JOURNAL OF PARTIAL DIFFERENTIAL EQUATIONS
J. Part. Diff. Eq., Vol. **24**, No. 1, pp. 83-96

doi: 10.4208/jpde.v24.n1.6
February 2011

Regularity Theorems for Elliptic and Hypoelliptic Operators via the Global Relation

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Received 9 September 2010; Accepted 6 December 2010

Abstract. In this paper we give a new proof regarding the regularity of solutions to hypoelliptic partial differential equations with constant coefficients. On the assumption of existence, we provide a spectral representation for the solution and use this spectral representation to deduce regularity results. By exploiting analyticity properties of the terms within the spectral representation, we are able to give simple estimates for the size of the derivatives of the solutions and interpret them in terms of Gevrey classes.

AMS Subject Classifications: 46Fxx, 32Axx, 35H10, 35C15

Chinese Library Classifications: O175.27, O177.4

Key Words: Hypoelliptic operators; constant coefficient; distributions.

1 Introduction

Studies concerning the regularity of solutions to partial differential equations (PDEs) with constant coefficients have a long and rich history. Much of this work originated from interest in the smoothness of solutions to the Laplace's equation, for which Weyl [1] showed that all weak solutions are in fact smooth solutions using his method of orthogonal projection. Regarding higher regularity conditions, Petrowsky [2] proved the analyticity of classical solutions to homogeneous, elliptic PDEs with constant coefficients. One of the most beautiful results in the study of regularity of solutions to constant coefficient PDEs is due to Hörmander [3], who gave a complete algebraic classification of operators $P(D)$ which have the hypoelliptic property:

$$P(D)u\in C^\infty \Rightarrow u\in C^\infty.$$

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We refer the reader to [4–7] for an in depth account of this result, whereas a less detailed but more accessible discussion can be found in [8].

Hörmander's theorem theorem tells us that if $P\in\mathbf{C}[\lambda]$ is a polynomial in $\lambda\in\mathbf{C}^n$ such that $|\mathrm{Im}\lambda|\to\infty$ as $\lambda\to\infty$ in the algebraic variety Z_P:

$$Z_P=\{\lambda\in\mathbf{C}^n:P(\lambda)=0\},$$

then $P(D)$ with $D=-\mathrm{i}\partial$ is hypoelliptic, i.e. $\mathrm{singsupp}\,u=\mathrm{singsupp}\,Pu$. Hörmander's original proof relies on the construction of a parametrix for the operator $P(D)$ and deriving properties of this parametrix using very precise results regarding the variety Z_P and the growth of the functions $P(\lambda)^{-1}D^\alpha P(\lambda)$. In this paper, we give an alternate proof of the theorem. Using some relatively low-tech methods, we provide a spectral representation of the function u which satisfies

$$P(D)u=f,\qquad \text{in } \Omega\subset\mathbf{R}^n,$$

from which the regularity of u can readily be seen. The theorem is precisely stated as follows.

Theorem 1.1. *Let $P\in\mathbf{C}[\lambda]$ be such that*

$$|\mathrm{Im}\lambda|\to\infty,\qquad \text{as } \lambda\to\infty$$

in the variety Z_P. Suppose $u\in\mathcal{D}'(\Omega)$ satisfies $P(D)u=f$ in the distributional sense. Then P is hypoelliptic and for each $x\in\Omega$, u has a spectral representation of the form

$$u(x)=E*f(x)+\sum_I\int_{\Sigma_I}e^{\mathrm{i}\lambda\cdot x}U_I(\lambda)\,\mathrm{d}\lambda, \tag{1.1}$$

where $E\in\mathcal{S}'(\mathbf{R}^n)$ is a known function given in terns of $P(\lambda)$ and the finite collection of known surfaces $\Sigma_I\subset\mathbf{C}^n$ are such that the known functions $e^{\mathrm{i}\lambda\cdot x}U_I(\lambda)$ have rapid decay, i.e.,

$$e^{\mathrm{i}\lambda\cdot x}U_I(\lambda)\in\mathcal{S}(\Sigma_I),$$

for fixed $x\in\Omega$. The derivatives of u are given by

$$D^\alpha u(x)=E*D^\alpha f(x)+\sum_I\int_{\Sigma_I}\lambda^\alpha e^{\mathrm{i}\lambda\cdot x}U_I(\lambda)\,\mathrm{d}\lambda,$$

for each fixed $x\in\Omega$.

The spectral representation (1.1) is explicit. Both E and the $\{U_I\}$ are known, and are expressible in terms of the distributions $u,f\in\mathcal{D}'(\Omega)$. The exact description of these objects is given in Proposition 2.1. Clearly the function $E\in\mathcal{S}'(\mathbf{R}^n)$ represents a parametrix for $P(D)$, since (1.1) implies:

$$C^\infty\ni P(D)E*f-P(D)u=(P(D)E-\delta)*f,$$

for all $f \in \mathcal{D}'(\Omega)$, from which it follows that

$$P(D)E - \delta \in C^\infty.$$

We note also that the spectral representation (1.1) is in accordance to the Fundamental Principle of Palamodov and Ehrenpreis [9–11], in that the x-dependence is solely contained inside the exponential.

