Traveling Waves and Capillarity Driven Spreading of Shear-Thinning Fluids

Joseph A. Iaia*

P.O. Box 311430, Department of Mathematics, University of North Texas, Denton, TX 76203-1430, USA.

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Abstract. We study capillary spreadings of thin films of liquids of power-law rheology. These satisfy

\[ u_t + (u^{\lambda+2} |u_{xxx}|^{\lambda-1} u_{xxx})_x = 0, \]

where \( u(x,t) \) represents the thickness of the one-dimensional liquid and \( \lambda > 1 \). We look for traveling wave solutions so that \( u(x,t) = g(x + ct) \) and thus \( g \) satisfies

\[ g''' = \frac{|g - \epsilon|^{\lambda}}{g^{1+\lambda}} \text{sgn}(g - \epsilon). \]

We show that for each \( \epsilon > 0 \) there is an infinitely oscillating solution, \( g_\epsilon \), such that

\[ \lim_{t \to \infty} g_\epsilon = \epsilon \]

and that \( g_\epsilon \to g_0 \) as \( \epsilon \to 0 \), where \( g_0 \equiv 0 \) for \( t \geq 0 \) and

\[ g_0 = c_\lambda |t|^\frac{\lambda}{\lambda+1} \quad \text{for} \quad t < 0 \]

for some constant \( c_\lambda \).

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1 Introduction

In this work, we study capillary spreadings of thin films of liquids of power-law rheology, also known as Ostwald-de Waele fluids. The following equation for one-dimensional
motion was derived in [1, 2] and is

\[ u_t + \left( u^{\lambda+2} |u_{xxx}|^{\lambda-1} u_{xxx} \right)_x = 0, \]

where \( \lambda \) is a real constant and \( u(x,t) \) represents the thickness of the one-dimensional liquid film at position \( x \) and time \( t \). See also [3, 4]. When \( \lambda > 1 \), the fluid is called shear thinning and the viscosity tends to zero at high strain rates [5]. Typical values for \( \lambda \) are between 1.7 and 6.7 [6].

For gravity driven spreadings studied in [7], \( u(x,t) \) satisfies

\[ u_t - \left( u^{\lambda+2} |u_x|^{\lambda-1} u_x \right)_x = 0. \]

If we look for traveling wave solutions of the above equation so that \( u(x,t) = g(x+ct) \) for some nonzero \( c \in \mathbb{R} \), we obtain

\[ cg' = \left( g^{\lambda+2} |g'|^{\lambda-1} g' \right)' \]

and thus

\[ c(g-K) = g^{\lambda+2} |g'|^{\lambda-1} g' \]

for some constant \( K \). In the case \( K = 0 \) we obtain

\[ g(z) = d(z-z_0)^{\frac{1}{\lambda+1}} \]

for some constant \( d \) which represents a current advancing with constant speed, \( c \), and front located at \( x = -ct-z_0 \). In particular, this differential equation has no oscillatory traveling wave solutions. Similarly, in the case \( K \neq 0 \) there are no oscillatory traveling wave solutions. If \( g'(m_1) = g'(m_2) = 0 \) with \( m_1 < m_2 \), then it follows from the differential equation that \( g(m_1) = K = g(m_2) \). Now let \( M \) be the maximum (or minimum) of \( g \) on \([m_1,m_2] \). Then \( g'(M) = 0 \) and thus \( g(M) = K \). Thus \( g \equiv K \) on \([m_1,m_2] \).

In this paper, we will study traveling wave solutions for capillarity-driven spreadings in which case we obtain

\[ cg' + \left( g^{\lambda+2} |g'''|^{\lambda-1} g''' \right)' = 0 \]

and so

\[ cg + g^{\lambda+2} |g'''|^{\lambda-1} g''' = K. \]

If we expect that \( g \) will be essentially constant as \( t \to \infty \), say \( \epsilon > 0 \), then this gives the equation

\[ c(g-\epsilon) + g^{\lambda+2} |g'''|^{\lambda-1} g''' = 0. \]

This reduces to

\[ g''' = d \frac{|g-\epsilon|^{\frac{1}{\lambda}}}{g^{\lambda+2} |g'''|^{\frac{1}{\lambda}}} \text{sgn}(g-\epsilon), \quad \text{where} \quad d = -\frac{c}{|c|^{1-\frac{1}{\lambda}}}. \]
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Letting \( y(t) = g\left(\frac{1}{\sqrt[3]{t}}\right) \) gives

\[
y'' = \frac{|y-e|^{\frac{1}{\lambda}}}{y^{1+\frac{1}{\lambda}}} \text{sgn}(y-e).
\]

We now consider

\[
y'''(t) = f_\epsilon(y(t)), \quad (1.1)
\]

\[
y(t_0) = y_0 > 0, \quad y'(t_0) = y'_0, \quad y''(t_0) = y''_0,
\]

where

\[
f_\epsilon(y) \equiv \frac{|y-e|^{\frac{1}{\lambda}}}{y^{1+\frac{1}{\lambda}}} \text{sgn}(y-e), \quad y, \epsilon, \lambda \in \mathbb{R}, \quad y > 0, \quad \epsilon > 0, \quad \lambda > 1. \quad (1.3)
\]

We note that \( f_\epsilon \) is increasing for \( 0 < y < (1 + \frac{1}{\lambda+1}) \epsilon \), decreasing for \( (1 + \frac{1}{\lambda+1}) \epsilon < y < \infty \), and has an absolute maximum at \( y = (1 + \frac{1}{\lambda+1}) \epsilon \). We also see that \( f_\epsilon(y) \) is not integrable at \( y = 0 \) and is integrable at \( y = \infty \). Next we define

\[
F_\epsilon(y) = \int_y^\infty f_\epsilon(t)dt \quad \text{for} \quad y > 0.
\]

We see that \( F_\epsilon(y) \geq 0, \) \( F_\epsilon \) is decreasing on \( (0, \epsilon) \), increasing on \( (\epsilon, \infty) \),

\[
\lim_{y \to 0^+} F_\epsilon(y) = +\infty, \quad (1.4a)
\]

and there exists \( 0 < F_\epsilon,\infty < \infty \) such that

\[
\lim_{y \to \infty} F_\epsilon(y) = F_\epsilon,\infty. \quad (1.4b)
\]

Also we see that there exists \( 0 < L_\epsilon < \epsilon \) such that

\[
F_\epsilon(L_\epsilon) = F_\epsilon,\infty. \quad (1.5)
\]

We now define the following “energy” type functions which will be useful in analyzing solutions of Eq. (1.1). Let

\[
E_{1,y} = \frac{1}{2}(y')^2 - (y-e)y'', \quad (1.6a)
\]

\[
E_{2,y} = F_\epsilon(y) - y'y''', \quad (1.6b)
\]

\[
E_{3,y} = \frac{1}{2}(y'')^2 - f_\epsilon(y)y'. \quad (1.6c)
\]

Note that

\[
E'_{1,y} = -(y-e)y'' = -(y-e)f_\epsilon(y) = -\frac{|y-e|^{1+\frac{1}{\lambda}}}{y^{1+\frac{1}{\lambda}}} \leq 0, \quad (1.7a)
\]

\[
E'_{2,y} = -(y'')^2 \leq 0, \quad (1.7b)
\]

\[
E'_{3,y} = -f_\epsilon'(y)(y')^2. \quad (1.7c)
\]
It can be verified that
\[ E_{3,y}' \leq 0 \quad \text{for} \quad 0 < y \leq \left(1 + \frac{1}{\lambda + 1}\right) \epsilon \]
and
\[ E_{3,y}' \geq 0 \quad \text{for} \quad y \geq \left(1 + \frac{1}{\lambda + 1}\right) \epsilon. \]

In this paper we prove the following:

Main Theorem. Let \( \epsilon > 0 \) and \( \lambda > 1 \). There exists a solution of (1.1) with \( y(0) = L\epsilon, y'(0) = 0, \) and \( y''(0) = b \epsilon > 0 \) and \( y_b \) is decreasing on \((-\infty,0)\), oscillates infinitely often on \([0,\infty)\) and
\[
\lim_{t \to -\infty} y_{b\epsilon}(t) = \epsilon. \tag{1.8}
\]

In addition,
\[
\lim_{\epsilon \to 0} y_{b\epsilon}(t) = y_0(t), \tag{1.9}
\]
where
\[
y_0 = \begin{cases} 
0, & \text{for } t \geq 0, \\
c_{\lambda} |t|^{\frac{2\lambda}{3\lambda + 1}}, & \text{for } t < 0,
\end{cases} \tag{1.10a}
\]
where
\[
c_{\lambda} = \left[\frac{(2\lambda + 1)^3}{3\lambda(\lambda - 1)(\lambda + 2)}\right]^{\frac{1}{3\lambda + 1}}. \tag{1.10b}
\]

Note that \( y_0 \) satisfies the limiting differential equation
\[
y''' = \frac{1}{y^{1+\frac{2}{3}}} \quad \text{for} \quad t < 0.
\]

Also, since \( \lambda > 1 \) then \( 3\lambda / (2\lambda + 1) > 1 \) so that \( y_0 \) has zero contact angle at \( t = 0 \). According to [3], there are other solutions to
\[
y''' = \frac{1}{y^{1+\frac{2}{3}}} \]
with nonzero contact angle at \( t = 0 \) which grow like \( |t|^{3\lambda/(2\lambda + 1)} \) at \(-\infty\). However, zero contact angle is more physically reasonable.

2 Preliminaries

In this section, we fix \( \epsilon > 0 \) and write \( f, F, E_1, E_2, \) and \( E_3 \) instead of \( f_\epsilon, F_\epsilon, E_{1,y}, E_{2,y}, \) and \( E_{3,y} \).

Lemma 2.1. Let \( t_0 \in \mathbb{R} \). There is a solution of (1.1)-(1.2) on \((t_0 - \delta, t_0 + \delta)\) for some \( \delta > 0 \). Also, for
\[
y_0 > 0, \quad |y_0 - \epsilon| + |y_0'| + |y_0''| > 0,
\]
the solution is unique and the solution varies continuously with respect to the parameters \((y_0, y'_0, y''_0)\).
**Proof.** The standard existence-uniqueness-continuous-dependence theorem applies for all \( y_0 > 0 \) with \( y_0 \neq e \).

If \( y_0 = e \) then we still have existence by the Peano existence theorem. Now suppose \( y_0 = e \) but that \( y'_0 \neq 0 \). Then near \( t_0 \) we have that

\[
|y-e-y'_0(t-t_0)| \leq C|t-t_0|^2,
\]

which implies

\[
\frac{1}{2}|y'_0||t-t_0| \leq |y-e| \leq 2|y'_0||t-t_0| \quad \text{near } t_0.
\]

Assuming without loss of generality that \( y'_0 > 0 \) then we see that this means

\[
\frac{1}{2}y'_0|t-t_0| \leq (y-e) \leq 2y'_0|t-t_0| \quad \text{for } t \text{ near } t_0 \text{ and } t > t_0.
\] (2.1)

Similarly, if \( z \) is another solution (1.1)-(1.2) with \( z_0 = e_0, z'_0 = y'_0, \) and \( z''_0 = y''_0', \) then

\[
\frac{1}{2}y'_0|t-t_0| \leq (z-e) \leq 2y'_0|t-t_0| \quad \text{for } t \text{ near } t_0 \text{ and } t > t_0.
\] (2.2)

Now

\[
[y-z] = \int_{t_0}^{t} \int_{t_0}^{s} \int_{t_0}^{w} |f(y(x)) - f(z(x))| \, dx \, dw \, ds,
\]

so for any fixed \( x \) we have by the Mean-Value Theorem that

\[
f(y(x)) - f(z(x)) = f'(\mu y(x) + (1-\mu)z(x))[y(x)-z(x)]
\]

for some \( 0 < \mu < 1 \). Using (2.1) and that \( \lambda > 1 \) gives for some constant \( C > 0 \)

\[
|f'(\mu y(x) + (1-\mu)z(x))| \leq C|\mu y + (1-\mu)z - e|^{\lambda-1}
\]

\[
= C|\mu(y-e) + (1-\mu)(z-e)|^{\lambda-1}
\]

\[
\leq C\left(\frac{1}{2}y'_0\right)^{\lambda-1}|x-t_0|^{\lambda-1}.
\]

Therefore

\[
[y-z] \leq \int_{t_0}^{t} \int_{t_0}^{s} \int_{t_0}^{w} |f(y) - f(z)| \, dx \, dw \, ds
\]

\[
\leq C\left(\frac{1}{2}y'_0\right)^{\lambda-1} \int_{t_0}^{t} \int_{t_0}^{s} \int_{t_0}^{w} |x-t_0|^{\lambda-1} |y-z| \, dx \, dw \, ds
\]

\[
\leq \left(\frac{1}{2}y'_0\right)^{\lambda-1} (t-t_0)^2 \int_{t_0}^{t} |s-t_0|^{\lambda-1} |y-z| \, ds.
\]
It follows from (2.1) and (2.2) that the last integral on the right-hand side is defined. Thus for some constant $C > 0$

$$|y - z| \leq C(t - t_0)^2 \int_{t_0}^t |s - t_0|^{-\frac{1}{\lambda} - 1}|y - z| \, ds. \quad (2.3)$$

Letting

$$w = \int_{t_0}^t |s - t_0|^{-\frac{1}{\lambda} - 1}|y - z| \, ds \geq 0.$$  

Then

$$w' = |t - t_0|^{-\frac{1}{\lambda} - 1}|y - z|.$$  

Consequently, (2.3) becomes

$$w'|t - t_0|^{-\frac{1}{\lambda}} \leq C(t - t_0)^2 w$$

so that

$$w' \leq C|t - t_0|^{1 + \frac{1}{\lambda}} w \leq Cw \quad \text{for } t \text{ near } t_0.$$  

Therefore,

$$\int_{t_0}^t (we^{-ct})' \leq 0$$

which implies $w \equiv 0$ on $(t_0, t)$. Hence $y \equiv z$ on $(t_0, t)$. A similar argument shows $y \equiv z$ on $(t, t_0)$.

