}$$

This gives the equality $A^t_a v = \mathcal{F}A_{a^{\#}}\mathcal{F}^{-1}v$ for any $v \in \mathcal{S}$; by a density argument it immediately extends to any $v \in \mathcal{S}'$.

We are now able to prove the following

Proposition 9.1 *Any pseudodifferential operator $A \in M\Gamma^0_{\rho,\Lambda}$ extends to a linear bounded operator from B_p to itself for every $1 < p < \infty$.*

Proof Let $a(x,\xi) \in M\Gamma^m_{\rho,\Lambda}$ be the left symbol of A.
In view of Lemma 9.2, for any $u \in \mathcal{S}$ we have

$$\|Au\|_{B_p} := \|\hat{Au}\|_{L^p} = \|A^t_{a^\#}\hat{u}\|_{L^p}, \tag{64}$$

where $A_{a^\#}$ is the pseudodifferential operator in $ML^0_{\rho,\Lambda^\#}$ with left symbol $a^\#$ defined by (59). By Proposition 4.4 the operator $A^t_{a^\#}$ belongs to $ML^0_{\rho,\Lambda^\#}$ and then it is L^p-continuous in view of Proposition 7.1.

So $\|A^t_{a^\#}\hat{u}\|_{L^p} \le C\|\hat{u}\|_{L^p}$, for some positive constant C independent of u. This proves the B_p-continuity of A, by observing that $\mathcal{S}$ is dense in B_p, for $1 < p < \infty$.

For any weight function $\Lambda(x,\xi)$, we set $Q^{s,p}_\Lambda$ for the inverse image under the Fourier transform of the Banach space $H^{s,p}_\Lambda$; that is

$$Q^{s,p}_\Lambda := \{u \in \mathcal{S}' : \hat{u} \in H^{s,p}_\Lambda\}. \tag{65}$$

Since the Fourier transform is an automorphism of $\mathcal{S}'$, $Q^{s,p}_\Lambda$ turns to be a Banach space with respect to the norm

$$\|u\|_{Q^{s,p}_\Lambda} := \|\hat{u}\|_{s,p,\Lambda}. \tag{66}$$

Using again the continuity of the Fourier transform and its inverse on $\mathcal{S}$ and $\mathcal{S}'$, we can also state the following

Proposition 9.2 *For any weight function $\Lambda(x,\xi)$, $t < s$ and $1 < p < \infty$ the inclusions*

$$\mathcal{S} \subset Q^{s,p}_\Lambda \subset Q^{t,p}_\Lambda \subset \mathcal{S}',$$

hold with continuous embedding.

As a consequence of Proposition 9.1 we may show the following

Proposition 9.3 *If $A \in ML^m_{\rho,\Lambda}$, then*

$$A : Q^{s+m,p}_{\Lambda^\#} \to Q^{s,p}_{\Lambda^\#} \tag{67}$$

continuously for all $s \in \mathbb{R}$ and $1 < p < \infty$, where $\Lambda^\#(x,\xi)$ is defined in (59).

To prove the above result, we use the following characterizations of the spaces $Q^{s,p}_{\Lambda^\#}$.

Proposition 9.4 *Let $s \in \mathbb{R}$ and $1 < p < \infty$ be given and $\Lambda(x,\xi)$ a weight function. Then the following hold:*

1. *for any $T \in EML^s_{\rho,\Lambda}$*
$$Q^{s,p}_{\Lambda^\#} = \{u \in \mathcal{S}' : Tu \in B_p\}. \tag{68}$$
Moreover if $T' \in EML^{-s}_{\rho,\Lambda}$ is a parametrix of T and $T'T = I + R$, $R \in L^{-\infty}_{\rho,\Lambda}$, then an equivalent norm on $Q^{s,p}_{\Lambda^\#}$ is given by
$$\|Tu\|_{B_p} + \|Ru\|_{B_p}; \tag{69}$$

2. *if A is any pseudodifferential operator with positive Anti - Wick symbol $a(x,\xi)\in EM\Gamma^s_{\rho,\Lambda}$, then*

$$Q^{s,p}_{\Lambda^\#} = A^{-1}(B_p) = \{u \in \mathcal{S}' : Au \in B_p\} \tag{70}$$

and $\|Au\|_{B_p}$ is an equivalent norm on $Q^{s,p}_{\Lambda^\#}$.