The approach we take in this paper is motivated by a new, unified approach to boundary value problems introduced by Fokas [12]. One of the core ideas behind this approach is to invoke the powerful tools of complex analysis to obtain spectral representations for solutions to boundary value problems that have particularly striking properties. In particular, Fokas has been able to construct spectral representations that completely solve a large class of linear and integrable nonlinear boundary value problems [13–15]. Even in the case of boundary value problems for non-integrable PDEs, the new approach has proved extremely advantageous [16,17]. The novelty of the spectral representations arising in this approach is that they only contain known boundary data, thus providing an explicit solution to the boundary value problem. In many cases, these spectral representations are constructed by deforming contours of integration in the complex plane in such a way that the unknown boundary data is eliminated. In this paper we adapt this philosophy so it can be used to deduce regularity of solutions to boundary value problems on the assumption of existence.

The use of spectral analysis to solve boundary value problems is a well-established area of functional analysis. It is widely used in applied mathematics and dates back to Fourier's treatise on the transport of heat [18]. When studying a linear problem on a Hilbert space $\mathcal{H}$ involving a self-adjoint operator $A \in \mathcal{L}(\mathcal{H})$, one often looks for a unitary operator $U : \mathcal{H} \to L^2(M, \mathrm{d}\mu)$ such that the linear operator $A \in \mathcal{L}(\mathcal{H})$ becomes multiplicative, i.e.,

$$(UAU^{-1}f)(x) = \lambda(x) f(x).$$

Such an operator is guaranteed by the celebrated Spectral Theorem (see [19] for an excellent account). The beauty of this approach is that problems of the form $Au = f$ in $\mathcal{H}$ become particularly simple on $L^2(M, \mathrm{d}\mu)$. Spectral theory allows the *description* of the problem to take a particularly simple form. Within this framework, the spectral representation of the solution can be found and the problem is solved, modulo inverting the spectral mapping to give a representation of the solution in the "physical" Hilbert space $\mathcal{H}$. Sadly, this last operation often hides many of the properties of the solution as an element of $\mathcal{H}$. For instance, the Fourier transform

$$U : L^2(\mathbf{R}, \mathrm{d}x) \to L^2(\mathbf{R}, \mathrm{d}\lambda) : f \mapsto Uf = \hat{f}$$

makes the operation $D = -\mathrm{i}\partial$ multiplicative and so the description of problems of the form $P(D)u = f$ are particularly simple in spectral space and we can solve for $\hat{u} = \hat{u}(\lambda)$. *However*, the properties of the function $u = u(x)$ are not necessarily transparent using the spectral representation

$$u(x) = (U^{-1}\hat{u})(x).$$

For instance, the spectral representation of the indicator function on $\Omega=(-1,1)$ is given by:

$$\chi_\Omega(x)=\frac{1}{\pi}\int e^{\mathrm{i}\lambda x}\left(\frac{\sin\lambda}{\lambda}\right)\mathrm{d}\lambda, \qquad x\in\Omega,$$

where the improper integral is understood. The right hand side is actually analytic for $x\in\Omega$, but using the spectral representation it is not even immediately clear this function is differentiable at any point $x\in\Omega$. Even in this particularly simple case, the spectral representation has hidden a lot of important information.

In this paper, we provide a spectral representation for the solutions to elliptic and hypoelliptic PDEs with constant coefficients for which many properties of the solutions are transparent. We do this by using elementary tools from complex analysis to deform contours in the spectral representation so that the resulting integrand has rapid decay. The legitimacy of the contour deformations is a direct consequence of the fact that $|\mathrm{Im}\lambda|\to\infty$ as $\lambda\to\infty$ in the algebraic variety Z_P. The main advantage of the integral representation (1.1) is that it allows one to immediately see the hypoelliptic properties of the operator $P(D)$. In fact, much more can be said about the smoothness of solutions to hypoelliptic problems. In §3 we use the spectral representation of Theorem 1.1 to give a short proof of the regularity of solutions to hypoelliptic problems in terms of Gevrey spaces.

2 The integral representation

In this section we prove the result appearing in Theorem 1.1. Here and throughout, $\Omega\subset\mathbf{R}^n$ denotes a domain with compact closure, $P\in\mathbf{C}[\lambda]$ denotes an element of the ring of polynomials in $\lambda\in\mathbf{C}^n$, Z_P denotes the algebraic variety defined by the zero set of P. We use the standard notation for distributions so that

$$\mathcal{E}'(\mathbf{R}^n)\subset\mathcal{S}'(\mathbf{R}^n)\subset\mathcal{D}'(\mathbf{R}^n)$$

denote the spaces of distributions with compact support, the tempered distributions and continuous distributions respectively. The corresponding spaces of test functions being $\mathcal{D}(\mathbf{R}^n)\subset\mathcal{S}(\mathbf{R}^n)\subset\mathcal{E}(\mathbf{R}^n)$, the space of smooth functions with compact support, the space of Schwartz functions and the space smooth functions respectively. The action of a distribution on a test function will be denoted by a pairing in the bracket $\langle\cdot,\cdot\rangle$. The Fourier transform will be used extensively and is defined for $f\in\mathcal{S}(\mathbf{R}^n)$ by:

$$f\mapsto\hat{f}(\lambda)=\int e^{-\mathrm{i}\lambda\cdot x}f(x)\,\mathrm{d}x,$$