Now suppose $y_0 = \epsilon$ and $y_0' = 0$ but $y_0'' \neq 0$. Then a similar argument as above shows that

$$\frac{1}{4}|y_0''|(t - t_0)^2 \leq |y - \epsilon| \leq |y_0''|(t - t_0)^2 \quad \text{for } t \text{ near } t_0.$$  

Assuming without loss of generality that $y_0'' > 0$, we see that this means

$$\frac{1}{4}y_0''(t - t_0)^2 \leq y - \epsilon \leq y_0''(t - t_0)^2 \quad \text{for } t \text{ near } t_0 \text{ and } t > t_0. \quad (2.4)$$

Similarly if $z$ is another solution then

$$\frac{1}{4}y_0''(t - t_0)^2 \leq z - \epsilon \leq y_0''(t - t_0)^2 \quad \text{for } t \text{ near } t_0 \text{ and } t > t_0. \quad (2.5)$$

Again by the Mean-Value Theorem we have for each fixed $x$

$$|f(y) - f(z)| = |f'(\mu y(x) + (1 - \mu) z(x))||y(x) - z(x)| \leq C|\mu y + (1 - \mu) z - \epsilon|^{\frac{1}{\lambda} - 1}$$

$$= C|\mu(y - \epsilon) + (1 - \mu)(z - \epsilon)|^{\frac{1}{\lambda} - 1}$$

$$\leq C\left(\frac{1}{4}y_0''\right)^{\frac{1}{\lambda} - 1}|x - t_0|^{\frac{1}{\lambda} - 2}. \quad (2.6)$$
Therefore
\[ |y-z| \leq \int_{t_0}^{t} \int_{t_0}^{s} \int_{t_0}^{w} |f(y) - f(z)| \, dx \, dw \, ds \]
\[ \leq C \left( \frac{1}{4} y''_0 \right)^{\frac{1}{\lambda} - 1} \int_{t_0}^{t} \int_{t_0}^{s} \int_{t_0}^{w} |x-t_0|^\frac{\lambda}{2} - 2 |y-z| \, dx \, dw \, ds \]
\[ \leq C \left( \frac{1}{2} y''_0 \right)^{\frac{1}{\lambda} - 1} (t-t_0)^2 \int_{t_0}^{t} |s-t_0|^\frac{\lambda}{2} - 2 |y-z| \, ds. \]

It follows from (2.4) and (2.5) that the last integral is defined. Therefore we have for some constant \( C \)
\[ |y-z| \leq C(t-t_0)^2 \int_{t_0}^{t} |s-t_0|^\frac{\lambda}{2} - 2 |y-z| \, ds. \quad (2.6) \]

Letting
\[ w = \int_{t_0}^{t} |s-t_0|^\frac{\lambda}{2} - 2 |y-z| \, ds \geq 0. \]

Then
\[ w' = (t-t_0)|s-t_0|^\frac{\lambda}{2} - 2 |y-z| \]
and thus (2.6) becomes
\[ w'|t-t_0|^\frac{\lambda}{2} - 4 \leq C(t-t_0)^2 w. \]

Consequently,
\[ w' \leq C|t-t_0|^\frac{\lambda}{2} w \leq Cw \text{ for } t \text{ near } t_0. \]

Therefore,
\[ \int_{t_0}^{t} (we^{-C})' \leq 0, \]
which implies that \( w \equiv 0 \) on \((t_0, t)\). Hence \( y \equiv z \) on \((t_0, t)\). A similar argument shows \( y \equiv z \) on \((t, t_0)\).

Thus we have shown that the solution is unique if \( y_0 = \epsilon \) and either \( y'_0 = 0 \) or \( y''_0 = 0 \) but not both.

Remark: If \( y_0 = \epsilon \) and \( y'_0 = y''_0 = 0 \), then there are nonlinearities \( f \) for which there is more than one solution of (1.1)-(1.3). For example, if
\[ f(y) = |y-\epsilon|^{\frac{1}{\lambda}} \text{sgn}(y-\epsilon) \]
then \( y = \epsilon \) is a solution and
\[ y = \epsilon + a_\lambda t^{\frac{3}{\lambda-1}}, \]
where
\[ a_\lambda = \left[ \frac{3\lambda(2\lambda+1)(\lambda+2)}{(\lambda-1)^3} \right]^{\frac{1}{\lambda-1}}, \]
is also a solution.

Suppose now that there is a triple \((y_0, y'_0, y''_0)\) with
\[
y_0 > 0, \quad |y_0 - \epsilon| + |y'_0| + |y''_0| > 0
\]
and suppose \(y_0(t)\) is the solution of (1.1) with
\[
y_0(t_0) = y_0, \quad y'_0(t_0) = y'_0, \quad y''_0(t_0) = y''_0.
\]
Let \((y_{0,n}, y'_{0,n}, y''_{0,n})\) be a sequence that converges to \((y_0, y'_0, y''_0)\) and let \(y_n\) be the solution of (1.1) with
\[
y_n(t_0) = y_{0,n}, \quad y'_n(t_0) = y'_{0,n}, \quad y''_n(t_0) = y''_{0,n}.
\]
By the existence proof all of the \(y_n\)'s are defined on \((t_0 - \delta, t_0 + \delta)\) for some \(\delta > 0\) which is independent of \(n\). On this set we have that \(|f(y_n(t))|\) is bounded by a constant \(M\) so that \(|y''_n| \leq M\) and so \(y_n, |y'_n|, |y''_n|\) are all bounded by a constant on \([t_0 - \delta/2, t_0 + \delta/2]\). By the Arzela-Ascoli theorem a subsequence (denoted by \(y_{n_k}\)) along with its first and second derivatives converges uniformly to a function \(y\) with initial condition (2.8). From Eq. (1.1) we see that \(y''_n\) converges uniformly to \(y''\) and \(y\) solves (1.1). With (2.7), by the uniqueness part of the proof established earlier we must have \(y(t) \equiv y_0(t)\) and hence \(y_{n_k}\) converges uniformly to \(y_0\). It then follows from this that \(y_n\) converges uniformly to \(y_0\) for if not then there would be an \(\eta > 0\) and a sequence \(t_{n_k} \in [t_0 - \delta/2, t_0 + \delta/2]\) with \(t_{n_k} \to t^*\) such that
\[
|y_{n_k}(t_{n_k}) - y_0(t^*)| \geq \eta > 0.
\]
However, we could proceed through the same argument as above and find a subsequence \(y_{n_k}\) of \(y_{n_k}\) such that \(y_{n_k}\) converges uniformly to \(y_0\) on \([t_0 - \delta/2, t_0 + \delta/2]\) contradicting the above inequality. This completes the proof of the lemma.

**Lemma 2.2.** Let \(y(t)\) be any solution of (1.1)-(1.2). Then there is a maximal open interval \((T_1, T_2)\) with \(T_1 < t_0 < T_2\) where \(y(t)\) is defined. In addition, if \(T_1 > -\infty\) then \(y\) is increasing near \(T_1\) and
\[
\lim_{t \to T_1^-} y(t) = 0,
\]
and if \(T_2 < \infty\) then \(y\) is decreasing near \(T_2\) and
\[
\lim_{t \to T_2} y(t) = 0.
\]

**Proof.** Let \((T_1, T_2)\) with \(T_1 < t_0 < T_2\) be the maximal open interval where \(y(t)\) is defined (and \(y(t) > 0\)). We now let
\[
c_1 = \inf_{(t_1, t_0]} y(t) \quad \text{and} \quad c_2 = \inf_{[t_0, T_2]} y(t).
\]
Clearly, \(c_1 \geq 0, c_2 \geq 0\). If \(c_2 > 0\) then from the definition of \(f\) we see that \(y''(t)\) is uniformly bounded on \([t_0, T_2]\). Thus if \(T_2 < \infty\) then \(y, y', y''\) are also uniformly bounded on
\( t_0, T_2 \) and so the solution \( y \) could be extended to \( (T_1, T_2 + \delta) \) for some \( \delta > 0 \) contradicting the definition of \( T_2 \). Thus \( T_2 = \infty \) if \( c_2 > 0 \). A similar argument shows that \( T_1 = -\infty \) if \( c_1 > 0 \).

So now suppose that \( c_2 = 0 \). Then either there is a \( T < T_2 \) such that \( y(t) \) is decreasing on \((T, T_2)\) or there is an increasing sequence of local minima, \( m_k \), of \( y \) converging to \( T_2 \) such that \( y(m_{k+1}) < y(m_k) \) and \( \lim_{k \to \infty} y(m_k) = 0 \). However, if the latter is true then by (1.7b) we would have

\[
F(y(m_{k+1})) = E_2(m_{k+1}) \leq E_2(m_k) = F(y(m_k)).
\]

But also for large \( k \), \( y(m_k) < \epsilon \) and since \( F \) is decreasing for \( 0 < \epsilon < \epsilon \), we would have

\[
F(y(m_{k+1})) \geq F(y(m_k))
\]

a contradiction. Thus there is a \( T < T_2 \) such that \( y(t) \) is decreasing on \((T, T_2)\). Thus (2.10) holds. Similarly, if \( c_1 = 0 \) then there is \( T > T_1 \) such that \( y(t) \) is increasing on \((T_1, T)\) and (2.9) holds. This completes the proof of the lemma.

**Lemma 2.3.** If there is an \( m \) such that \( 0 < y(m) \leq L_c \), \( y'(m) = 0 \), and \( y''(m) \geq 0 \), then \( T_1 = -\infty \), \( y' < 0 \) and \( y'' > 0 \) for \( t < m \), and

\[
\lim_{t \to -\infty} y(t) = \infty.
\]

**Proof.** If \( y''(m) > 0 \), then there exists \( \delta > 0 \) such that \( y' < 0 \) on \((m-\delta, m)\). If \( y''(m) = 0 \), then since \( y'''(m) = f(y(m)) < 0 \), it follows that there exists \( \delta > 0 \) such that \( y'' > 0 \) on \((m-\delta, m)\). Since \( y'(m) = 0 \) it then follows that \( y' < 0 \) on \((m-\delta, m)\). Thus we see that if \( y''(m) \geq 0 \) then there exists \( \delta > 0 \) such that \( y' < 0 \) on \((m-\delta, m)\).

Now suppose there exists an \( m^* < m \) such that \( y'(m^*) = 0 \) and \( y' < 0 \) on \((m^*, m)\). Then \( y(m^*) > y(m) \) and since \( E_2 \) is decreasing we see that

\[
F(y(m^*)) = E_2(m^*) \geq E_2(m) = F(y(m)) \geq F_\infty.
\]

Now if \( y(m^*) \leq L_c \), then since \( F \) is strictly decreasing on \((0, L_c)\) we see that \( F(y(m^*)) < F(y(m)) \) which contradicts (2.12). On the other hand, if \( y(m^*) > L_c \), then we see that \( F(y(m^*)) < F_\infty \) which again contradicts (2.12). Thus, no such \( m^* \) can exist and therefore \( y \) is decreasing for \( t < m \). Then from Lemma 2.2 it follows that \( T_1 = -\infty \).

Next, we show that \( y \) has no inflection points for \( t < m \). First we show that if \( y \) has an inflection point, \( p \), then \( y(p) > \epsilon \). So suppose there is a \( p < m \) with \( y''(p) = 0 \) and \( y'' > 0 \) on \((p, m)\) and \( y(p) < \epsilon \). Then on \([p, m]\) we have by (1.7c)

\[
E_3 = -f'(y)(y')^2 \leq 0 \quad \text{since} \quad y < \left(1 + \frac{1}{\lambda+1}\right)\epsilon \quad \text{on} \quad [p, m].
\]

Also

\[
E_3(m) = \frac{1}{2}(y''(m))^2 \geq 0
\]
so
\[ \frac{1}{2}(y'')^2 - f(y)y' \geq 0 \quad \text{on } [p,m]. \]
Evaluating at \( p \) we obtain \( f(y(p))y'(p) \leq 0 \) and since \( y'(p) < 0 \) it follows then that \( f(y(p)) \geq 0 \). Consequently, \( y(p) \geq \epsilon \). Since we assumed \( y(p) \leq \epsilon \) we see that the only possibility is \( y(p) = \epsilon \). However, if \( y(p) = \epsilon \) then \( y''' < 0 \) on \( (p,m) \) and since \( y'''(p) = 0 \) this implies \( y'' < 0 \) on \( (p,m) \), which is a contradiction. Thus, \( y(p) > \epsilon \). Since \( y' < 0 \) for \( t < m \) it follows that \( y''' > 0 \) for \( t < p \) so if \( t < q < p \) then
\[ y''(t) < y''(q) < 0. \]
Integrating on \((t,q)\) gives
\[ y'(q) - y'(t) < y''(q)(q-t). \]
Thus,
\[ y'(q) - y''(q)(q-t) < y'(t) \]
and the left-hand side goes to \( +\infty \) as \( t \to -\infty \) contradicting with \( y' < 0 \) for \( t < m \). Thus \( y'' > 0 \) for \( t < m \). Since we also have that \( y' < 0 \) for \( t < m \) we then see that (2.11) holds. This completes the proof of the lemma.

\[ \square \]

3 Existence of a solution with \( \lim_{t \to -\infty} y(t) = \epsilon \)

We now fix \( \epsilon > 0 \) and \( b \geq 0 \). Let \( y_b \) be the solution of:
\[ \begin{align*}
    y'''(t) &= f_\epsilon(y(t)), \\
    y(0) &= L_\epsilon, \quad y'(0) = 0, \quad y''(0) = b,
\end{align*} \tag{3.1} \tag{3.2} \]
where \( L_\epsilon \) is defined in the statement after (1.4b).

We denote the maximal open interval of existence of (3.1)-(3.2) as \( (T_{1,b}, T_{2,b}) \). From Lemma 2.3 it follows that \( T_{1,b} = -\infty \).

**Lemma 3.1.** If \( b = 0 \), then \( T_{2,b} < \infty \).

**Proof.** We see that \( E_{1,y_b}(0) = 0 \) and since \( E'_{1,y_b}(t) \leq 0 \) (by (1.7a)) and \( E'_{1,y_b}(0) < 0 \) it follows that
\[ E_{1,y_b}(t) < 0 \quad \text{on } (0,T_{2,b}). \]
Hence
\[ 0 \leq \frac{1}{2}(y_b')^2 < (y_b - \epsilon)^2 \quad \text{on } (0,T_{2,b}). \]
Then since \( y_b(0) = L_\epsilon < \epsilon \), we see that \( y_b < \epsilon \) and \( y_b' < 0 \) for \( t > 0 \). Since \( y_b'(0) = 0 \) it follows then that \( y_b' < 0 \) for \( t > 0 \) and therefore \( y_b \) is decreasing and concave down on \((0,T_{2,b})\). Hence \( y_b \) must become zero at some finite value of \( t \). Thus, \( T_{2,b} < \infty \). This completes the proof of the lemma.

\[ \square \]
Lemma 3.2. If \( b > 0 \) is sufficiently large, then \( T_{2,b} = \infty \) and \( y'_b(t) > 0 \) for all \( t > 0 \) (and hence \( y_b(t) > 0 \) for all \( t \in \mathbb{R} \) by Lemma 2.3).