Proof Let $t(x,\xi) \in EM\Gamma^s_{\rho,\Lambda}$ be the left symbol of T and set $T^\#$ for the pseudodifferential operator with left symbol $t^\#(x,\xi) \in EM\Gamma^s_{\rho,\Lambda^\#}$.

Let us consider now $u \in Q^{s,p}_{\Lambda^\#}$. Since the transposed of an operator in $EML^s_{\rho,\Lambda^\#}$ is also in $EML^s_{\rho,\Lambda^\#}$, in view of Remark 4 we have that $T^{\#,t}\hat{u} \in L^p(\mathbb{R}^n)$. But from Lemma 9.2, $T^{\#,t}\hat{u} = \hat{Tu}$ and then $Tu \in B_p$. This proves the equality (68).

It is easy to check that (69) defines a norm on $Q^{s,p}_{\Lambda^\#}$.

In order to show that $\|T.\|_{B_p} + \|R.\|_{B_p}$ is equivalent to the norm (66), it suffices now to observe that $T'T = I + R$ implies $T^\#(T')^\# = I + R^\#$ and then $(T')^{\#,t}T^{\#,t} = I + R^{\#,t}$ with $R^{\#,t} \in L^{-\infty}_{\rho,\Lambda^\#}$; thus by Remark 4 and Lemma 9.2 $\|u\|_{Q^{s,p}_{\Lambda^\#}} = \|T^{\#,t}\hat{u}\|_{L^p} + \|R^{\#,t}\hat{u}\|_{L^p} = \|Tu\|_{B_p} + \|Ru\|_{B_p}$.

Statement *2* is an immediate consequence of *1*; indeed it suffices to observe that any pseudodifferential operator A with positive Anti-Wick symbol $a(x,\xi)$ of order s is in $EML^s_{\rho,\Lambda}$ and a parametrix of A is given by the inverse operator A^{-1}.

We are now able to prove Proposition 9.3.

Proof of Proposition 9.3 We set $G_t := \mathrm{Op}_{AW}[\Lambda^t]$ for the pseudodifferential operator with Anti- Wick symbol $\Lambda(x,\xi)^t$. By Proposition 9.4 $Q^{t,p}_{\Lambda^\#} = G_t^{-1}(B_p)$ and we may assume $\|u\|_{Q^{t,p}_{\Lambda^\#}} = \|G_t u\|_{B_p}$, for any $t \in \mathbb{R}$.

For $u \in Q^{s+m,p}_{\Lambda^\#}$, $1 < p < \infty$, write $G_sAu = G_sAG^{-1}_{s+m}(G_{s+m}u)$. Since $G_sAG^{-1}_{s+m} \in ML^0_{\rho,\Lambda}$, in view of Proposition 9.1 we can find a positive constant $C_{s,m,p}$ only depending on s, m and p such that

$$\|Au\|_{Q^{s,p}_{\Lambda^\#}} \le C_{s,m}\|G_{s+m}u\|_{B_p} = C_{s,m}\|u\|_{Q^{s+m,p}_{\Lambda^\#}}. \tag{71}$$

Remark 7 When the weight function $\Lambda(x,\xi)$ is the multi-quasi-elliptic polynomial $\Lambda_{\mathcal{P}}(x,\xi)$ corresponding to a complete polyhedron $\mathcal{P}$, it is not difficult to prove that the space $Q^{1,p}_{\mathcal{P}} := Q^{1,p}_{\Lambda_{\mathcal{P}}}$ can be described as

$$Q^{1,p}_{\mathcal{P}} = \{u \in \mathcal{S}' : x^\alpha\partial^\beta u \in B_p, \quad (\alpha,\beta) \in V(\mathcal{P}^\#)\}, \tag{72}$$

where $\mathcal{P}^\#$ is the polyhedron defined in Remark 6. Moreover the expression $\sum_{(\alpha,\beta)\in V(\mathcal{P}^\#)} \|x^\alpha\partial^\beta u\|_{B_p}$ gives an equivalent norm on $Q^{1,p}_{\mathcal{P}}$.