and then on $\mathcal{S}'(\mathbf{R}^n)$ and $\mathcal{D}'(\mathbf{R}^n)$ by duality. The Fourier transform for $T\in\mathcal{E}'(\mathbf{R}^n)$ is defined by $\hat{T}(\lambda)=\langle T,e^{-\mathrm{i}\lambda\cdot x}\rangle$. Multi-index notation will be used throughout so that

$$\begin{aligned}&\partial^\alpha=\partial^{\alpha_1}_{x^1}\cdots\partial^{\alpha_n}_{x^n},\\&\alpha!=\alpha_1!\cdots\alpha_n!,\\&|\alpha|=\alpha_1+\cdots+\alpha_n,\end{aligned}$$

and so on. We adopt the standard notation $D = -\mathrm{i}\partial$ so that

$$(D^\alpha T)\hat{} = \lambda^\alpha \hat{T}.$$

2.1 Proof to Theorem 1.1

Suppose $u \in \mathcal{D}'(\Omega)$ is a distributional solution to the equation $P(D)u = f$, where $f \in \mathcal{D}'(\Omega)$. We introduce the cut-off function $\rho_\epsilon \in \mathcal{D}(\Omega)$ with the property that $\rho_\epsilon(x) = 1$ in the set

$$\Omega_\epsilon = \Big\{ y \in \Omega : \mathrm{dist}(\partial\Omega, y) > \epsilon \Big\}.$$

Then the distribution $u_\epsilon = \rho_\epsilon u$ is in $\mathcal{E}'(\mathbf{R}^n)$ and the Fourier transform $\hat{u}_\epsilon$ is well defined. We observe the following lemma.

Lemma 2.1. *The Fourier transform of the distribution $u_\epsilon \in \mathcal{E}'(\mathbf{R}^n)$ is given by:*

$$\hat{u}_\epsilon(\lambda) = \frac{1}{P(\lambda)} \Big(\Big\langle T_P^\epsilon(u), e^{-\mathrm{i}\lambda\cdot x} \Big\rangle + \hat{f}_\epsilon \Big), \tag{2.1}$$

where $f_\epsilon = \rho_\epsilon f$ and $T_P^\epsilon(u) \in \mathcal{E}'(\mathbf{R}^n)$ is a distribution depending linearly on u and its derivatives, with compact support contained in $\Omega \backslash \Omega_\epsilon$. In addition, $T_P^\epsilon(u)$ satisfies the estimate:

$$\Big| \Big\langle T_P^\epsilon(u), e^{-\mathrm{i}\lambda\cdot x} \Big\rangle \Big| \leq C(1+|\lambda|)^N \sup_{x \in \Omega\backslash\Omega_\epsilon} e^{\mathrm{Im}\lambda\cdot x}, \tag{2.2}$$

for some constants $C, N > 0$.

Proof. The following equalities are immediate:

$$\begin{aligned} P(\lambda)\hat{u}_\epsilon(\lambda) &= \Big\langle P(\lambda)u_\epsilon, e^{-\mathrm{i}\lambda\cdot x} \Big\rangle \\ &= \Big\langle u_\epsilon, P(-D)e^{-\mathrm{i}\lambda\cdot x} \Big\rangle \\ &= \Big\langle u_\epsilon, {}^tP(D)e^{-\mathrm{i}\lambda\cdot x} \Big\rangle - \Big\langle \rho_\epsilon P(D)u, e^{-\mathrm{i}\lambda\cdot x} \Big\rangle + \hat{f}_\epsilon \\ &= \Big\langle P(D)u_\epsilon - \rho_\epsilon P(D)u, e^{-\mathrm{i}\lambda\cdot x} \Big\rangle + \hat{f}_\epsilon. \end{aligned}$$

If we define the distribution $T_P^\epsilon(u) \in \mathcal{D}'(\Omega)$ by:

$$T_P^\epsilon(u) = P(D)u_\epsilon - \rho_\epsilon P(D)u,$$

it is easy to show that $\mathrm{supp}\, T_P^\epsilon(u) \subset \Omega \backslash \Omega_\epsilon$. What is more, since $u \in \mathcal{D}'(\Omega)$ we have the estimate:

$$|\langle u, \phi \rangle| \leq C \sum_{|\alpha| \leq N} \sup |D^\alpha \phi|, \qquad \forall \phi \in \mathcal{D}(\Omega),$$

for some $N\geq 0$. Using the fact that $T_P(u)$ is distribution defined via distributional derivatives of $u\in\mathcal{D}'(\Omega)$ and multiplication by smooth functions with compact support, we deduce that the following estimate holds:

$$\left|\left\langle T_P^\epsilon(u), e^{-\mathrm{i}\lambda\cdot x}\right\rangle\right| \leq C(1+|\lambda|)^N \sup_{x\in\Omega\backslash\Omega_\epsilon} e^{\mathrm{Im}\lambda\cdot x},$$

for some constants $C,N>0$. □

Since $\hat{u}_\epsilon(\lambda)$ is the Fourier transform of a distribution with compact support, the well known theorem of Paley, Wiener and Schwartz implies that it is an entire function of $\lambda\in\mathbf{C}^n$. So a necessary constraint on u for $P(D)u=f$ to hold is

$$\left\langle T_P^\epsilon(u), e^{-\mathrm{i}\lambda\cdot x}\right\rangle + \hat{f}_\epsilon = 0, \qquad \text{on } Z_P.$$

In the limiting case when $\epsilon\to 0$, this becomes what Fokas calls the global relation [12] in the context of boundary value problems. The fact that this expression vanishes on Z_P means it is necessarily a multiple of $P(\lambda)$.