Proof. Since \( y'_b(0) = 0 \) and \( y''_b(0) = b > 0 \), we see that \( y'_b > 0 \) on \((0, \delta)\) for some \( \delta > 0 \). Suppose first that \( T_{2,b} < \infty \). Then by Lemma 2.2, there is an \( M > 0 \) such that \( y'_b(M) = 0 \) and \( y'_b > 0 \) on \((0, M)\). So we see that on \((0, M)\) we have

\[
y_b(t) > y_b(0) = L_\varepsilon
\]

and therefore

\[
y''_b = f(x)(y_b) > f(x)(L_\varepsilon).
\]

Integrating on \((0, t)\) gives

\[
y''_b > b + f(x)(L_\varepsilon)t \quad \text{on} \quad (0, M).
\]

Integrating again on \((0, t)\) gives

\[
y'_b > bt + \frac{f(x)(L_\varepsilon)t^2}{2} \quad \text{on} \quad (0, M).
\]

Taking the limit as \( t \to M^- \) we get \( M \geq \frac{2b}{|f(x)(L_\varepsilon)|} \). Therefore we see that

\[
y'_b > 0 \quad \text{for} \quad 0 < t < \frac{b}{|f(x)(L_\varepsilon)|}.
\]

After another integration we see that

\[
y_b > L_\varepsilon + \frac{b}{2}t^2 + \frac{f(x)(L_\varepsilon)t^3}{6} \quad \text{on} \quad (0, M).
\]

Evaluating this inequality and the \( y''_b \) inequality at \( t = \frac{b}{|f(x)(L_\varepsilon)|} \) we see that

\[
y_b\left(\frac{b}{|f(x)(L_\varepsilon)|}\right) > L_\varepsilon + \frac{b^3}{3|f(x)(L_\varepsilon)|^2}, \quad y''_b\left(\frac{b}{|f(x)(L_\varepsilon)|}\right) > 0.
\]

Therefore, we see that

\[
y_b\left(\frac{b}{|f(x)(L_\varepsilon)|}\right) > \varepsilon \quad \text{if} \quad b \text{ is chosen sufficiently large}.
\]

Now since we already know that \( y'_b > 0 \) on \((0, M)\) so in particular this inequality is true on the interval \((b/|f(x)(L_\varepsilon)|, M)\), we see that

\[
y''_b = f(x)(y_b) > 0 \quad \text{on} \quad \left(\frac{b}{|f(x)(L_\varepsilon)|}, M\right)
\]

so that \( y''_b \) is increasing on this interval and since \( y''_b(b/|f(x)(L_\varepsilon)|) > 0 \), this implies \( y''_b(M) > 0 \).

On the other hand, \( y'_b(M) = 0 \) and \( y'_b > 0 \) on \((0, M)\) which implies \( y''_b(M) \leq 0 \) and so we obtain a contradiction. Thus we see that \( T_{2,b} = \infty \).
So we now assume that $T_{2,b} = \infty$ but that $y_b$ is not increasing for all $t > 0$. So suppose there is an $M$ so that $y'_b > 0$ on $(0, M)$ and $y'_b(M) = 0$. Then repeating the same argument as at the beginning of the proof of this lemma, we will obtain again a contradiction. Thus this completes the proof of the lemma.

Now we define

$$S = \{ b \geq 0 \mid T_{2,b} < \infty \}. \quad (3.3)$$

It follows that $S$ is nonempty (since $0 \in S$ by Lemma 3.1) and bounded above (by Lemma 3.2). Thus we define

$$b_\epsilon = \text{sup} S \quad (3.4)$$

and note that $b_\epsilon \geq 0$.

**Lemma 3.3.** $y_{b_\epsilon}(t) > 0$ for all $t$. (That is, $T_{2,b_\epsilon} = \infty$ and hence $b_\epsilon > 0$ by Lemma 3.1).

**Proof.** Suppose not. Then $T_{2,b_\epsilon} < \infty$ and so by Lemma 2.2 it follows that $y_{b_\epsilon}$ is decreasing on $(T_{2,b_\epsilon} - \delta, T_{2,b_\epsilon})$ for some $\delta > 0$ and

$$\lim_{t \to T_{2,b_\epsilon}^{-}} y_{b_\epsilon}(t) = 0. \quad (3.5)$$

Since $E_{2,y_{b_\epsilon}}$ is decreasing (by (1.7b)) we have

$$F_\epsilon(y_{b_\epsilon}) - y'_{b_\epsilon} y''_{b_\epsilon} = E_{2,y_{b_\epsilon}}(t) \leq E_{2,y_{b_\epsilon}}(0) = F_\epsilon(L_\epsilon) \quad \text{for } 0 \leq t \leq T_{2,b_\epsilon}. \quad (3.6)$$

Now it follows from (1.4a) and Lemma 2.2 that

$$\lim_{t \to T_{2,b_\epsilon}} F_\epsilon(y_{b_\epsilon}(t)) = +\infty. \quad (3.7)$$

Therefore since the right hand side of (3.6) is bounded (since $\epsilon$ is fixed), it follows that

$$\lim_{t \to T_{2,b_\epsilon}} y'_{b_\epsilon}(t) y''_{b_\epsilon}(t) = +\infty.$$ 

From this and Lemma 2.2 it follows that there exists a neighborhood of $T_{2,b_\epsilon}$, $(T_{2,b_\epsilon} - \delta, T_{2,b_\epsilon})$ (where we decrease the size of the $\delta$ chosen at the beginning of the proof if necessary), such that

$$0 < y_{b_\epsilon}(t) < \epsilon, \quad y'_{b_\epsilon}(t) < 0, \quad y''_{b_\epsilon}(t) < 0 \quad \text{for all } t \in (T_{2,b_\epsilon} - \delta, T_{2,b_\epsilon}).$$

Now by Lemma 2.1, it follows that

$$0 < y_b < \epsilon, \quad y_b' < 0, \quad y_b'' < 0 \quad \text{on } \left( T_{2,b_\epsilon} - \frac{2}{3} \delta, T_{2,b_\epsilon} - \frac{1}{3} \delta \right).$$
if $b$ is sufficiently close to $b_c$. If we also require $b > b_c$, then $T_{2,b} = \infty$ (by definition of $b_c$) and so $y_b(t) > 0$ for all $t$. Let us now denote $(T_{2,b_k} - \frac{2}{3}\delta, A_k)$ as the maximal interval for which

$$0 < y_b < \epsilon, \quad y_b' < 0, \quad y_b'' < 0.$$  \hfill (3.8)

From (1.1) we see that $y_b''' < 0$ on $(T_{2,b_k} - \frac{2}{3}\delta, A_k)$. Thus, $0 < y_b < \epsilon$, $y_b$ is decreasing, concave down, and $y_b'$ is decreasing on $(T_{2,b_k} - \frac{2}{3}\delta, A_k)$. Now $A_k$ must be finite for if $A_k$ were infinite then $y_b$ would be decreasing and concave down for $t$ large forcing $y_b$ to become zero in a finite value of $t$ contradicting the fact that $y_b > 0$ for all $t$ (since $b > b_c$). Thus, $A_k$ is finite. Thus, either

$$y_b(A_k) = 0 \quad \text{or} \quad y_b'(A_k) = 0 \quad \text{or} \quad y_b''(A_k) = 0.$$  \hfill (3.9)

However, since $b > b_c$, $y_b > 0$ for all $t$, the first condition is impossible. Also

$$y_b\left(T_{2,b_k} - \frac{2}{3}\delta\right) < \epsilon, \quad y_b'\left(T_{2,b_k} - \frac{2}{3}\delta\right) < 0, \quad y_b''\left(T_{2,b_k} - \frac{2}{3}\delta\right) < 0,$$  

and so from (3.8) we see that $y_b$ is decreasing, concave down, and $y_b''$ is decreasing on $(T_{2,b_k} - \frac{2}{3}\delta, A_k)$. Thus

$$y_b'(A_k) < y_b'\left(T_{2,b_k} - \frac{2}{3}\delta\right) < 0,$$

and

$$y_b''(A_k) < y_b''\left(T_{2,b_k} - \frac{2}{3}\delta\right) < 0,$$

which contradict (3.9). Thus the assumption that $T_{2,b} = \infty$ must be false and so $T_{2,b_k} = \infty$. This completes the proof of the lemma. \hfill \Box

**Lemma 3.4.** $y_{b_k}(t)$ has a first critical point, $m_{1,e} > 0$, which is a local maximum, and $y_{b_k}' > 0$ on $(0, m_{1,e})$. Also,

$$y_{b_k}\left(m_{1,e}\right) > \epsilon, \quad y_{b_k}''\left(m_{1,e}\right) < 0,$$  \hfill (3.10)

and

$$F_{e}\left(y_{b_k}\left(m_{1,e}\right)\right) < F_{e}\left(L_{e}\right).$$  \hfill (3.11)

**Proof.** If not then $y_{b_k}'(t) > 0$ for all $t > 0$. We will now show that this implies $y_{b_k}$ increases without bound. If not then

$$\lim_{t \to \infty} y_{b_k}(t) = B_{\epsilon} < \infty.$$

In this case, we see that

$$\lim_{t \to \infty} y_{b_k}''(t) = \frac{|B_{\epsilon} - \epsilon|}{B_{\epsilon}^{1 + \frac{1}{4}}} \text{sgn}(B_{\epsilon} - \epsilon) \equiv C_{\epsilon}.$$  \hfill (3.12)

If $B_{\epsilon} > \epsilon$ then $y_{b_k}''' \geq C_{\epsilon} > 0$ for large $t$ and integrating three times we see that this would imply that $y_{b_k}$ would be increasing without bound contradicting the fact that

$$\lim_{t \to \infty} y_{b_k}(t) = B_{\epsilon}.$$  \hfill (3.13)

"
On the other hand if $0 \leq B_\epsilon < \epsilon$ then $y_{b_\epsilon}''' \leq C_\epsilon < 0$ for large $t$ and integrating twice we see that this would imply that $y_{b_\epsilon}$ is decreasing for large $t$ contradicting the fact that we are assuming that $y_{b_\epsilon}'(t) > 0$ for all $t > 0$. Thus it must be $B_\epsilon = \epsilon$ so that $y_{b_\epsilon}' > 0$ and $y_{b_\epsilon} < \epsilon$ for all $t > 0$.

Next since $y_{b_\epsilon}'(0) = b_\epsilon > 0$, we see that $y_{b_\epsilon}$ must have a first inflection point $p_\epsilon > 0$ and $y_{b_\epsilon}'' > 0$ on $(0, p_\epsilon)$. Then from (1.1) we see that $y_{b_\epsilon}'$ is decreasing for $t > 0$ so it follows that $y_{b_\epsilon} < 0$ for $t > p_\epsilon$, and it also follows that there is a $q_\epsilon > p_\epsilon$ such that

$$y_{b_\epsilon}'' < y_{b_\epsilon}''(q_\epsilon) < 0 \quad \text{for} \quad t > q_\epsilon.$$ Integrating on $(q_\epsilon, t)$ gives

$$y_{b_\epsilon}' < y_{b_\epsilon}'(q_\epsilon) + y_{b_\epsilon}''(q_\epsilon)(t - q_\epsilon)$$

which implies that $y_{b_\epsilon}' < 0$ for large enough $t$ which contradicts that $y_{b_\epsilon}' > 0$ for $t > 0$. Thus, we see that if $y_{b_\epsilon}' > 0$ for all $t > 0$ then it must be the case that $y_{b_\epsilon}$ does not stay bounded on $[0, \infty)$.

In particular, then there is a $z_\epsilon > 0$ with $y_{b_\epsilon}'(z_\epsilon) = \epsilon$ and $y_{b_\epsilon}$ is increasing for all $t > 0$. Thus from (1.1), $y_{b_\epsilon}'' > 0$ for $t > z_\epsilon$. So there is a $q_\epsilon > z_\epsilon$ and a $c_\epsilon > 0$ such that $y_{b_\epsilon}''' > c_\epsilon$ for $t > q_\epsilon$ hence

$$y_{b_\epsilon}''(t) > y_{b_\epsilon}''(q_\epsilon) + c_\epsilon(t - q_\epsilon) \quad \text{for} \quad t > q_\epsilon$$

and so we see that there is an $r_\epsilon$ such that $y_{b_\epsilon}'(t) > 0$ for $t > r_\epsilon$. Integrating again we see that $y_{b_\epsilon}'(t) > 0$ for $t > r_\epsilon$ and another integration gives that $y_{b_\epsilon}(t) > \epsilon$ for $t > r_\epsilon$.

Now if $b < b_\epsilon$ and $b$ is sufficiently close to $b_\epsilon$ then by Lemma 2.1 $y_b > \epsilon$, $y_b' > 0$ and $y_b''' > 0$ for $r_\epsilon < t < r_\epsilon + 1$. Then from (1.1) $y_b''' > 0$ for $r_\epsilon < t < r_\epsilon + 1$. Therefore, $y_b$, $y_b'$, and $y_b''$ are increasing and $y_b > \epsilon$ for $r_\epsilon < t < r_\epsilon + 1$ and so we see that these conditions continue to hold for $r_\epsilon < t < \infty$, but this contradicts the fact that for $b < b_\epsilon$, $y_b$ must have a zero. Thus we finally see that $y_{b_\epsilon}$ cannot be increasing for all $t > 0$ and so we see that there exists $m_{1,\epsilon} > 0$ such that

$$y_{b_\epsilon}' > 0 \quad \text{on} \quad (0, m_{1,\epsilon}) \quad \text{and} \quad y_{b_\epsilon}'(m_{1,\epsilon}) = 0.$$ From calculus, it also follows that $y_{b_\epsilon}''(m_{1,\epsilon}) \leq 0$.

We next claim that $y_{b_\epsilon}(m_{1,\epsilon}) > \epsilon$. First we suppose that $y_{b_\epsilon}(m_{1,\epsilon}) < \epsilon$. Then

$$E_{1, y_{b_\epsilon}}(m_{1,\epsilon}) \leq 0 \quad \text{and} \quad E_{1, y_{b_\epsilon}}'(m_{1,\epsilon}) < 0$$

so that since $E_{1, y_{b_\epsilon}}$ is decreasing (by (1.7a)), we see that $E_{1, y_{b_\epsilon}} < 0$ for $t > m_{1,\epsilon}$. Thus

$$0 \leq \frac{1}{2}(y_{b_\epsilon}')^2 < (y_{b_\epsilon} - \epsilon)y_{b_\epsilon}''' \quad \text{for} \quad t > m_{1,\epsilon}$$

and since $y_{b_\epsilon}(m_{1,\epsilon}) < \epsilon$ we see that

$$y_{b_\epsilon}(t) < \epsilon \quad \text{for} \quad t > m_{1,\epsilon} \quad \text{and} \quad y_{b_\epsilon}''(t) < 0 \quad \text{for} \quad t > m_{1,\epsilon}.$$
Thus there is a \( y'_{b_e}(m_{1,\varepsilon}) = 0 \), this implies \( y_{b_e}(t) \) will become 0 at some finite value of \( t \) contradicting Lemma 3.3. Thus we see that \( y_{b_e}(m_{1,\varepsilon}) \geq \varepsilon \).

Next we suppose that \( y_{b_e}(m_{1,\varepsilon}) = c \). In this case either \( y''_{b_e}(m_{1,\varepsilon}) = 0 \) or \( y''_{b_e}(m_{1,\varepsilon}) < 0 \).