10. Regularity Results for Λ-Elliptic Operators in L^p and B_p Spaces

Proposition 10.1 *Let $\Lambda(x,\xi)$ be a weight function, $m \in \mathbb{R}$, $A \in EML^m_{\rho,\Lambda}$ and $u \in \mathcal{S}'$. Then the following holds*

1. *if* $Au \in H^{s,p}_{\Lambda}$, *then* $u \in H^{s+m,p}_{\Lambda}$ *for all* $1 < p < \infty$ *and* $s \in \mathbb{R}$. *Moreover for any* $t < s+m$ *there is a positive constant* $C = C_{t,s,m,p}$, *independent of* u, *for which*

$$\|u\|_{s+m,p,\Lambda} \leq C(\|Au\|_{s,p,\Lambda} + \|u\|_{t,p,\Lambda}); \tag{73}$$

2. *if* $Au \in Q^{s,p}_{\Lambda\#}$, *then* $u \in Q^{s+m,p}_{\Lambda\#}$ *for all* $1 < p < \infty$ *and* $s \in \mathbb{R}$. *Moreover for any* $t < s+m$ *there exists a constant* $C = C_{t,s,m,p} > 0$, *independent of* u, *such that*

$$\|u\|_{Q^{s+m,p}_{\Lambda\#}} \leq C(\|Au\|_{Q^{s,p}_{\Lambda\#}} + \|u\|_{Q^{t,p}_{\Lambda\#}}). \tag{74}$$

Proof From Proposition 6.2 there exists an operator $B \in EML^{-m}_{\rho,\Lambda}$ such that $BA = I + R$ with $R \in L^{-\infty}_{\rho,\Lambda}$. For any $u \in \mathcal{S}'$ we have then $u = BAu - Ru$.

If $Au \in H^{s,p}_{\Lambda}$ as in Statement *1* we get

$$G_{s+m}u = G_{s+m}BG_s^{-1}(G_sAu) - G_{s+m}Ru, \tag{75}$$

where G_t, $t \in \mathbb{R}$, has the same meaning as in Proposition 8.3; this shows that $u \in H^{s+m,p}_{\Lambda}$, since $G_{s+m}BG_s^{-1} \in ML^0_{\rho,\Lambda}$, $G_sAu \in L^p(\mathbb{R}^n)$ and $G_{s+m}R \in L^{-\infty}_{\rho,\Lambda}$.

In view of the L^p-continuity of $ML^0_{\rho,\Lambda}$ we can find a positive constant C_1 such that

$$\|G_{s+m}BG_s^{-1}(G_sAu)\|_{L^p} \leq C_1\|G_sAu\|_{L^p} = C_1\|Au\|_{s,p,\Lambda}. \tag{76}$$

Moreover for any $t < s+m$ we may write $G_{s+m}Ru = G_{s+m}RG_t^{-1}(G_tu)$; so there is a constant $C_2 > 0$ such that

$$\|G_{s+m}Ru\|_{L^p} \leq C_2\|G_tu\|_{L^p} = C_2\|u\|_{t,p,\Lambda}, \tag{77}$$

as $G_{s+m}RG_t^{-1} \in L^{-\infty}_{\rho,\Lambda}$ and $G_tu \in L^p(\mathbb{R}^n)$ (cf. Remark 5).

Thus the estimate (73) follows from the estimates (76), (77).

The proof of the second statement follows the same arguments, by using the continuity of operators in $ML^0_{\rho,\Lambda}$ on the $Q^{s,p}_{\Lambda\#}$ spaces instead of the continuity on the Sobolev spaces.

As an example of the previous regularity result, let us consider now the linear partial differential operator of Schrödinger type

$$P(x,D) = -\Delta + V(x), \tag{78}$$

where $\Delta := \sum_{j=1}^n \partial^2_{x_j}$ is the Laplacian operator in dimension n, the "potential" $V(x) = \sum_{\alpha\in\mathcal{R}} a_\alpha x^\alpha$ is a polynomial with constant coefficients $a_\alpha \in \mathbb{C}$ and $\mathcal{R}$ is a complete polyhedron of $\mathbb{R}^n_x$.