Since $\hat{u}_\epsilon(\lambda)$ is an entire function, the Fourier inversion theorem on $\mathcal{S}'(\mathbf{R}^n)$ gives us the following implicit formula for u_ϵ:

$$\langle u_\epsilon,\phi\rangle = \frac{1}{(2\pi)^n}\int \hat{u}_\epsilon(\lambda)\hat{\check{\phi}}(\lambda)\,\mathrm{d}\lambda, \qquad \forall\phi\in\mathcal{S}(\mathbf{R}^n),$$

where $\check{\phi}(x)=\phi(-x)$. In particular, we restrict our attention to test functions in $\mathcal{D}(\Omega_\epsilon)$, since the distributions u and u_ϵ coincide on $\mathcal{D}'(\Omega_\epsilon)$. Now treating u as a distribution in $\mathcal{D}'(\Omega_\epsilon)$, we have the inversion formula

$$\langle u,\phi\rangle = \frac{1}{(2\pi)^n}\int \hat{u}_\epsilon(\lambda)\left(\int e^{\mathrm{i}\lambda\cdot x}\phi(x)\,\mathrm{d}x\right)\mathrm{d}\lambda, \qquad \forall\phi\in\mathcal{D}(\Omega_\epsilon), \tag{2.3}$$

where the x-integral is taken over $\mathrm{supp}\phi\subset\Omega_\epsilon$.

Proposition 2.1. *Let $u\in\mathcal{D}'(\Omega_\epsilon)$ defined by* (2.3) *and suppose $P\in\mathbf{C}[\lambda]$ has the property that $|\mathrm{Im}\lambda|\to\infty$ as $\lambda\to\infty$ in the variety Z_P. We denote by $\{\mathcal{O}_i(x)\}$ an open covering of* $\mathrm{cl}(\Omega\backslash\Omega_\epsilon)$ *defined by:*

$$\mathcal{O}_i(x) = \left\{y\in\Omega\backslash\Omega_\epsilon : |x^i - y^i| > \delta\right\},$$

for some small $\delta>0$ and define $\{\vartheta_i\}$ to be a partition of unity subordinate to this open cover. Let $R>0$ such that

$$Z_P\cap\mathbf{R}^n\subset B_R,$$

where B_R is the ball $\{\lambda \in \mathbf{R}^n : |\lambda| \leq R\}$ and let the cut-off function $\chi(\lambda) \in \mathcal{D}(\mathbf{R}^n)$ be such that $\chi = 1$ on B_R and $\chi = 0$ on B_{R+1}. Then u has the following spectral representation:

$$u(x) = E * f_\epsilon(x) \tag{2.4a}$$
$$+ \frac{1}{(2\pi)^n} \int e^{\mathrm{i}\lambda\cdot x} \hat{u}_\epsilon(\lambda)\chi(\lambda)\,\mathrm{d}\lambda \tag{2.4b}$$
$$+ \frac{1}{(2\pi)^n} \int_{B_{R+1}\setminus B_R} \frac{\left\langle T_P^\epsilon(u), e^{\mathrm{i}\lambda\cdot(x-y)}\right\rangle}{P(\lambda)} (1-\chi(\lambda))\,\mathrm{d}\lambda \tag{2.4c}$$
$$+ \frac{1}{(2\pi)^n} \sum_I \int_{\Sigma_I} \frac{\left\langle T_P^\epsilon(u), \vartheta_I(y) e^{\mathrm{i}\lambda\cdot(x-y)}\right\rangle}{P(\lambda)}\,\mathrm{d}\lambda, \tag{2.4d}$$

where $E \in \mathcal{S}'(\mathbf{R}^n)$ is defined by

$$\hat{E}(\lambda) = (1-\chi(\lambda))\,P(\lambda)^{-1}$$

and the Σ_I are known surfaces in $\mathbf{C}^n$ on which the integrand has rapid decay.

Proof. First note that by the Paley-Wiener-Schwartz theorem $\left\langle T_P^\epsilon(u), e^{-\mathrm{i}\lambda\cdot y}\right\rangle$ is an entire function of $\lambda \in \mathbf{C}^n$, so each integral is well-defined. It is clear from (2.1) that the distribution $\hat{u}_\epsilon$ can be split as follows:

$$\begin{aligned} \hat{u}_\epsilon &= \chi(\lambda)\hat{u}_\epsilon + (1-\chi(\lambda))\,\hat{u}_\epsilon \\ &= \chi(\lambda)\hat{u}_\epsilon + \hat{E}(\lambda)\hat{f}_\epsilon + \frac{(1-\chi(\lambda))}{P(\lambda)} \left\langle T_P^\epsilon(u), e^{-\mathrm{i}\lambda\cdot x}\right\rangle. \end{aligned} \tag{2.5}$$