If \( y''_{b_e}(m_{1,\varepsilon}) < 0 \) then \( y_{b_e} < \varepsilon \) on \( (m_{1,\varepsilon}, m_{1,\varepsilon} + \delta) \) for some \( \delta > 0 \). Hence \( E'_{1,y_{b_e}} < 0 \) on \( (m_{1,\varepsilon}, m_{1,\varepsilon} + \delta) \) and by (1.7a) since \( E_{1,y_{b_e}}(m_{1,\varepsilon}) = 0 \) we see that \( E_{1,y_{b_e}}(t) < 0 \) for \( t > m_{1,\varepsilon} \). Then as in the previous paragraph this implies \( y_{b_e}(t) \) will become 0 at some finite value of \( t \) again contradicting Lemma 3.3.

Finally, we suppose that \( y_{b_e}(m_{1,\varepsilon}) = \varepsilon \) and \( y''_{b_e}(m_{1,\varepsilon}) = 0 \). Since \( y_{b_e}(t) < \varepsilon \) for \( 0 < t < m_{1,\varepsilon} \), we have \( y''_{b_e}(t) < 0 \) for \( 0 < t < m_{1,\varepsilon} \). Thus, \( y_{b_e}(t) \) is decreasing for \( 0 < t < m_{1,\varepsilon} \). Since \( y''_{b_e}(m_{1,\varepsilon}) = 0 \) this implies \( y''_{b_e} > 0 \) for \( 0 < t < m_{1,\varepsilon} \). However, the mean value theorem implies that there exists a \( c \) with \( 0 < c < m_{1,\varepsilon} \) such that

\[
0 = y'_{b_e}(m_{1,\varepsilon}) - y'_{b_e}(0) = y''_{b_e}(c)m_{1,\varepsilon}
\]

which contradicts with \( y''_{b_e} > 0 \) for \( 0 < t < m_{1,\varepsilon} \).

Thus we demonstrate that \( y_{b_e}(m_{1,\varepsilon}) > \varepsilon \).

Next we show that \( y''_{b_e}(m_{1,\varepsilon}) < 0 \). From calculus it follows that \( y''_{b_e}(m_{1,\varepsilon}) \leq 0 \), so we assume now by way of contradiction that \( y''_{b_e}(m_{1,\varepsilon}) = 0 \). This implies that \( E_{1,y_{b_e}}(m_{1,\varepsilon}) = 0 \). Also, since \( y_{b_e}(m_{1,\varepsilon}) > \varepsilon \) we see that \( E'_{1,y_{b_e}}(m_{1,\varepsilon}) < 0 \) and since \( E_{1,y_{b_e}} \) is decreasing (by (1.7a)) we see that

\[
\frac{1}{2}(y'_{b_e})^2 - y_{b_e}y''_{b_e} = E_{1,y_{b_e}} < 0 \quad \text{for } t > m_{1,\varepsilon}.
\]

Thus there is a \( \delta > 0 \) such that \( E_{1,y_{b_e}} < 0 \) for \( t \geq m_{1,\varepsilon} + \delta \). Thus for \( b < b_e \) and \( b \) sufficiently close to \( b_e \) we also have \( E_{1,y_{b}} < 0 \) for \( t \geq m_{1,\varepsilon} + \delta \).

Also, perhaps by choosing a smaller \( \delta \) if necessary, we see that \( y_{b_e} > 0 \) on \( (0, m_{1,\varepsilon} - \delta] \) and \( y_{b_e} > \varepsilon \) on \( [m_{1,\varepsilon} - \delta, m_{1,\varepsilon} + \delta] \).

So by Lemma 2.1 and since \( b_e > 0 \), if \( b \) is sufficiently close to \( b_e \) then \( y'_{b} > 0 \) on \( (0, m_{1,\varepsilon} - \delta] \) and \( y_{b} > \varepsilon \) on \( [m_{1,\varepsilon} - \delta, m_{1,\varepsilon} + \delta] \). Now if we choose \( b > b_e \), then by definition of \( b_e \) we see there exists an \( r_b > m_{1,\varepsilon} + \delta \) such that \( y_{b}(r_b) = 0 \). Therefore by the intermediate value theorem there is a \( z_b \) with \( m_{1,\varepsilon} + \delta < z_b < r_b \) such that \( y_{b}(z_b) = \varepsilon \). Hence

\[
E_{1,y_{b}}(z_b) = \frac{1}{2}(y'_{b}(z_b))^2 \geq 0.
\]

On the other hand, we know from earlier that since \( z_b > m_{1,\varepsilon} + \delta \) then \( E_{1,y_{b}}(z_b) < 0 \). Thus we obtain a contradiction. Therefore it must be that \( y''_{b_e}(m_{1,\varepsilon}) < 0 \).

Finally, since \( E_{2,y_{b_e}} \) is decreasing (by (1.7b)) and \( E'_{2,y_{b_e}}(0) < 0 \) we have

\[
E_{2,y_{b_e}}(m_{1,\varepsilon}) < E_{2,y_{b_e}}(0)
\]

and hence (3.11) holds. This completes the proof of the lemma. \[\Box\]
Lemma 3.5. $y_{b_\epsilon}(t)$ has a second critical point at $m_{2,\epsilon} > 0$ which is a local minimum, and $y'_{b_\epsilon} < 0$ on $(m_{1,\epsilon}, m_{2,\epsilon})$. Also,

$$y_{b_\epsilon}(m_{2,\epsilon}) < \epsilon \quad \text{and} \quad y''_{b_\epsilon}(m_{2,\epsilon}) > 0$$

and

$$F_\epsilon(y_{b_\epsilon}(m_{2,\epsilon})) < F_\epsilon(y_{b_\epsilon}(m_{1,\epsilon})).$$

Proof. The proof of this lemma is nearly identical to the proof of Lemma 3.4 and we omit it here. \[\Box\]

In order to simplify notation a bit we now write $E_1, E_2, E_3$ instead of $E_{1,y_{b_\epsilon}}, E_{2,y_{b_\epsilon}},$ and $E_{3,y_{b_\epsilon}}$, respectively.

Continuing in this way we see that there is a sequence of extrema with

$$m_{1,\epsilon} < m_{2,\epsilon} < m_{3,\epsilon} < m_{4,\epsilon} < \cdots$$

such that the $m_{2k,\epsilon}$ are local minima, the $m_{2k-1,\epsilon}$ are local maxima, $y$ is monotone of $(m_{n,\epsilon}, m_{n+1,\epsilon})$, and since $E_{2,\epsilon}$ is decreasing, we have

$$F_\epsilon(y_{b_\epsilon}(m_{k+1,\epsilon})) < F_\epsilon(y_{b_\epsilon}(m_{k,\epsilon})).$$

Note that this implies

$$y_{b_\epsilon}(m_{2k,\epsilon}) < y_{b_\epsilon}(m_{2k+2,\epsilon}) < \epsilon \quad \text{and} \quad \epsilon < y_{b_\epsilon}(m_{2k+1,\epsilon}) < y_{b_\epsilon}(m_{2k-1,\epsilon}).$$

(3.16)

We now let

$$M_\epsilon = \lim_{n \to \infty} m_{n,\epsilon}$$

(3.17)

and note that $M_\epsilon \leq \infty$.

Lemma 3.6. $y_{b_\epsilon}(t)$ oscillates infinitely often, and

$$\lim_{t \to M_\epsilon^-} y_{b_\epsilon}(t) = \epsilon, \quad \lim_{t \to M_\epsilon^+} y'_{b_\epsilon}(t) = 0, \quad \lim_{t \to M_\epsilon^+} y''_{b_\epsilon}(t) = 0.$$

Proof. We have $0 \equiv m_{0,\epsilon} < m_{1,\epsilon} < m_{2,\epsilon} < m_{3,\epsilon} < \cdots$ and

$$F_\epsilon(L_\epsilon) > F_\epsilon(y_{b_\epsilon}(m_{1,\epsilon})) > F_\epsilon(y_{b_\epsilon}(m_{2,\epsilon})) > F_\epsilon(y_{b_\epsilon}(m_{3,\epsilon})) > \cdots.$$

Also, there exists $z_{k,\epsilon}$ such that

$$0 < z_{1,\epsilon} < m_{1,\epsilon} < z_{2,\epsilon} < m_{2,\epsilon} < z_{3,\epsilon} < \cdots, \quad y_{b_\epsilon}(z_{n,\epsilon}) = \epsilon, \quad \lim_{n \to \infty} z_{n,\epsilon} = M_\epsilon.$$

Next we observe that since $y'_{b_\epsilon}(m_k) = y'_{b_\epsilon}(m_{k+1}) = 0$ the extrema of $y'_{b_\epsilon}$ on $(m_{k,\epsilon}, m_{k+1,\epsilon})$ must occur at points $p$ where $y''_{b_\epsilon}(p) = 0$ so

$$\frac{1}{2}[y'_{b_\epsilon}(p)]^2 = E_{1,\epsilon}(p) \leq E_{1,\epsilon}(0) = (\epsilon - L_\epsilon) b_\epsilon.$$
Thus for every $k \geq 0$

$$|y''_b(t)| \leq \sqrt{2(e-L_e)\epsilon} = K_e \quad \text{on} \quad [m_{k,e}, m_{k+1,e}].$$

Then since $m_{k,e} \to M_e$ as $k \to \infty$ we obtain

$$|y''_b(t)| \leq \sqrt{2(e-L_e)\epsilon} = K_e \quad \text{on} \quad [0,M_e]. \quad (3.18)$$

Next, since $E_{1,e}$ is decreasing, $E_{1,e}(z_{k,e}) = \frac{1}{2}|y'_b(z_{k,e})|^2 \geq 0$, and $z_{k,e} \to M_e$ we see that

$$\lim_{t \to M_e^-} E_{1,e}(t) = e_{1,e} \geq 0. \quad (3.19)$$

Integrating (1.7a) on $(0,t)$ we obtain

$$E_{1,e}(t) = (e-L_e)\epsilon - \int_0^t (y'_b - \epsilon) f_e(y'_b) dt.$$ Using (3.19) and taking limits as $t \to M_e^-$ give

$$(e-L_e)\epsilon = e_{1,e} + \int_0^{M_e} (y'_b - \epsilon) f_e(y'_b).$$

Thus we see that

$$\int_0^{M_e} (y'_b - \epsilon) f_e(y'_b) \quad \text{is finite.} \quad (3.20)$$

We have $y''''_{b,e} > 0$ on $(z_{1,e}, m_{1,e})$ so that $y''_{b,e}$ is increasing on $(z_{1,e}, m_{1,e})$. Also from Lemma 3.4 we know that $y''''_{b,e} (m_{1,e}) < 0$ therefore it follows that $y''_{b,e} < 0$ on $(z_{1,e}, m_{1,e})$. Therefore, $y'_b$ is concave down on $(z_{1,e}, m_{1,e})$ and so it follows that

$$y'_b - \epsilon \geq \frac{y_{b,e}(m_{1,e}) - \epsilon}{m_{1,e} - z_{1,e}} (t - z_{1,e}) \quad \text{on} \quad (z_{1,e}, m_{1,e}). \quad (3.21)$$

Similarly, since $y''''_{b,e} > 0$ on $(z_{2,e}, m_{2,e})$ we see that

$$y'_b - \epsilon \leq \frac{y_{b,e}(m_{2,e}) - \epsilon}{m_{2,e} - z_{2,e}} (t - z_{2,e}) \quad \text{on} \quad (z_{2,e}, m_{2,e}). \quad (3.22)$$

Thus, it follows from (3.21) that

$$\int_{z_{1,e}}^{m_{1,e}} (y'_b - \epsilon) f(y'_b) dt \geq \int_{z_{1,e}}^{m_{1,e}} \frac{|y'_b - \epsilon|^{1+\frac{1}{\lambda}}}{y'_b^{1+\frac{1}{\lambda}}} dt$$

$$\geq \frac{1}{y_{b,e}(m_{1,e})^{1+\frac{1}{\lambda}}} \int_{m_{1,e} - z_{1,e}}^{y_{b,e}(m_{1,e}) - \epsilon} \int_{z_{1,e}}^{m_{1,e}} (t - z_{1,e})^{1+\frac{1}{\lambda}} dt$$

$$= \frac{\lambda}{2\lambda + 1} \frac{|y_{b,e}(m_{1,e}) - \epsilon|^{1+\frac{1}{\lambda}}}{y_{b,e}(m_{1,e})^{1+\frac{1}{\lambda}}} (m_{1,e} - z_{1,e}).$$
Also, by the mean value theorem and (3.18) we have
\[ |y_{b_{\epsilon}}(m_{1,\epsilon}) - \epsilon| = |y_{b_{\epsilon}}(m_{1,\epsilon}) - y_{b_{\epsilon}}(z_{1,\epsilon})| = |y'_{b_{\epsilon}}(c_{1,\epsilon})||(m_{1,\epsilon} - z_{1,\epsilon})| \leq K_{\epsilon}|m_{1,\epsilon} - z_{1,\epsilon}|. \]

Thus
\[ \int_{z_{1,\epsilon}}^{m_{1,\epsilon}} (y_{b_{\epsilon}} - \epsilon) f_{\epsilon}(y_{b_{\epsilon}}) \geq \frac{\lambda |y_{b_{\epsilon}}(m_{1,\epsilon}) - \epsilon|^{2+\frac{1}{z}}}{(2\lambda + 1)K_{\epsilon}y_{b_{\epsilon}}(m_{1,\epsilon})^{1+\frac{1}{z}}}. \tag{3.23} \]

A similar inequality holds over \((z_{2,\epsilon}, m_{2,\epsilon})\) and thus
\[ \int_{z_{2,\epsilon}}^{m_{2,\epsilon}} (y_{b_{\epsilon}} - \epsilon) f_{\epsilon}(y_{b_{\epsilon}}) \geq \frac{\lambda |y_{b_{\epsilon}}(m_{2,\epsilon}) - \epsilon|^{2+\frac{1}{z}}}{(2\lambda + 1)K_{\epsilon}y_{b_{\epsilon}}(m_{2,\epsilon})^{1+\frac{1}{z}}}. \]

Now using (3.16) we see that
\[ \int_{z_{2,\epsilon}}^{m_{2,\epsilon}} (y_{b_{\epsilon}} - \epsilon) f_{\epsilon}(y_{b_{\epsilon}}) \geq \frac{\lambda |y_{b_{\epsilon}}(m_{2,\epsilon}) - \epsilon|^{2+\frac{1}{z}}}{(2\lambda + 1)K_{\epsilon}y_{b_{\epsilon}}(m_{1,\epsilon})^{1+\frac{1}{z}}}. \]

Similarly we can show
\[ \int_{z_{1,\epsilon}}^{m_{1,\epsilon}} (y_{b_{\epsilon}} - \epsilon) f_{\epsilon}(y_{b_{\epsilon}}) \geq \frac{\lambda |y_{b_{\epsilon}}(m_{k,\epsilon}) - \epsilon|^{2+\frac{1}{z}}}{(2\lambda + 1)K_{\epsilon}y_{b_{\epsilon}}(m_{1,\epsilon})^{1+\frac{1}{z}}}. \tag{3.24} \]