We may easily check that the symbol $P(x,\xi) = |\xi|^2 + V(x)$ of the operator $P(x,D)$ belongs to the class $ML^1_{\rho,\lambda}$; here $\lambda(x,\xi)$ is the multi-quasi-elliptic weight function associated to the complete polyhedron of $\mathbb{R}^n_x \times \mathbb{R}^n_\xi$ defined as the convex hull of the set $\{(\alpha,0), \alpha \in V(\mathcal{R})\} \cup \{(0,2e_j), e_j = (0,...,0,1,0,...,0), j = 1,2,...,n\}$, namely

$$\lambda(x,\xi) = \sqrt{\sum_{\alpha\in V(\mathcal{R})} x^{2\alpha} + \sum_{j=1}^n \xi_j^4} = \sqrt{\Lambda_{\mathcal{R}}(x)^2 + \sum_{j=1}^n \xi_j^4}. \tag{79}$$

As a consequence of Proposition 10.1 we may state the following

Proposition 10.2 *Let $P(x,D) = -\Delta + V(x)$ be the previous linear partial differential operator and assume $V(x)$ is positive and $\mathcal{R}$-multi-quasi-elliptic, that is*

$$V(x) \geq C\Lambda_{\mathcal{R}}(x), \qquad |x| \geq R,$$

with suitable positive constants C, R. Then for all $s \in \mathbb{R}$ and $1 < p < \infty$

1. *if $-\Delta u + V(x)u \in H_\lambda^{s,p}$, then $u \in H_\lambda^{s+1,p}$ and for any $t < s+1$ there is a constant $C_{t,s,p} > 0$ such that*

$$\|u\|_{s+1,p,\lambda} \leq C(\| -\Delta u + V(.)u\|_{s,p,\lambda} + \|u\|_{t,p,\lambda}), \tag{80}$$

   where $\lambda(x,\xi)$ is defined by (79);

2. *if $-\Delta u + V(x)u \in Q_{\lambda\#}^{s,p}$, then $u \in Q_{\lambda\#}^{s+1,p}$. Moreover for any $t < s+1$ there exists $C_{t,s,p} > 0$ such that*

$$\|u\|_{Q_{\lambda\#}^{s+1,p}} \leq C(\| -\Delta u + V(.)u\|_{Q_{\lambda\#}^{s,p}} + \|u\|_{Q_{\lambda\#}^{t,p}}), \tag{81}$$

   where $\lambda^{\#}(x,\xi) = \sqrt{\sum_{j=1}^{n} x_j^4 + \Lambda_{\mathcal{R}}(\xi)^2}$.

For the proof it suffices to notice that under the assumptions of Proposition 10.2 the operator $P(x,D)$ is $\lambda(x,\xi)$ elliptic of order 1.

When in particular we put $s = 0$ in the previous statement, we have in view of Remark 7 that $-\Delta u + V(x)u \in L^p(\mathbb{R}^n)$ yields $u \in L^p(\mathbb{R}^n)$, $\partial_{x_j}^2 u \in L^p(\mathbb{R}^n)$, $x^\alpha u \in L^p(\mathbb{R}^n)$ for any $\alpha \in \mathcal{R}$ and $j = 1, ..., n$; on the other hand, when $-\Delta u + V(x)u \in B_p$, $u \in B_p$, $\partial_{x_j}^2 u \in B_p$ and $x^\alpha u \in B_p$ for all $\alpha \in \mathcal{R}$ and $j = 1, ..., n$.

Coming back to the examples in two variables (x_1, x_2) given in the Introduction, put for instance $V(x) = V_1(x_1, x_2) = x_1^h + x_2^k$ for $h = 4$ and $k = 6$.

Since $V_1(x_1, x_2) \geq 0$ and it is multi-quasi-elliptic with respect to its *Newton polyhedron* (that is the convex hull of the points $(0,0)$, $(4,0)$ and $(0,6)$)

1. if $-\Delta u + x_1^4 u + x_2^6 u = f(x_1, x_2) \in L^p(\mathbb{R}^2)$, then $u, \partial_{x_1}^2 u, \partial_{x_2}^2 u, x_1^4 u, x_2^6 u \in L^p(\mathbb{R}^2)$ and we can find a positive constant C such that

$$\|u\|_{L^p} + \sum_{j=1}^{2} \|\partial_{x_j}^2 u\|_{L^p} + \|x_1^4 u\|_{L^p} + \|x_2^6 u\|_{L^p} \leq C(\|f\|_{L^p} + \|u\|_{L^p}). \tag{82}$$

2. if $-\Delta u + x_1^4 u + x_2^6 u = f(x_1, x_2) \in B_p$, then $u, \partial_{x_1}^2 u, \partial_{x_2}^2 u, x_1^4 u, x_2^6 u \in B_p$ and we can find $C > 0$ such that the a priori estimate (82) holds with the B_p norm instead of the L^p norm.

Similarly we can argue for $V(x) = V_2(x_1, x_2)$ given by (2) in the Introduction.

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