By appealing to Fubini's theorem to interchange the order of integration, it is clear that first two terms in (2.5) correspond to (2.4b) and (2.4a) respectively. The final term in (2.5) corresponds to the sum of the term in (2.4c) and a distribution $u_\infty \in \mathcal{D}'(\Omega_\epsilon)$ defined by

$$\begin{aligned} \langle u_\infty, \phi\rangle &= \frac{1}{(2\pi)^n} \int_{B_{R+1}^c} \frac{\left\langle T_P^\epsilon(u), e^{\mathrm{i}\lambda\cdot(x-y)}\right\rangle}{P(\lambda)} \hat{\phi}(-\lambda)\,\mathrm{d}\lambda \\ &= \frac{1}{(2\pi)^n} \int_{B_{R+1}^c} \frac{\left\langle T_P^\epsilon(u), e^{\mathrm{i}\lambda\cdot(x-y)}\right\rangle}{(-\lambda)^\alpha P(\lambda)} (D^\alpha\phi)\hat{}(-\lambda)\,\mathrm{d}\lambda, \end{aligned}$$

where A^c denotes the complement of the set A. We proceed to show this distribution can be identified with (2.4d).

Choosing $|\alpha|$ sufficiently large so that fraction is integrable (in the sense of Lebesgue), we can once again apply Fubini's theorem to show that

$$\langle u_\infty, \phi\rangle = \langle \tilde{u}_\infty, D^\alpha\phi\rangle,$$

i.e., in the sense of distributions $u_\infty = (-D)^\alpha \tilde{u}_\infty$, where

$$\tilde{u}_\infty(x) = \frac{1}{(2\pi)^n} \int_{B^c_{R+1}} \frac{\left\langle T^\epsilon_P(u), e^{\mathrm{i}\lambda\cdot(x-y)} \right\rangle}{(-\lambda)^\alpha P(\lambda)} \mathrm{d}\lambda, \qquad x \in \Omega_\epsilon, \tag{2.6}$$

which is clearly well-defined since the denominator is bounded away from zero and $|\alpha|$ can be chosen as large as we please. Now we introduce an open cover of $\mathrm{cl}(\Omega \backslash \Omega_\epsilon)$ so that on each open set, $|x^i - y^i| > \delta$ for some $i \in \{1, \cdots, n\}$. Such a cover certainly exists since for fixed $x \in \Omega_\epsilon$, each $y \in \Omega \backslash \Omega_\epsilon$ must differ in at least one component to x. Define the open sets $\{\mathcal{O}_i(x)\}$ so that

$$\mathcal{O}_i(x) = \left\{ y \in \mathrm{cl}(\Omega \backslash \Omega_\epsilon) : |x^i - y^i| > \delta \right\}.$$

Then $\{\mathcal{O}_i(x)\}$ provide an open cover of $\mathrm{cl}(\Omega \backslash \Omega_\epsilon)$ for $\delta > 0$ sufficiently small. In addition, owing to the compactness $\mathrm{cl}(\Omega \backslash \Omega_\epsilon)$ we may assume, without loss of generality, that this cover is finite. Introducing a partition of unity $\{\vartheta_i\}$ subordinate to this covering, we can split the integral (2.6) into a sum, on the parts of which $|x_i - y_i| > \delta$ for some $i \in \{1, \cdots, n\}$

$$\sum_i \frac{1}{(2\pi)^n} \int_{B^c_{R+1}} \frac{\left\langle T^\epsilon_P(u), \vartheta_i(y) e^{\mathrm{i}\lambda\cdot(x-y)} \right\rangle}{(-\lambda)^\alpha P(\lambda)} \mathrm{d}\lambda.$$

Let us consider a generic term in this sum of integrals for which $x^1 - y^1 > \delta$ (for example)

$$\frac{1}{(2\pi)^n} \int_{B^c_{R+1}} \frac{\left\langle T^\epsilon_P(u), \vartheta_1(y) e^{\mathrm{i}\lambda_1(x^1-y^1)} e^{\mathrm{i}\lambda'\cdot(x'-y')} \right\rangle}{(-\lambda)^\alpha P(\lambda)} \mathrm{d}\lambda_1 \wedge \mathrm{d}\lambda', \tag{2.7}$$

where $x = (x^1, x')$ and $\lambda = (\lambda_1, \lambda')$ etc. Within this integral we make the elementary change of variables:

$$(\lambda_1, \lambda') \mapsto (\theta_1, \theta') : \begin{cases} \theta_1 = \lambda_1 - \beta' \cdot \lambda', \\ \theta' = \lambda', \end{cases} \tag{2.8}$$

where $\beta' \in \mathbf{R}^{n-1}$. This obviously corresponds to a simple shear with

$$\mathrm{d}\lambda_1 \wedge \mathrm{d}\lambda' \mapsto \mathrm{d}\theta_1 \wedge \mathrm{d}\theta'.$$

The ball B_{R+1} becomes Π_R in the new coordinates, where Π_R is obtained from the shear of B_{R+1} induced by the coordinate change. The integral in (2.7) becomes

$$\frac{1}{(2\pi)^n} \int_{\Pi^c_R} \frac{\left\langle T^\epsilon_P(u), \vartheta_1(y) e^{\mathrm{i}\theta_1(x^1-y^1)} e^{\mathrm{i}\theta'\cdot(x'-y'+\beta'(x^1-y^1))} \right\rangle}{(-\lambda(\theta))^\alpha P(\lambda(\theta))} \mathrm{d}\theta_1 \wedge \mathrm{d}\theta'.$$