Next using (3.20) and the fact that \((y_{b_{\epsilon}} - \epsilon) f_{\epsilon}(y_{b_{\epsilon}}) \geq 0\) for all \(t\) we obtain
\[ \infty > \int_{0}^{M_{\epsilon}} (y_{b_{\epsilon}} - \epsilon) f_{\epsilon}(y_{b_{\epsilon}}) \, dt \geq \sum_{k=1}^{\infty} \int_{z_{k,\epsilon}}^{m_{k,\epsilon}} (y_{b_{\epsilon}} - \epsilon) f_{\epsilon}(y_{b_{\epsilon}}) \, dt \geq \frac{\lambda}{(2\lambda + 1)K_{\epsilon}y_{b_{\epsilon}}(m_{1,\epsilon})^{1+\frac{1}{z}}} \sum_{k=1}^{\infty} |y_{b_{\epsilon}}(m_{k,\epsilon}) - \epsilon|^{2+\frac{1}{z}}. \]

Thus
\[ \sum_{k=1}^{\infty} |y_{b_{\epsilon}}(m_{k,\epsilon}) - \epsilon|^{2+\frac{1}{z}} < \infty. \]

Consequently,
\[ \lim_{k \to \infty} |y_{b_{\epsilon}}(m_{k,\epsilon}) - \epsilon| = 0 \]

and since \(m_{k,\epsilon} \to M_{\epsilon}^{-}\) and the \(m_{k,\epsilon}\) are extrema of \(y_{b_{\epsilon}}\) we see that
\[ \lim_{t \to M_{\epsilon}} |y_{b_{\epsilon}}(t) - \epsilon| = 0. \tag{3.25} \]
Then by (1.1) we obtain
\[ \lim_{t \to M^-} y''_{b_\epsilon}(t) = 0. \] (3.26)

We also know that \( E_{2,\epsilon}' \leq 0 \) (by (1.7b)) and by (1.7c) and (3.25) we know that \( E_{3,\epsilon}' \leq 0 \) for \( t \) close to \( M_\epsilon \) so that
\[ \lim_{t \to M^-} E_{2,\epsilon}(t) = e_{2,\epsilon}, \quad \lim_{t \to M^-} E_{3,\epsilon}(t) = e_{3,\epsilon}. \] (3.27)

Also since \( E_{2,\epsilon}(m_{k,\epsilon}) \geq 0 \) and \( E_{3,\epsilon}(m_{k,\epsilon}) \geq 0 \) and since \( m_{k,\epsilon} \to M_\epsilon \) we see that
\[ e_{2,\epsilon} \geq 0 \quad \text{and} \quad e_{3,\epsilon} \geq 0. \] (3.28)

From (3.18) and (3.25) it follows that
\[ f_\epsilon(y_{b_\epsilon})y_{b_\epsilon} \to 0 \quad \text{as} \quad t \to M^-_\epsilon. \]

Combining this with (3.27) we see that
\[ \lim_{t \to M^-} \frac{1}{2} (y''_{b_\epsilon})^2 = e_{3,\epsilon}. \]

Since \( y'_{b_\epsilon} \) is bounded (by (3.18)) we see that the only possibility is that \( e_{3,\epsilon} = 0 \) thus
\[ \lim_{t \to M^-} y''_{b_\epsilon} = 0. \] (3.29)

Now using (3.19), (3.25), and (3.29) we see that
\[ \lim_{t \to M^-} \frac{1}{2} (y'_{b_\epsilon})^2 = \lim_{t \to M^-} E_{1,\epsilon} = e_{1,\epsilon}. \] (3.30)

Since \( y_{b_\epsilon} \) is bounded (by (3.25)) we see that the only possibility is that \( e_{1,\epsilon} = 0 \) and so
\[ \lim_{t \to M^-} y'_{b_\epsilon}(t) = 0. \] (3.31)

Using (3.25), (3.29), and (3.31) completes the proof of the lemma. \( \square \)

One final note, if \( M_\epsilon < \infty \) then since
\[ \lim_{t \to M^-} y_{b_\epsilon}(t) = \epsilon, \quad \lim_{t \to M^-} y'_{b_\epsilon}(t) = 0, \quad \lim_{t \to M^-} y''_{b_\epsilon}(t) = 0, \]
we see that we may extend \( y_{b_\epsilon}(t) \) for \( t \geq M_\epsilon \) by simply defining
\[ y_{b_\epsilon}(t) \equiv \epsilon \quad \text{for} \quad t \geq M_\epsilon. \]

Then whether \( M_\epsilon < \infty \) or \( M_\epsilon = \infty \) we see that
\[ \lim_{t \to \infty} y_{b_\epsilon}(t) = \epsilon. \]
4 Determination of $\lim_{\epsilon \to 0^+} y_b(\epsilon)(t)$

Lemma 4.1. Let $L_\epsilon$ be defined by (1.5). Then

$$L_\epsilon = L_1 \epsilon \quad \text{where} \quad 0 < L_1 < 1.$$ (4.1)

Proof. First we denote

$$I = \int_1^\infty \frac{(t-1)^{\frac{\lambda}{2}}}{t^{1+\frac{\lambda}{2}}} \, dt. \quad (4.2)$$

Next, by definition we have

$$F_\epsilon(y) = \int_y^\infty \frac{|s-y|^{\frac{\lambda}{2}} \text{sgn}(s-y)}{s^{1+\frac{\lambda}{2}}} \, ds.$$ (4.3)

Making the change of variables $s = \epsilon t$ we obtain

$$F_\epsilon(y) = \epsilon^{-\frac{\lambda}{2}} F_1(y/\epsilon).$$ (4.3)

Hence, by (1.4b), (4.2), and (4.3) we see that

$$F_{\epsilon,\infty} = \lim_{\epsilon \to \infty} F_\epsilon(y) = e^{-\frac{\lambda}{2}} \int_1^\infty \frac{(t-1)^{\frac{\lambda}{2}}}{t^{1+\frac{\lambda}{2}}} \, dt = e^{-\frac{\lambda}{2}} I.$$ (4.5)

Also, by the statement after (1.4b) and (4.3) we see that

$$e^{-\frac{\lambda}{2}} \int_\frac{1}{\epsilon}^1 \frac{(1-t)^{\frac{\lambda}{2}}}{t^{1+\frac{\lambda}{2}}} \, dt = F_\epsilon(L_\epsilon) = F_{\epsilon,\infty} = e^{-\frac{\lambda}{2}} I.$$ (4.5)

So we see from (4.2) and the above line that

$$\int_1^\infty \frac{(t-1)^{\frac{\lambda}{2}}}{t^{1+\frac{\lambda}{2}}} \, dt = \int_\frac{1}{\epsilon}^1 \frac{(1-t)^{\frac{\lambda}{2}}}{t^{1+\frac{\lambda}{2}}} \, dt,$$

which implies that $L_\epsilon/\epsilon$ is independent of $\epsilon$ since $I$ does not depend on $\epsilon$ (by (4.2)). Thus $L_\epsilon/\epsilon = L_1$. Also, from the statement after (1.4b) we see that $0 < L_\epsilon < \epsilon$ and thus $0 < L_1 < 1$. This completes the proof of the lemma.

Lemma 4.2. If

$$b > \left[3 f_\epsilon^2 (L_\epsilon)(\epsilon-L_\epsilon) \right]^{\frac{1}{4}},$$ (4.4)

then $y_b(t) > 0$ for all $t \geq 0$ (and thus $b \not\in S$ (see (3.3))). Hence,

$$b_\epsilon \leq \left[3 f_\epsilon^2 (L_\epsilon)(\epsilon-L_\epsilon) \right]^{\frac{1}{4}}.$$ (4.5)
Proof. Since
\[ y_b(0) = L_\epsilon, \quad y_b'(0) = 0, \quad y_b''(0) = b > 0, \]
it follows that \( y_b(t) \) is initially increasing and so \( y_b(t) > L_\epsilon \) on \((0, \delta)\) for some \( \delta > 0 \). So on this interval we have
\[ y_b'' > f_\epsilon(L_\epsilon). \]
Successively integrating on \((0,t]\) we get
\[ y_b'' > b + tf_\epsilon(L_\epsilon), \quad y_b' > bt + \frac{t^2 f_\epsilon(L_\epsilon)}{2}, \quad y_b > L_\epsilon + \frac{bt^2}{2} + \frac{t^3 f_\epsilon(L_\epsilon)}{6}. \]

Next, we observe that
\[ y_b' > 0, \quad y_b'' > 0 \quad \text{for} \quad 0 < t \leq \frac{b}{|f_\epsilon(L_\epsilon)|}. \]
From the inequality for \( y_b \) and (4.4) we see that
\[ y_b\left(\frac{b}{|f_\epsilon(L_\epsilon)|}\right) > L_\epsilon + \frac{b^3}{3|f_\epsilon(L_\epsilon)|^2} > L_\epsilon + \epsilon - L_\epsilon = \epsilon. \]
Then since
\[ y_b'\left(\frac{b}{|f_\epsilon(L_\epsilon)|}\right) > 0, \quad y_b''\left(\frac{b}{|f_\epsilon(L_\epsilon)|}\right) > 0, \]
it follows from (1.1) that
\[ y_b'''\left(\frac{b}{|f_\epsilon(L_\epsilon)|}\right) > 0. \]
This in fact implies hence \( y_b' > 0 \) and \( y_b'' > 0 \) for all \( t > b/|f_\epsilon(L_\epsilon)| \) so that in fact \( y_b(t) > 0 \) for all \( t \geq 0 \). This completes the proof of the lemma. \( \square \)

Lemma 4.3.
\[ b_\epsilon \leq \frac{Q}{\epsilon^{\frac{1}{3} + \frac{2}{\alpha}}}, \quad \text{where} \quad Q = \left(\frac{3(1-L_1)^{\frac{1}{2} + \frac{2}{\alpha}}}{L_1^{2 + \frac{2}{\alpha}}}\right)^{\frac{3}{4}}. \]

Proof. We know that \( L_\epsilon = L_1\epsilon \) by Lemma 4.1 so that
\[ |f_\epsilon(L_\epsilon)| = |f_\epsilon(L_1\epsilon)| = \frac{(1-L_1)^{\frac{1}{2} + \frac{2}{\alpha}} 1}{L_1^{1 + \frac{2}{\alpha}} \epsilon^{1 + \frac{2}{\alpha}}}. \]
Substituting this equation and that \( L_\epsilon = L_1\epsilon \) into the consequence of Lemma 4.2 we see that
\[ b_\epsilon^3 \leq 3f_\epsilon^2(L_\epsilon)(\epsilon - L_\epsilon) = \frac{3(1-L_1)^{\frac{1}{2} + \frac{2}{\alpha}} 1}{L_1^{2 + \frac{2}{\alpha}}} \frac{1}{\epsilon^{1 + \frac{2}{\alpha}}}(1-L_1)\epsilon = \frac{Q^3}{\epsilon^{1 + \frac{2}{\alpha}}}. \]
Taking cube roots we see that this completes the proof of the lemma. \( \square \)
Lemma 4.4. $y_{b_\varepsilon} \to 0$ and $y_{b_\varepsilon}' \to 0$ uniformly on compact subsets of $[0, \infty)$.

Proof. Since $E_{1,\varepsilon}$ is decreasing by (1.7a), for $t \geq 0$ we have by Lemma 4.3 that

$$\frac{1}{2}(y_{b_\varepsilon}')^2 - (y_{b_\varepsilon} - e)y_{b_\varepsilon}'' = E_{1,\varepsilon} \leq E_{1,\varepsilon}(0) = (e - L_{\varepsilon})b_{\varepsilon} \leq eb_{\varepsilon} \leq Qe^{\frac{\varepsilon}{2}(1 - \frac{1}{\lambda})}. \quad (4.6)$$

Also, since $y_{b_\varepsilon}'(0) = 0$ and $\lim_{t \to -M_{\varepsilon}} y_{b_\varepsilon}'(t) = 0$ (by Lemma 3.6) we see that the maximum of $|y_{b_\varepsilon}'|$ occurs at some point $p$ where $y_{b_\varepsilon}''(p) = 0$. Evaluating (4.6) at $p$ gives

$$\frac{1}{2}(y_{b_\varepsilon}'(p))^2 \leq Qe^{\frac{\varepsilon}{2}(1 - \frac{1}{\lambda})}.$$ 

Thus

$$|y_{b_\varepsilon}'(t)| \leq \sqrt{2Qe^{\frac{\varepsilon}{2}(1 - \frac{1}{\lambda})}} \quad \text{for all} \quad t \geq 0.$$ 

Consequently,

$$|y_{b_\varepsilon}'(t)| \to 0 \quad \text{uniformly on} \quad [0, \infty).$$

Now letting $P > 0$ and integrating on $[0, P]$ we see that

$$|y_{b_\varepsilon}(t) - L_{\varepsilon}| \leq P\sqrt{2Qe^{\frac{\varepsilon}{2}(1 - \frac{1}{\lambda})}}$$

and since $L_{\varepsilon} \to 0$ as $\varepsilon \to 0$ (by Lemma 4.1) we see that $y_{b_\varepsilon}(t) \to 0$ uniformly on compact subsets of $[0, \infty)$. This completes the proof of the lemma.

We now investigate the behavior of $y_{b_\varepsilon}(t)$ as $t \to -\infty$. From Lemma 2.3 we know that

$$y_{b_\varepsilon}'(t) < 0, \quad y_{b_\varepsilon}''(t) > 0 \quad \text{for} \quad t < 0 \quad \text{and} \quad \lim_{t \to -\infty} y_{b_\varepsilon}(t) = \infty.$$ 

Thus, for $t$ sufficiently negative we have that

$$y_{b_\varepsilon}(t) > \left(1 + \frac{1}{\lambda + 1}\right)e$$

and thus by (1.7c) $E_{3,\varepsilon}' \geq 0$ if $t$ is sufficiently negative. Thus, there exists $t_{0,\varepsilon} < 0$ such that $E_{3,\varepsilon}(t) \leq E_{3,\varepsilon}(t_{0,\varepsilon})$ for $t < t_{0,\varepsilon}$. Thus,

$$\frac{1}{2}(y_{b_\varepsilon}'(t))^2 - f_{\varepsilon}(y_{b_\varepsilon})y_{b_\varepsilon}'' \leq E_{3,\varepsilon}(t_{0,\varepsilon}) \quad \text{for} \quad t < t_{0,\varepsilon}.$$ 

Since $y_{b_\varepsilon}' < 0$ for $t < 0$ and $y_{b_\varepsilon} > (1 + \frac{1}{\lambda + 1})e > e$ for $t < t_{0,\varepsilon}$ we see that

$$0 \leq \frac{1}{2}(y_{b_\varepsilon}'(t))^2 \leq E_{3,\varepsilon}(t_{0,\varepsilon}), \quad 0 \leq -f_{\varepsilon}(y_{b_\varepsilon})y_{b_\varepsilon}' \leq E_{3,\varepsilon}(t_{0,\varepsilon}) \quad \text{for} \quad t < t_{0,\varepsilon}.$$
Thus $E_{3,\varepsilon}(t) \geq 0$ for $t < t_{0,\varepsilon}$ and since $E_{3,\varepsilon}(t)$ is increasing for $t < t_{0,\varepsilon}$ it follows that
\[ \lim_{t \to -\infty} E_{3,\varepsilon}(t) = e_{3,\varepsilon} \geq 0. \]

Since $y_{b_{\varepsilon}}'' = f_{\varepsilon}(y_{b_{\varepsilon}}) > 0$ for $t < t_{0,\varepsilon}$, we see that $y_{b_{\varepsilon}}''$ is increasing for $t < t_{0,\varepsilon}$ and since we also have $y_{b_{\varepsilon}}'' > 0$ for $t < 0$, it follows that
\[ \lim_{t \to -\infty} y_{b_{\varepsilon}}''(t) = A_{\varepsilon} \geq 0. \]

Combining this with the fact that $E_{3,\varepsilon}$ has a limit as $t \to -\infty$ it follows that
\[ \lim_{t \to -\infty} -f_{\varepsilon}(y_{b_{\varepsilon}}) y_{b_{\varepsilon}}' = G_{\varepsilon} \geq 0. \]

**Lemma 4.5.**

\[ \lim_{t \to -\infty} f_{\varepsilon}(y_{b_{\varepsilon}}) y_{b_{\varepsilon}}' = 0. \]

**Proof.** Suppose that $G_{\varepsilon} > 0$. Then there exists a sufficiently negative $t_{1,\varepsilon}$ such that
\[ -f_{\varepsilon}(y_{b_{\varepsilon}}) y_{b_{\varepsilon}}' \geq \frac{G_{\varepsilon}}{2} \quad \text{for} \quad t < t_{1,\varepsilon}. \]

Therefore
\[ \int_{t}^{t_{1,\varepsilon}} -f_{\varepsilon}(y_{b_{\varepsilon}}) y_{b_{\varepsilon}}' \, ds \geq \int_{t}^{t_{1,\varepsilon}} \frac{G_{\varepsilon}}{2} \, ds \]
so that
\[ \infty > F_{\varepsilon,\infty} \geq F_{\varepsilon}(y_{b_{\varepsilon}}(t)) \geq -f_{\varepsilon}(y_{b_{\varepsilon}}(t)) + F_{\varepsilon}(y_{b_{\varepsilon}}(t)) \geq \frac{G_{\varepsilon}}{2} \quad \text{for} \quad t < t_{1,\varepsilon}. \]

However, as $t \to -\infty$ the right hand side goes to $\infty$ as $t \to -\infty$ which is a contradiction to the above inequality. Hence it must be that $G_{\varepsilon} = 0$. This completes the proof of the lemma.

**Lemma 4.6.**

\[ \lim_{t \to -\infty} -\frac{y_{b_{\varepsilon}}'}{\sqrt{y_{b_{\varepsilon}} - \varepsilon}} = \sqrt{2A_{\varepsilon}}. \]

**Proof.** Since $E_{1,\varepsilon}' \leq 0$ and $E_{1,\varepsilon}(0) = (e - L_{\varepsilon}) b_{\varepsilon} \geq 0$, it follows that $E_{1,\varepsilon} \geq 0$ for $t \leq 0$. Since $y_{b_{\varepsilon}}'(t) < 0$ for $t < 0$ and $y_{b_{\varepsilon}}(t) > \varepsilon$ for $t$ sufficiently negative we see that
\[ \left( \frac{-y_{b_{\varepsilon}}'}{\sqrt{y_{b_{\varepsilon}} - \varepsilon}} \right)' = \frac{E_{1,\varepsilon}}{(y_{b_{\varepsilon}} - \varepsilon)^{3/2}} > 0 \]
for $t$ sufficiently negative. Thus the function within the bracket above is positive and increasing for $t$ sufficiently negative. Consequently,
\[ \lim_{t \to -\infty} -\frac{y_{b_{\varepsilon}}'}{\sqrt{y_{b_{\varepsilon}} - \varepsilon}} = V_{\varepsilon} \geq 0. \]
Also, since
\[ 0 \leq E_{1,\varepsilon} = \frac{1}{2} (y_{b_{\varepsilon}}')^2 - (y_{b_{\varepsilon}} - \varepsilon)y''_{b_{\varepsilon}} \quad \text{for} \ t < 0 \]
and \( y_{b_{\varepsilon}}(t) > \varepsilon \), for \( t \) sufficiently negative we have
\[ \frac{(y_{b_{\varepsilon}}')^2}{y_{b_{\varepsilon}} - \varepsilon} \geq 2y''_{b_{\varepsilon}}. \]
Taking limits as \( t \to -\infty \) we obtain \( V_{e}^2 \geq 2A_{\varepsilon} \). Thus, if \( V_{e} = 0 \) then \( A_{\varepsilon} = 0 \). If \( V_{e} > 0 \), then since \( y_{b_{\varepsilon}}(t) \to \infty \) as \( t \to -\infty \) then also \( -y_{b_{\varepsilon}}' \to \infty \) as \( t \to -\infty \). Thus we may apply L’Hopital’s rule and obtain
\[ V_{e}^2 = \lim_{t \to -\infty} \frac{(y_{b_{\varepsilon}}')^2}{y_{b_{\varepsilon}} - \varepsilon} = \lim_{t \to -\infty} \frac{2y_{b_{\varepsilon}}' y''_{b_{\varepsilon}}}{y_{b_{\varepsilon}}'} = 2A_{\varepsilon}. \]
Thus in all cases we obtain \( V_{e} = \sqrt{2A_{\varepsilon}} \). This completes the proof of the lemma.

We now define
\[ w_{\varepsilon}(t) = \frac{1}{\varepsilon} y_{b_{\varepsilon}}(e^{\frac{2}{3\lambda_{\varepsilon}}} t) \quad (4.7) \]
and observe that \( w_{\varepsilon} \) satisfies
\[ \frac{w_{\varepsilon}(t)}{|t|^{\frac{2}{3\lambda_{\varepsilon}}}} = y_{b_{\varepsilon}}(s), \quad \frac{w_{\varepsilon}'(t)}{|t|^{\frac{2}{3\lambda_{\varepsilon}}}} = y_{b_{\varepsilon}}'(s), \quad |t|^{\frac{4\lambda_{\varepsilon}}{3\lambda_{\varepsilon} + 2}} w_{\varepsilon}''(t) = |s|^{\frac{4\lambda_{\varepsilon}}{3\lambda_{\varepsilon} + 2}} y_{b_{\varepsilon}}''(s), \quad (4.8) \]
where \( s = e^{\frac{2}{3\lambda_{\varepsilon}}} t \). Also, we see that \( w_{\varepsilon} \) satisfies
\[ w_{\varepsilon}'' = \frac{|w_{\varepsilon} - 1|^{\frac{4}{3\lambda_{\varepsilon} + 2}}}{w_{\varepsilon}^{\frac{2}{3\lambda_{\varepsilon} + 2}}} \text{sgn}(w_{\varepsilon} - 1) = f_{1}(w_{\varepsilon}), \quad (4.9) \]
\[ w_{\varepsilon}(0) = \frac{L_{\varepsilon}}{\varepsilon} = L_{1} \text{ by Lemma 4.1}, \]
\[ w_{\varepsilon}'(0) = 0, \quad w_{\varepsilon}''(0) = \varepsilon^{\frac{1}{3}} + \frac{2}{3\lambda_{\varepsilon}} b_{\varepsilon}. \]
We also define
\[ \hat{E}_{1,\varepsilon} = \frac{1}{2} (w_{\varepsilon}')^2 - (w_{\varepsilon} - 1)w_{\varepsilon}'' = F_{1}(w_{\varepsilon}) - w_{\varepsilon}'w_{\varepsilon}'', \quad (4.10) \]
\[ \hat{E}_{2,\varepsilon} = \frac{1}{2} (w_{\varepsilon}'')^2 - f_{1}(w_{\varepsilon})w_{\varepsilon}'. \quad (4.11) \]
Note that
\[ \hat{E}_{1,\varepsilon}' = - (w_{\varepsilon} - 1)w_{\varepsilon}'' = - (w_{\varepsilon} - 1) f_{1}(w_{\varepsilon}) = - \frac{|w_{\varepsilon} - 1|^{1 + \frac{4}{3\lambda_{\varepsilon} + 2}}}{w_{\varepsilon}^{\frac{2}{3\lambda_{\varepsilon} + 2}}} \leq 0, \quad (4.12) \]
\[ \hat{E}_{2,\varepsilon}' = - (w_{\varepsilon}'')^2 \leq 0, \quad (4.13) \]
\[ \hat{E}_{3,\varepsilon}' = - f_{1}'(w_{\varepsilon})(w_{\varepsilon}')^2 \quad (4.14) \]
so that
\[ \dot{E}_{3,\epsilon}^t \leq 0 \quad \text{for} \quad 0 < w_\epsilon \leq 1 + \frac{1}{\lambda + 1} \quad \text{and} \quad \dot{E}_{3,\epsilon}^t \geq 0 \quad \text{for} \quad w_\epsilon \geq 1 + \frac{1}{\lambda + 1}. \]

In Lemma 4.3 we showed that \( e^{\frac{1}{2} + \frac{2}{22}b_\epsilon} \leq Q \), where \( Q \) is independent of \( \epsilon \). Thus there is a subsequence of the \( \epsilon \) (still denoted \( \epsilon \)) such that
\[ \lim_{\epsilon \to 0} e^{\frac{1}{2} + \frac{2}{22}b_\epsilon} = c_0 \geq 0 \]
and for which \( w_\epsilon \) converges uniformly on compact sets to \( w_0 \) and \( w_0 \) satisfies
\[ w_0''' = \frac{|w_0 - 1|^3}{1 + w_0^2} \text{sgn}(w_0 - 1) = f_1(w_0), \quad (4.15a) \]
\[ w_0(0) = L_1, \quad w_0'(0) = 0, \quad w_0''(0) = c_0 \geq 0. \quad (4.15b) \]

We note in fact that \( c_0 > 0 \) for if \( c_0 = 0 \) then since \( w_0'''(0) < 0 \) we see that \( w_0'' \) is decreasing near \( t = 0 \) so that \( w_0'' < 0 \) for \( t > 0 \) and \( t \) small. From (4.10) it follows that \( w_0 \) continues to be concave down and decreasing so that \( w_0 \) becomes 0 at some finite value of \( t \), say \( t_0 \). Since \( w_\epsilon \to w_0 \) uniformly on compact sets and since \( w_\epsilon > 0 \) (since \( y_{b_\epsilon} > 0 \) by Lemma 3.3) then \( w_\epsilon \)
must have a local minimum, \( t_\epsilon \), near \( t_0 \) and \( w_\epsilon(t_\epsilon) < L_1 \). However, this implies from (4.13)
\[ F_1(w_\epsilon(t_\epsilon)) = \dot{E}_{2,\epsilon}(t_\epsilon) \leq \dot{E}_2(0) = F_1(L_1). \]
On the other hand, since \( 0 < w_\epsilon(t_\epsilon) < L_1 \) and \( F_1 \) is decreasing on \((0, L_1)\) we have \( F_1(w_\epsilon(t_\epsilon)) > F_1(L_1) \) which is a contradiction. Thus \( c_0 > 0 \).

**Lemma 4.7.**
\[ \lim_{t \to -\infty} w_\epsilon''(t) = 0 \quad \text{for} \quad \epsilon > 0. \]

**Proof.** From Lemma 2.3 it follows that \( y_{b_\epsilon}' < 0 \) and \( y_{b_\epsilon}'' > 0 \) for \( t < 0 \) and also that \( y_{b_\epsilon} \to \infty \) as \( t \to -\infty \). Hence from (4.7) we see that \( w_\epsilon' < 0 \) and \( w_\epsilon'' > 0 \) for \( t < 0 \) and also that \( w_\epsilon \to \infty \) as \( t \to -\infty \). Thus, \( w_0' \leq 0, \quad w_0'' \geq 0, \quad \text{and} \quad w_0 \to -\infty \) as \( t \to -\infty \).

Thus from (4.14) we see that \( \dot{E}_{3,\epsilon} \geq 0 \) for \( t \) sufficiently negative. Thus \( \dot{E}_{3,\epsilon} \) defined by (4.11) is increasing for \( t \) sufficiently negative and since \( -f_1(w_\epsilon)w_\epsilon' \geq 0 \) for \( t \) sufficiently negative we see that \( 0 \leq \frac{1}{2}(w_\epsilon'')^2 \) and \( 0 \leq -f_1(w_\epsilon)w_\epsilon' \) are both bounded above for \( t \) sufficiently negative. Also, \( w_\epsilon'' > 0 \) for \( t \) sufficiently negative and since \( w_\epsilon'' > 0 \) for \( t \) sufficiently negative, it follows that
\[ \lim_{t \to -\infty} w_\epsilon''(t) = H_\epsilon \quad \text{for some} \quad H_\epsilon > 0. \]

Assume now by the way of contradiction that \( H_\epsilon > 0 \). Then it follows that
\[ \lim_{t \to -\infty} w_\epsilon'(t) = -\infty. \]
and it follows then from L’Hopital’s rule that

\[
\lim_{t \to -\infty} \frac{w'_e(t)}{t} = H_e, \quad \lim_{t \to -\infty} \frac{w_e(t)}{t^2} = \frac{H_e}{2}, \quad \lim_{t \to -\infty} \frac{(w'_e)^2}{w_e - 1} = 2H_e. \tag{4.16}
\]

Integrating (4.9) for \(t\) sufficiently negative when \(w_e \geq 1\) we obtain

\[
w''_e - H_e = \int_{-\infty}^{t} \left[ \frac{w_e - 1}{w_e} \right]^\frac{1}{\lambda} \, dt = \int_{-\infty}^{t} \frac{1}{w_e} \left( 1 - \frac{1}{w_e} \right)^\frac{1}{\lambda} \, dt.
\]

Using L’Hopital’s rule and (4.16) it follows that

\[
\lim_{t \to -\infty} \left| t \left[ \frac{1}{w_e} \right]^\frac{1}{\lambda} \right| = \frac{\lambda}{\lambda + 2} \left( \frac{2}{H_e} \right)^{1 + \frac{1}{\lambda}}. \tag{4.17}
\]