Now we focus attention on the terms arising in the exponents within this integral. We are free to choose each component of β' to be larger than $\delta^{-1}\mathrm{diam}(\Omega)$, and this ensures that the imaginary parts of each of the exponents are strictly positive. Now since $|\mathrm{Im}\lambda|\to\infty$ as $\lambda\to\infty$ in Z_P, it is straightforward to show that the same is true in the variety Z_{λ^*P} for the map $\lambda(\theta)$ given in (2.8), where $\lambda^*P(\theta)=P(\lambda(\theta))$. It follows [4] (Th. 4.1.3) that the distance $d(\theta)$ between $\theta\in\mathbf{R}^n$ and Z_{λ^*P} satisfies $d(\theta)>C|\theta|^c$ for positive constants $C>0$, $c\le 1$ and for $|\theta|$ sufficiently large.

Choosing $|\alpha|$ larger still if needs be, each of the θ_i integrals has enough decay so that the θ_i contours can be deformed into the upper half θ_i complex plane via Cauchy's theorem. More precisely, for $|\theta_i|$ sufficiently large, we can deform the θ_i-contour towards the curve $\mathrm{Im}\theta_i=D|\mathrm{Re}\theta_i|^{d_i}$ for some positive constants D, d_i and $|\theta|$ sufficiently large, since the integrand is analytic in

$$\mathrm{Im}\theta_i\le D|\mathrm{Re}\theta_i|^{d_i}.$$

The validity of this step follows from the positivity of the imaginary parts of the exponents within the integral and some elementary estimates of the form (2.2). We label the new surface of integration by γ. In summary, the integral in (2.7) is the same as:

$$\frac{1}{(2\pi)^n}\int_\gamma \frac{\left\langle T_P^\epsilon(u),\vartheta_1(y)e^{\mathrm{i}\theta_1(x^1-y^1)}e^{\mathrm{i}\theta'\cdot(x'-y'+\beta'(x^1-y^1))}\right\rangle}{(-\lambda(\theta))^\alpha P(\lambda(\theta))}\mathrm{d}\theta_1\wedge\mathrm{d}\theta',$$

in which the integrand has rapid decay. The remaining integrals corresponding to the other $\mathcal{O}_i(x)$ can be treated analogously, by deforming contours in the lower or upper θ_i planes appropriately. Finally, we note that the covering $\mathcal{O}_i(x)$ and the corresponding partition of unity are stable with respect to small perturbations in x. That is to say, one can use the same covering of $\mathrm{cl}(\Omega\backslash\Omega_\epsilon)$ and the same partition of unity for any $X=x+h$ with $|h|$ sufficiently small. It follows that the spectral representation is valid in a neighbourhood of each x, and as such we can differentiate under the integral sign.

$$\begin{aligned}&\frac{(-D)^\alpha}{(2\pi)^n}\int_\gamma \frac{\left\langle T_P^\epsilon(u),\vartheta_1(y)e^{\mathrm{i}\theta_1(x^1-y^1)}e^{\mathrm{i}\theta'\cdot(x'-y'+\beta'(x^1-y^1))}\right\rangle}{(-\lambda(\theta))^\alpha P(\lambda(\theta))}\mathrm{d}\theta_1\wedge\mathrm{d}\theta'\\ =&\frac{1}{(2\pi)^n}\int_\gamma \frac{\left\langle T_P^\epsilon(u),\vartheta_1(y)e^{\mathrm{i}\theta_1(x^1-y^1)}e^{\mathrm{i}\theta'\cdot(x'-y'+\beta'(x^1-y^1))}\right\rangle}{P(\lambda(\theta))}\mathrm{d}\theta_1\wedge\mathrm{d}\theta'.\end{aligned}$$

Now we can see that $u_\infty(x)$ corresponds to the final term (2.4d) after inverting the map $\lambda\mapsto\theta(\lambda)$ and labelling the corresponding λ surfaces by Σ_i. $\square$

The proof to this proposition gives rise to the result in Theorem 1.1. What is more, since the terms appearing in (2.4b)–(2.4d) are clearly smooth, we have the immediate corollary regarding hypoellipticity.

Corollary 1. *If $P\in\mathbf{C}[\lambda]$ is such that $|\mathrm{Im}\lambda|\to\infty$ as $\lambda\to\infty$ in Z_P, then $P(D)$ is hypoelliptic, i.e.* $\mathrm{singsupp}\,u=\mathrm{singsupp}\,Pu$.

Although it is clear from the construction, the spectral representation (1.1) also shows us that u satisfies $P(D)u=f$. Indeed, the action of $P(D)$ on u in (2.4a)–(2.4d) yields

$$P(D)u=\frac{1}{(2\pi)^n}\int e^{i\lambda\cdot x}\hat{f}_\epsilon(\lambda)\mathrm{d}\lambda=f_\epsilon(x).$$

The additional terms involving u and $T_P(u)$ all disappear since the factor of $P(\lambda)^{-1}$ is removed so the contours may be closed up in the upper/lower half λ_i planes owing to analyticity considerations.