Also, we know from (4.12) that \( \tilde{E}_{1,e} \) defined by (4.10) satisfies

\[
\tilde{E}'_{1,e} = -\frac{|w_e - 1|^{1 + \frac{1}{\lambda}}}{w_e} = -\frac{1}{w_e} \left| 1 - \frac{1}{w_e} \right|^{1 + \frac{1}{\lambda}}
\]

and so integrating on \((t,0)\) gives:

\[
\tilde{E}_{1,e} = \frac{1}{2} (w'_e)^2 - (w_e - 1)w''_e = \tilde{E}_{1,e}(0) + \int_{0}^{t} \frac{1}{w_e} \left| 1 - \frac{1}{w_e} \right|^{1 + \frac{1}{\lambda}} \, dt.
\]

We now first consider the case where \(1 < \lambda < 2\). The integral on the right converges as \(t \to -\infty\) since \(\lim_{t \to -\infty} w_e / t^2 = H_e / 2\) and \(\lambda < 2\) (by (1.3)). Thus, \(\tilde{E}_{1,e}(t) \to J_e\) for some \(J_e\) as \(t \to -\infty\) and thus for \(t\) sufficiently negative

\[
\frac{1}{2} (w'_e)^2 - (w_e - 1)w''_e - J_e = -\int_{-\infty}^{t} \frac{1}{w_e} \left( 1 - \frac{1}{w_e} \right)^\frac{1}{\lambda} \, dt.
\]

Also, since \(w_e(0) = L_1 < 1\) and \(w_e \to \infty\) as \(t \to -\infty\) it follows then that there exists a \(t_{1,e} < 0\) such that \(w_e(t_{1,e}) = 1\). Then we see since \(\tilde{E}'_{1,e} \leq 0\) (by (4.12)) that

\[
J_e \geq \tilde{E}_{1,e}(t_{1,e}) = \frac{1}{2} (w'_e(t_{1,e}))^2 \geq 0.
\]

Thus

\[
J_e \geq 0. \tag{4.18}
\]

Moreover, by L’Hopital’s rule it follows that

\[
\lim_{t \to -\infty} \left| t \right|^{\frac{1}{\lambda} - 1} \left( \frac{1}{2} (w'_e)^2 - (w_e - 1)w''_e - J_e \right) = -\frac{\lambda}{2 - \lambda} \left( \frac{2}{H_e} \right)^{\frac{1}{\lambda}}. \tag{4.19}
\]
Combining (4.17) and (4.19) we obtain

\[
\lim_{t \to -\infty} |t|^{-\frac{1}{2}} \left( \frac{1}{2}(w'_e)^2 - H_e w_e - (J_e - H_e) \right) = -\frac{2\lambda^2}{4-\lambda^2} \left( \frac{2}{H_e} \right)^{\frac{1}{2}}. \tag{4.20}
\]

It follows from (4.20) that

\[
\lim_{t \to -\infty} \left( \frac{1}{2}(w'_e)^2 - H_e w_e - (J_e - H_e) \right) = 0. \tag{4.21}
\]

We also know that when \( w_e > 1 \)

\[
\left( -\frac{w'_e}{\sqrt{w_e-1}} \right)' = \frac{E_{1,e}}{(w_e-1)^{\frac{3}{2}}},
\]

and since \( E_{1,e} \to J_e \) as \( t \to -\infty \) we see that

\[
\lim_{t \to -\infty} \left[ (w_e-1)^{\frac{3}{2}} \left( -\frac{w'_e}{\sqrt{w_e-1}} \right)' \right] = J_e
\]

and from the second result of (4.16) it follows that

\[
\lim_{t \to -\infty} \left[ t^3 \left( -\frac{w'_e}{\sqrt{w_e-1}} \right)' \right] = \frac{2\sqrt{2J_e}}{H_e^{\frac{3}{2}}}. \tag{4.22}
\]

Using (4.16) again and applying L'Hopital's rule we see that

\[
\lim_{t \to -\infty} \left[ t^2 \left( \frac{w'_e}{\sqrt{w_e-1}} + \sqrt{2H_e} \right) \right] = \frac{\sqrt{2J_e}}{H_e^{\frac{3}{2}}}. \tag{4.22}
\]

Now let \( \delta > 0 \). Then for \( t \) sufficiently negative we have by (4.22)

\[
0 \leq -w'_e \leq \left[ \sqrt{2H_e} + \left( \frac{-\sqrt{2J_e}}{H_e^{\frac{3}{2}}} + \delta \right) \frac{1}{t^2} \right] \sqrt{w_e-1}.
\]

Squaring both sides and simplifying we obtain

\[
\frac{1}{2}(w'_e)^2 \leq H_e(w_e-1) + \frac{\sqrt{2H_e}(w_e-1)}{t^2} \left( -\frac{\sqrt{2J_e}}{H_e^{\frac{3}{2}}} + \delta \right) + \frac{1}{2} \left( \frac{-\sqrt{2J_e}}{H_e^{\frac{3}{2}}} + \delta \right)^2 \frac{(w_e-1)}{t^4}
\]

and then

\[
\frac{1}{2}(w'_e)^2 - H_e w_e - (J_e - H_e) \leq \frac{\sqrt{2H_e}(w_e-1)}{t^2} \left( -\frac{\sqrt{2J_e}}{H_e^{\frac{3}{2}}} + \delta \right) + \frac{1}{2} \left( \frac{-\sqrt{2J_e}}{H_e^{\frac{3}{2}}} + \delta \right)^2 \frac{(w_e-1)}{t^4} - J_e. \tag{4.23}
\]
Taking limits in (4.23) using (4.16) and (4.22) yields

\[ 0 \leq -2J_\varepsilon + \frac{H_\varepsilon^2}{\sqrt{2}} \delta. \]

This along with (4.18) gives

\[ 0 \leq J_\varepsilon \leq \frac{H_\varepsilon^2}{2\sqrt{2}} \delta. \]

Finally, since \( \delta > 0 \) is arbitrary we see therefore that \( J_\varepsilon = 0 \).

Therefore \( \lim_{t \to -\infty} \tilde{E}_{1,\varepsilon} = 0 \) but since \( \tilde{E}_{1,\varepsilon}' \leq 0 \) and \( \tilde{E}_{1,\varepsilon}(t_{1,\varepsilon}) \geq 0 \) it follows that \( \tilde{E}_{1,\varepsilon} \equiv 0 \) on \((-\infty, t_{1,\varepsilon})\). Thus

\[ -\frac{|w_\varepsilon - 1|^{1+\frac{1}{2}}}{w_\varepsilon^{1+\frac{1}{2}}} = \tilde{E}_{1,\varepsilon}' \equiv 0 \quad \text{on } (-\infty, t_{1,\varepsilon}) \]

and thus \( w_\varepsilon \equiv 1 \) on \((-\infty, t_{1,\varepsilon})\) contradicting that

\[ \lim_{t \to -\infty} \frac{w_\varepsilon}{t^2} = \frac{H_\varepsilon}{2} > 0. \]

Hence it must be the case that \( H_\varepsilon = 0 \) completing the proof of the lemma in the case where
\( 1 < \lambda < 2 \).

We now consider the case where \( \lambda \geq 2 \). We see from (4.16) and the equation after (4.17) that if \( \lambda \geq 2 \) then

\[ \lim_{t \to -\infty} \tilde{E}_{1,\varepsilon} = \infty. \] (4.24)

Next, we see that

\[ \frac{1}{2}(w_\varepsilon')^2 - H_\varepsilon (w_\varepsilon - 1) = \tilde{E}_{1,\varepsilon} + (w_\varepsilon - 1)(w_\varepsilon'' - H_\varepsilon). \]

Using (4.17) \( w_\varepsilon'' - H_\varepsilon \geq 0 \) for sufficiently negative \( t \) and (4.24), we obtain

\[ \lim_{t \to -\infty} \frac{1}{2}(w_\varepsilon')^2 - H_\varepsilon (w_\varepsilon - 1) = \infty. \] (4.25)

Also from the equation after (4.21) we see that

\[ -\frac{w_\varepsilon'}{\sqrt{w_\varepsilon - 1}}' = \frac{\tilde{E}_{1,\varepsilon}}{(w_\varepsilon - 1)^{\frac{3}{2}}}, \]

which gives

\[ \lim_{t \to -\infty} \left[ (w_\varepsilon - 1)^{\frac{3}{2}} \left( -\frac{w_\varepsilon'}{\sqrt{w_\varepsilon - 1}} \right)' \right] = \infty. \]
Also it follows from the second result of (4.16) that
\[
\lim_{t \to -\infty} \left[t^3 \left(-\frac{w'_e}{\sqrt{w_e-1}}\right)\right]' = \infty.
\]

Then by L’Hopital’s rule we see that
\[
\lim_{t \to -\infty} \left[t^2 \left(\frac{w'_e}{\sqrt{w_e-1}} + \sqrt{2H_e}\right)\right] = \infty.
\]

For \(M > 0\) large and \(t\) sufficiently negative we see from (4.26) that
\[
0 \leq -w'_e \leq \left(\sqrt{2H_e} - \frac{M}{t^2}\right) \sqrt{w_e-1}.
\]

Squaring both sides and rewriting gives
\[
\frac{1}{2} (w'_e)^2 - H_e (w_e - 1) \leq -M \sqrt{2H_e} \left(\frac{w_e-1}{t^2}\right) + \frac{M^2}{2t^2} \left(\frac{w_e-1}{t^2}\right).
\]

However, as \(t \to -\infty\) the left hand side goes to \(\infty\) by (4.25) and by (4.16) the right hand side goes to \(-MH_e^{3/2} / \sqrt{2} \leq 0\). This is a contradiction. As a result, if \(\lambda \geq 2\), then it also must have \(H_e = 0\). This completes the proof of the lemma.

**Lemma 4.8.** There are constants \(c_1 > 0\) and \(c_2 > 0\) with \(c_1, c_2\) independent of \(\epsilon\) and \(c_{1,\epsilon}, c_{2,\epsilon} > 0\) with
\[
\lim_{\epsilon \to 0} c_{1,\epsilon} = \lim_{\epsilon \to 0} c_{2,\epsilon} = 0
\]
such that
\[
\frac{y_b(s)}{|s|^{1/\lambda + 1}} \geq c_1 \text{ on } (-\infty, -c_{1,\epsilon}); \quad -\frac{y_b(s)}{|s|^{1/\lambda + 1}} \geq c_2 \text{ on } (-\infty, -c_{2,\epsilon}).
\]

**Proof.** Recall that
\[\bar{E}'_{2,\epsilon} = (F_1(w_{\epsilon}) - w'_\epsilon w''_{\epsilon})' = -(w''_{\epsilon})^2 \leq 0.\]
Integrating on \((t,0)\) and using (4.3) gives for \(t < 0\)
\[
\int_1^\infty f_1(s) \, ds = F_1(\epsilon) = F_1(1_{\epsilon}) \leq F_1(w_{\epsilon}) - w'_\epsilon w''_{\epsilon} = \int_{w_{\epsilon}}^{w_{\epsilon}} f_1(s) \, ds - w'_\epsilon w''_{\epsilon}.
\]
Thus
\[
\int_{w_{\epsilon}}^{\infty} f_1(s) \, ds \leq -w'_\epsilon w''_{\epsilon}.
\]
Recall from the remark at the beginning of Lemma 4.7 that \(\lim_{t \to -\infty} w_{\epsilon} = \infty\) and along with the fact that \(w_{\epsilon}(0) = L_1 < 1\) we see that there exists \(t_{2,\epsilon} < 0\) such that \(w_{\epsilon}(t_{2,\epsilon}) = 2\). Thus for \(t < t_{2,\epsilon}\) we have
\[
\int_{w_{\epsilon}}^{\infty} f_1(s) \, ds = \int_{w_{\epsilon}}^{\infty} \frac{|s-1|^{\frac{1}{2}}}{s^{1+\lambda}} \, ds \geq \frac{1}{2} \int_{w_{\epsilon}}^{\infty} \frac{1}{s^{1+\lambda}} \, ds = \frac{\lambda}{2^\frac{1}{2}} w_{\epsilon}^{-\frac{1}{2}}.
\]
Thus from (4.27)-(4.28) we see that
\[-w_e'w_e'' \geq \frac{\lambda}{2^\frac{1}{3}} w_e^{-\frac{4}{3}} \text{ when } t < t_{2,e}.
\]

Multiplying this by \(-w_e' > 0\) gives
\[(w_e')^2 w_e'' \geq \frac{\lambda}{2^\frac{1}{3}} w_e^{-\frac{4}{3}} (-w_e')
\]
and integrating on \((t,t_{2,e})\) and using that \(w_e' < 0\) gives
\[-(w_e')^3 \geq \frac{3\lambda^2}{2^\frac{1}{3}(\lambda-1)} (w_e^{-\frac{4}{3}} - 2^{1-\frac{4}{3}}).
\] (4.29)

Now let \(t_{3,e} < 0\) be such that \(w_e(t_{3,e}) = 3\). Then for \(t < t_{3,e}\) we have
\[w_e^{-\frac{1}{3}} - 2^{1-\frac{4}{3}} \geq \left(1 - \left(\frac{2}{3}\right)^{1-\frac{4}{3}}\right) w_e^{-\frac{1}{3}}.
\]

Thus, using this in (4.29) we obtain
\[\frac{1}{\left(1 - \left(\frac{2}{3}\right)^{1-\frac{4}{3}}\right)^\frac{1}{3}} \int_t^{t_{3,e}} \frac{-w_e'}{w_e^\frac{1}{3}(1-\frac{4}{3})} ds \geq \int_t^{t_{3,e}} \frac{-w_e'}{\left(w_e^{-\frac{4}{3}} - 2^{1-\frac{4}{3}}\right)^\frac{1}{3}} ds \geq \int_t^{t_{3,e}} \left(\frac{3\lambda^2}{2^\frac{1}{3}(\lambda-1)}\right)^\frac{1}{3} ds.
\]

Therefore, we have
\[w_e^{-\frac{24}{11}} \geq \left(\frac{24}{11} \lambda^2 - 3^{\frac{24}{11}}\right) \geq C_1(t_{3,e} - t),
\]
where
\[C_1 = \left(1 - \left(\frac{2}{3}\right)^{1-\frac{4}{3}}\right)^\frac{1}{3} \left(\frac{3\lambda^2}{2^\frac{1}{3}(\lambda-1)}\right)^\frac{1}{3} \left(\frac{2\lambda+1}{3\lambda}\right).
\]

Thus for \(t < 2t_{3,e}\),
\[\frac{w_e}{|t|^{\frac{31}{21}}} \geq C_1^{\frac{31}{21}} \left(1 - \left|\frac{t_{3,e}}{t}\right|\right)^\frac{31}{21} \geq \left(\frac{C_1}{2}\right)^{\frac{31}{21}} \equiv c_1.
\] (4.30)

Letting \(c_{1,e} = e^{\frac{31}{21}}(2|t_{3,e}|)\) and using the rescaling mentioned in (4.7)-(4.8) we see that
\[\frac{\|ar{w}_e(s)\|}{|s|^{\frac{31}{21}}} \geq c_1 \text{ on } (-\infty, -c_{1,e}).
\] (4.31)

Also, since \(w_e \to w_0\) uniformly on compact sets and \(w_0 \to \infty\) as \(t \to -\infty\) then \(t_{3,e} \to t_{3,0}\) where \(t_{3,0}\) is finite and \(t_{3,0} < 0\). Thus, \(\lim_{t \to -\infty} c_{1,e} = 0\). Substituting (4.30) into (4.29) gives for \(t < 2t_{3,e}\)
\[-(w_e')^3 \geq \frac{3\lambda^2}{2^\frac{1}{3}(\lambda-1)} \left(w_e^{-\frac{4}{3}} - 2^{1-\frac{4}{3}}\right) \geq \frac{3\lambda^2}{2^\frac{1}{3}(\lambda-1)} \left(c_1 |t|^{\frac{31}{21}}\right)^{\frac{1}{31}} - 2^{1-\frac{4}{3}}.
\]
Thus, for $t < 2t_{3,\varepsilon}$

$$\frac{w''_e}{|t|^{\frac{3\lambda+1}{2}}} \geq \left(\frac{3\lambda^2}{2^+(\lambda-1)}\right)^{\frac{1}{2}} \left(c_1^{1-\frac{1}{\lambda}} - \frac{2^{1-\frac{1}{\lambda}}}{|t|^{\frac{3\lambda-1}{2\lambda+1}}}\right)^{\frac{1}{2}}.$$ 

The right-hand side of the above is larger than

$$\frac{1}{2} \left(\frac{3\lambda^2}{2^+(\lambda-1)}\right)^{\frac{1}{2}} c_1^{\frac{1}{2}(1-\frac{1}{\lambda})} \equiv c_2$$

when

$$|t| \geq t^* \equiv 2^{-\frac{(2\lambda-1)(3\lambda+1)}{3\lambda(\lambda-1)}}/c_1^{2+\frac{1}{\lambda}}.$$ 

So letting $c_{2,\varepsilon} = e^{\frac{3\lambda+1}{2\lambda+1}} t^*$, we see that $c_{2,\varepsilon} \rightarrow 0$ as $\varepsilon \rightarrow 0$ and using the rescaling from (4.7)-(4.8) we see that

$$-\frac{y''_b(s)}{|s|^{\frac{3\lambda+1}{2\lambda+1}}} \geq c_2 \quad \text{on} \quad (-\infty, -c_{2,\varepsilon}).$$

This completes the proof of the lemma. □

**Lemma 4.9.** There are constants $c_3 > 0$, $c_4 > 0$, and $c_5 > 0$ with $c_3, c_4, c_5$ independent of $\varepsilon$ and $c_{3,\varepsilon} > 0$, $c_{4,\varepsilon} > 0$, $c_{5,\varepsilon} > 0$ with

$$\lim_{\varepsilon \to 0} c_{3,\varepsilon} = \lim_{\varepsilon \to 0} c_{4,\varepsilon} = \lim_{\varepsilon \to 0} c_{5,\varepsilon} = 0$$

such that

$$\frac{y_b(s)}{|s|^{\frac{3\lambda+1}{2\lambda+1}}} \leq c_3 \quad \text{on} \quad (-\infty, -c_{3,\varepsilon}), \quad -\frac{y''_t(s)}{|s|^{\frac{3\lambda+1}{2\lambda+1}}} \leq c_4 \quad \text{on} \quad (-\infty, -c_{4,\varepsilon}),$$

and

$$0 \leq |s|^{\frac{3\lambda+3}{2\lambda+1}} y''_t(s) \leq c_5 \quad \text{on} \quad (-\infty, -c_{5,\varepsilon}).$$

**Proof.** From Lemma 4.7 we know that $\lim_{t \to -\infty} w''_e = 0$ and from Lemma 2.3 we know that $w''_e \geq 0$ when $t < 0$. Thus, when $t < t_{2,\varepsilon}$ (defined in Lemma 4.8) we have

$$0 \leq w''_e(t) = \int_{-\infty}^{t} w'''_e \, ds = \int_{-\infty}^{t} \frac{|w_e - 1|^{\frac{1}{\lambda}}}{w_e^{1+\frac{1}{\lambda}}} \text{sgn}(w_e - 1) \, ds \leq \int_{-\infty}^{t} \frac{1}{w_e^{1+\frac{1}{\lambda}}} \, ds.$$

Then using (4.30) gives

$$0 \leq w''_e(t) \leq \frac{1}{c_1^{1+1/\lambda}} \int_{-\infty}^{t} |s|^{\frac{3\lambda+1}{2\lambda+1}} \, ds = \frac{1}{c_1^{1+1/\lambda}} |t|^{\frac{3\lambda+3}{2\lambda+1}} \quad \text{for} \quad t < 2t_{3,\varepsilon}.$$ 

Letting $c_5 = 1/c_1^{1+1/\lambda}$ we have

$$0 \leq |t|^{\frac{3\lambda+3}{2\lambda+1}} w''_e(t) \leq c_5 \quad \text{for} \quad t < 2t_{3,\varepsilon}. \quad (4.32)$$
Letting \( c_{5,e} = e^{\frac{2\lambda+1}{3\lambda}}(2|t_{3,e}|) \) and using the rescaling (4.7)-(4.8) gives
\[
0 \leq |s|^{\frac{2\lambda+1}{3\lambda}} y_{b_i}''(s) \leq c_5 \quad \text{on} \quad (-\infty, c_{5,e}).
\]

Also, as mentioned after Eq. (4.31), \( t_{3,e} \to t_{3,0} \) and \( t_{3,0} \) is finite so that \( c_{5,e} \to 0 \) as \( e \to 0 \).

Dividing (4.32) by \( |t|^{\frac{3\lambda+1}{3\lambda}} \) and integrating the resulting inequality on \((t, 2t_{3,e})\) gives
\[
0 \leq \frac{w_e'(2t_{3,e}) - w_e'(t)}{|t|^{\frac{3\lambda+1}{3\lambda}}} \leq c_5 \left( \frac{2\lambda+1}{\lambda-1} \right) |t|^{\frac{3\lambda+1}{3\lambda}}.
\]

Therefore
\[
0 \leq \frac{w_e'(t)}{|t|^{\frac{3\lambda+1}{3\lambda}}} = \frac{-w_e'(2t_{3,e})}{|t|^{\frac{3\lambda+1}{3\lambda}}} + c_5 \left( \frac{2\lambda+1}{\lambda-1} \right) \quad \text{for} \quad t < 2t_{3,e}.
\]

Since \( w_e' \to w_0' \) uniformly on compact sets and \( t_{3,e} \to t_{3,0} \), where \( t_{3,0} \) is finite and \( t_{3,0} < 0 \) as mentioned after (4.31), we have \( w_e'(t_{3,e}) \to w_0'(t_{3,0}) \) which is finite so we see for \( e \) small enough
\[
0 \leq \frac{w_e'(t)}{|t|^{\frac{3\lambda+1}{3\lambda}}} \leq -\frac{2w_0'(2t_{3,0})}{|t_{3,0}|^{\frac{3\lambda+1}{3\lambda}}} + c_5 \left( \frac{2\lambda+1}{\lambda-1} \right) \equiv c_4
\]
for \( t < 3t_{3,e_0} \). Then by the rescaling mentioned in (4.7) we see that
\[
0 \leq \frac{y_{b_i}'(s)}{|s|^{\frac{3\lambda+1}{3\lambda}}} \leq c_4 \quad \text{on} \quad (-\infty, -c_{4,e}),
\]
where \( c_{4,e} = e^{\frac{2\lambda+1}{3\lambda}}(3t_{3,e}) \to 0 \) as \( e \to 0 \). Multiplying (4.33) by \( |t|^{\frac{3\lambda+1}{3\lambda}} \) and integrating on \((s, 0)\) gives
\[
w_e(t) \leq w_e(3t_{3,0}) + \left( \frac{2\lambda+1}{3\lambda} \right) c_4 |t|^{\frac{3\lambda+1}{3\lambda}}.
\]

Consequently,
\[
\frac{w_e(t)}{|t|^{\frac{3\lambda+1}{3\lambda}}} \leq \frac{w_e(3t_{3,0})}{|t|^{\frac{3\lambda+1}{3\lambda}}} + \left( \frac{2\lambda+1}{3\lambda} \right) c_4 \equiv c_3.
\]

Then by the rescaling mentioned in (4.7) we see that
\[
\frac{y_{b_i}'}{|s|^{\frac{3\lambda+1}{3\lambda}}} \leq c_3 \quad \text{on} \quad (-\infty, -c_{3,e}),
\]
where \( c_{3,e} = e^{\frac{2\lambda+1}{3\lambda}}(3t_{3,e_0}) \to 0 \) as \( e \to 0 \). This completes the proof of the lemma. \( \square \)

It follows from Lemmas 4.8 and 4.9 that \( |y_{b_i}|, |y_{b_i}'|, |y_{b_i}''| \) are uniformly bounded on compact subsets of \((-\infty, 0)\) and from (3.1) we see that \( |y_{b_i}''| \) is also uniformly bounded on compact subsets of \((-\infty, 0)\). Consequently, \( y_{b_i}, y_{b_i}' \), and \( y_{b_i}'' \) converge uniformly on
compact subsets of \((-\infty,0)\) to a function \(y_0\) and from (3.1) we see that \(y_0^{(m)}\) converges uniformly on compact sets and that \(y_0\) satisfies:

\[
y_0''' = \frac{1}{y_0^{1+\frac{1}{\lambda}}},
\]

\[
\lim_{t \to 0^-} y_0(t) = 0, \quad \lim_{t \to 0^-} y_0'(t) = 0,
\]

\[
0 \leq |t| \frac{\lambda+2}{2\lambda+1} y_0''(t) \leq c_5 \quad \text{for} \ t < 0.
\]

Finally, we have the following result.

**Lemma 4.10.**

\[
y_0 = c_\lambda |t|^{\frac{3\lambda}{1+\lambda}}, \quad \text{where} \quad c_\lambda = \left(\frac{(2\lambda+1)^3}{3\lambda(\lambda-1)(\lambda+2)}\right)^{\frac{1}{\lambda+1}}.
\]

**Proof.** It is straightforward to show that \(y\) given above is a solution of

\[
y''' = \frac{1}{y^{1+\frac{1}{\lambda}}},
\]

\[
\lim_{t \to 0^-} y(t) = 0, \quad \lim_{t \to 0^-} y'(t) = 0,
\]

and

\[
0 \leq |t| \frac{\lambda+2}{2\lambda+1} y''(t) \leq C < \infty \quad \text{for} \ t < 0.
\]

Now we let \(v = y_0 - y\). From the Mean-Value Theorem we see that for any fixed \(t < 0\) there is an \(0 < \mu < 1\) such that

\[
v''' = y'''_0 - y''' = \frac{1}{y_0^{1+\frac{1}{\lambda}}} - \frac{1}{y^{1+\frac{1}{\lambda}}} = -\frac{(1+\frac{1}{\lambda})}{(\mu y + (1-\mu)y_0)^{2+\frac{1}{\lambda}}} [y_0 - y] = -p(t)v,
\]

where \(p(t) > 0\). Now we observe that

\[
\left(\frac{1}{2}(v')^2 - vv''\right)' = -vv''' = p(t)v^2 \geq 0.
\]

It follows from Lemmas 4.8 and 4.9, and (4.36)-(4.37) and (4.39)-(4.41) that

\[
\lim_{t \to 0^-} \frac{1}{2}(v')^2 - vv'' = 0,
\]

so we see that

\[
\frac{1}{2}(v')^2 - vv'' \leq 0 \quad \text{for} \ t < 0.
\]
Thus it follows that \( \nu \nu'' \geq 0 \) for \( t < 0 \). Then \((\nu v')' = \nu v'' + (v')^2 \geq 0\). Integrating on \((t, 0)\) and using Lemmas 4.8 and 4.9, (4.36) and (4.39) give \( \nu v' \leq 0 \) for \( t < 0 \). Suppose now that there is a \( t_0 < 0 \) for which \( \nu(t_0) = 0 \). Integrating on \((t_0, t)\) gives \( \nu^2(t) \leq 0 \) and so we see that \( \nu \equiv 0 \) on \((t_0, 0)\). Therefore either \( \nu \geq 0 \) for \( t < 0 \) or \( \nu \leq 0 \) for \( t < 0 \).

Suppose first that \( \nu \geq 0 \) for \( t < 0 \). Then we have

\[
y_0 \geq y \equiv c_\lambda |t|^{\frac{\lambda}{2\lambda+1}} \quad \text{for} \quad t < 0. \tag{4.41}
\]

Then by (4.37) and (4.39)

\[
y''_0 = \int_{-\infty}^t \frac{1}{y_0^{1+\frac{1}{\lambda}}} ds \leq \frac{1}{c_\lambda^{1+\frac{1}{\lambda}}} |s|^{\frac{\lambda}{2\lambda+1}} = \frac{1}{c_\lambda^{1+\frac{1}{\lambda}}} \left( \frac{2\lambda+1}{\lambda+2} \right) |t|^{\frac{\lambda}{2\lambda+1}}.
\]

Integrating on \((t, 0)\) gives

\[
y_0' = \int_t^0 \frac{2\lambda+1}{\lambda+2} |s|^{\frac{\lambda}{2\lambda+1}} ds \leq \frac{1}{c_\lambda^{1+\frac{1}{\lambda}}} \left( \frac{2\lambda+1}{\lambda+2} \right) \left( \frac{2\lambda+1}{\lambda-1} \right) |t|^{\frac{\lambda}{2\lambda+1}}
\]

and integrating again on \((t, 0)\) and using the definition of \( c_\lambda \) given in Lemma 4.10 we see that

\[
y_0 \leq \frac{1}{c_\lambda^{1+\frac{1}{\lambda}}} \left( \frac{2\lambda+1}{\lambda+2} \right) \left( \frac{2\lambda+1}{\lambda-1} \right) |t|^{\frac{\lambda}{2\lambda+1}} = c_\lambda |t|^{\frac{\lambda}{2\lambda+1}}. \tag{4.42}
\]

Thus combining (4.41)-(4.42) we see that

\[
y_0 \equiv c_\lambda |t|^{\frac{\lambda}{2\lambda+1}} \quad \text{for} \quad t < 0.
\]

Similarly if \( \nu \leq 0 \) for \( t < 0 \) then we have

\[
y_0 \leq c_\lambda |t|^{\frac{\lambda}{2\lambda+1}} \quad \text{for} \quad t < 0.
\]

Then as earlier we may go through a similar computation and show that

\[
y_0 \geq c_\lambda |t|^{\frac{\lambda}{2\lambda+1}} \quad \text{for} \quad t < 0
\]

and finally obtain

\[
y_0 \equiv c_\lambda |t|^{\frac{\lambda}{2\lambda+1}} \quad \text{for} \quad t < 0.
\]

This completes the proof of the lemma and the proof of the Main Theorem. \( \square \)
References