3 Interior regularity

Here we look at what conditions on the regularity of u that can be deduced from the spectral representation given in Theorem 1.1 when P is hypoelliptic. Since it is clear that the terms in (2.4b)–(2.4d), it is the term $E*f(x)$ that determines the degree of smoothness of u in $P(D)u=f$. Using a precise estimate due to Hörmander [5], it is a simple exercise to deduce the regularity of $u\in\mathcal{D}'(\Omega)$ when P is elliptic.

Theorem. (Hörmander [5], Th. 7.9.5-7.9.6) *If $\hat{E}\in\mathcal{S}'(\mathbf{R}^n)\cap L^1_{\mathrm{loc}}(\mathbf{R}^n)$ and obeys the estimate*

$$\sum_{|\alpha|\le s}\int_{B_{2R}\backslash B_{R/2}}\frac{R^{2|\alpha|}|D^\alpha\hat{E}(\lambda)|^2}{R^n}\mathrm{d}\lambda\le C<\infty,$$

for all $R>0$ and $s\ge\frac{n}{2}$ then:

$$\|E*f\|_{L^p}\le C_p\|f\|_{L^p},\qquad \text{for } 1<p<\infty,$$
$$\|E*f\|_{C^{k+\gamma}}\le C_\gamma\|f\|_{C^{k+\gamma}},$$

where C^γ is the Hölder space with exponent γ.

In particular, in the case when P is elliptic it is straightforward to derive the estimate:

$$|D^\alpha\hat{E}(\lambda)|=\mathcal{O}\left(|\lambda|^{-|\alpha|-m}\right)\quad\text{as } |\lambda|\to\infty,$$

for $\hat{E}$ as in Theorem 1.1. Using this in conjunction with Hörmander's precise results shows, for instance, that if $f\in C^{k+\gamma}(\Omega)$ and $P(D)$ is m-th order elliptic, then all distributional solutions to $P(D)u=f$ are in $C^{k+m+\gamma}(\Omega)$.

One particular advantage of the spectral representation comes when one wants to examine stronger regularity conditions than just C^∞. If we are interested in weaker regularity conditions than C^∞, then the (2.4b)–(2.4d) give us no new information over that obtained by working with a parametrix for the operator $P(D)$, since the analysis is done modulo smooth functions. However, if we are interested in analyticity properties, then we need to study the regularity of *each* of the expressions (2.4a)–(2.4d). An appropriate space of functions to work on is those of the Gevrey class.

Definition 3.1. *We denote by $\Gamma^\rho(\Omega)$ the ρ-th Gevrey class, consisting of the smooth functions on Ω such that for each compact $K \subset \Omega$ the following estimate is satisfied:*

$$\sup_{x \in K} |D^\alpha u(x)| \leq C^{|\alpha|+1} (\alpha!)^\rho,$$

for a positive constant C depending on u and K.

It is clear that if $\rho = 1$, then $\Gamma^\rho(\Omega)$ coincides with the class of analytic functions on Ω, since $f \in C^\infty(\Omega)$ is analytic if and only if $|D^\alpha f| = \mathcal{O}(C^\alpha \alpha!)$ for large α and some constant C. Hörmander [3] gave regularity results in terms of Gevrey classes using precise geometrical results and estimates of

$$\left| \frac{D^\alpha P(\lambda)}{P(\lambda)} \right|.$$

Very readable accounts of these results are given in [4,7]. Using the spectral representation in Theorem 1.1, we are able to give an alternate proof regarding the regularity of the solutions to $P(D)u = 0$ in terms of Gevrey classes, when P is hypoelliptic.

Theorem 3.1. *Let $P(D)$ be a hypoelliptic operator and suppose $u \in \mathcal{D}'(\Omega)$ satisfies $P(D)u = 0$. Then $u \in \Gamma^\rho(\Omega)$ for some $\rho \geq 1$.*

Proof. We estimate the size of each of the terms in (2.4b)-(2.4d) after an application of D^α. Since $\hat{u}$ and $\langle T_P^\epsilon(u), e^{-i\lambda\cdot x}\rangle$ are analytic and $|P(\lambda)| \geq C$ on $B_{R+1} \setminus B_R$, it is simple exercise to show the terms corresponding to (2.4b) and (2.4c) are dominated by $C^{|\alpha|+1}$, for an appropriate constant C depending on u and R. For instance

$$\frac{1}{(2\pi)^n} \left| D^\alpha \left(\int e^{i\lambda\cdot x} \hat{u}_\epsilon(\lambda) \chi(\lambda) \mathrm{d}\lambda \right) \right| \leq \frac{C}{(2\pi)^n} \int_{|\lambda|<R+1} |\lambda|^{|\alpha|} \mathrm{d}\lambda \leq C^{|\alpha|+1},$$

where here and throughout C represents a generic, positive constant, independent of the multi-index α. All that remains is to estimate the size of the derivatives of the remaining term (2.4d). To do this, it is favourable to use the the $\theta = (\theta_1, \theta')$ coordinates that were introduced in (2.8). It is straightforward to show that a generic term coming from D^α applied to the finite sum in (2.4d) is dominated by

$$C \int \left| (\theta_1 + \beta' \cdot \theta')^{\alpha_1} (\theta')^{\alpha'} \right| (1 + |\theta|)^M e^{-D_1|\theta_1|^{d_1}} \cdots e^{-D_n|\theta_n|^{d_n}} \mathrm{d}\theta.$$

Again we have used the notation $\alpha = (\alpha_1, \alpha')$ for the multi-index α. Using the (multi-index) binomial theorem and making an elementary change of variables, this integral is dominated by

$$C \sum_{|\gamma|=\alpha_1} \binom{\alpha_1}{\gamma} \Gamma\left(\frac{\gamma_1+1}{d_1}\right) \Gamma\left(\frac{\gamma_2+\alpha_2+1}{d_2}\right) \cdots \Gamma\left(\frac{\gamma_n+\alpha_n+1}{d_n}\right), \tag{3.1}$$

with C dependent on M and d. We have used Euler's definition of the Gamma function and the multi-nomial coefficient:

$$\Gamma(z)=\int_0^\infty e^{-t}t^{z-1}\mathrm{d}t, \qquad \binom{\alpha_1}{\gamma}=\frac{\Gamma(\alpha_1+1)}{\Gamma(\gamma_1+1)\cdots\Gamma(\gamma_n+1)}.$$

Setting $\rho=\max d_i^{-1}\geq 1$ and using the estimate (its proof is given in the appendix)

$$\Gamma\left(\frac{\mu+1}{\nu}\right)\leq B^{\mu+1}\Gamma(\mu+1)^{\frac{1}{\nu}}, \qquad \text{for fixed } \nu>0,\ \mu\geq 0 \text{ arbitrary},$$

for some constant $B=B(\nu)$, the sum in (3.1) can then be dominated by

$$C\sum_{|\gamma|=\alpha_1}\binom{\alpha_1}{\gamma}B^{\gamma_1+\alpha_2+\gamma_2+\cdots+\alpha_n+\gamma_n}\Gamma(\gamma_1+1)^\rho\Gamma(\gamma_2+\alpha_2+1)^\rho\cdots\Gamma(\gamma_n+\alpha_n+1)^\rho. \tag{3.2}$$

By repeatedly applying the well-known inequality [20]

$$\Gamma(a+x)\Gamma(b+x)\leq\Gamma(x)\Gamma(a+b+x),$$

which is valid for $a,b\geq 0$ and $x>0$, we find the sum in (3.2) bound above by

$$\begin{aligned}&C\sum_{|\gamma|=\alpha_1}\binom{\alpha_1}{\gamma}B^{|\gamma|+\alpha_2+\cdots+\alpha_n}\Gamma(\gamma_1+\gamma_2+\cdots+\gamma_n+1)^\rho\Gamma(\alpha_2+1)^\rho\cdots\Gamma(\alpha_n+1)^\rho\\ =&CB^{|\alpha|}\Gamma(\alpha_1+1)^\rho\Gamma(\alpha_2+1)^\rho\cdots\Gamma(\alpha_n+1)^\rho\sum_{|\gamma|=\alpha_1}\binom{\alpha_1}{\gamma},\end{aligned}$$

since $|\gamma|=\alpha_1$ throughout this sum. Finally we arrive at the following estimate:

$$\begin{aligned}&\left|\frac{D^\alpha}{(2\pi)^n}\sum_I\int_{\Gamma_I}\frac{\left\langle T_P^\epsilon(u),\vartheta_I(y)e^{\mathrm{i}\lambda\cdot(x-y)}\right\rangle}{P(\lambda)}\mathrm{d}\lambda\right|\\ \leq&C^{|\alpha|}(\alpha!)^\rho\sum_{|\gamma|=\alpha_1}\binom{\alpha_1}{\gamma}\\ \leq&C^{|\alpha|+1}(\alpha!)^\rho,\end{aligned}$$

since the sum is just n^{α_1}. And so we have the final estimate:

$$|D^\alpha u|\leq C^{|\alpha|+1}(\alpha!)^\rho,$$

for some appropriate constants $C>0$ and $\rho=\max_i d_i^{-1}\geq 1$. From this we deduce $u\in\Gamma^\rho(\Omega)$ for $\rho\geq 1$ and the theorem is proved. $\square$

4 Appendix

Here we give a proof to for the estimate used in the proof of Theorem 3.1, namely:

$$\Gamma\left(\frac{\mu+1}{\nu}\right) \le B^{\mu+1}\Gamma(\mu+1)^{\frac{1}{\nu}}, \tag{4.1}$$

for $\mu \in [0,\infty)$ and fixed $\nu > 0$. Although the result is not difficult to prove, we have been unable to find it anywhere in the literature. It is sufficient to prove

$$F_\nu(\mu) \equiv \frac{\log\Gamma\left(\frac{\mu+1}{\nu}\right) - \frac{1}{\nu}\log\Gamma(\mu+1)}{\mu+1} \le C(\nu),$$

for some constant $C(\nu)$. Since F_ν is clearly continuous on compact intervals and so necessarily bounded on each, it suffices to prove F_ν tends to some limit as $\mu \to \infty$. A application of Stirling's approximation for $\log\Gamma(z)$ shows

$$\lim_{\mu\to\infty} F_\nu(\mu) = \frac{1}{\nu}\log\left(\frac{1}{\nu}\right),$$

from which we deduce $F_\nu(\mu)$ is bounded for $\mu \ge 0$ and $\nu > 0$. The estimate in (4.1) follows.

Acknowledgments

The author would like to thank Emmanuel College, Cambridge for their support in the form of a Junior Research Fellowship.

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