ON OPPENHEIM’S INEQUALITY*

Yang Shangjun (杨尚俊) Cai Qian(蔡 茜)

Abstract We prove several inequalities for symmetric positive semidefinite, general $M$-matrices and inverse $M$-matrices which are generalization of the classical Oppenheim’s Inequality for symmetric positive semidefinite matrices.

Key words Hadamard’s inequality, Fischer’s inequality, Oppenheim’s inequality, $M$-matrices, inverse $M$-matrices, Hadamard product.

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For simplicity we denote the set of all $n \times n$ positive semidefinite, symmetric positive semidefinite, nonsingular $M$-matrices, general $M$-matrices, inverse $M$-matrices by $\mathcal{P}, \mathcal{SP}, \mathcal{M}, \mathcal{M}, \mathcal{M}^{-1}$, respectively; denote the Hadamard product of $A, B$ by $A \circ B$; denote the $(n - 1)$ th leading principal submatrix of the $n \times n$ matrix $A$ by $A(n)$.

The following inequality is known as Oppenheim’s inequality:

**Theorem OPP ([2],Theorem 7.8.6)** If $A, B \in \mathcal{SP}$, then

$$\prod_{i=1}^{n} b_{ii} = b_{11} \cdots b_{nn} \leq \det A \circ B. \quad (1)$$

We shall establish several inequalities which generalize Oppenheim’s inequality. First we give some lemmas.

**Lemma 1** $A, B \in M_n(R)$ satisfy inequality (1) if and only if for arbitrary positive diagonal matrices $D_1, D_2, \hat{A} = D_1 A, \hat{B} = BD_2$ satisfy (1).

**Proof** Suppose that the real matrices $A, B$ satisfy inequality (1). Then

$$(\det \hat{A})(\hat{b}_{11} \cdots \hat{b}_{nn}) = (\det D_1)(\det A)(b_{11} \cdots b_{nn})(\det D_2) \leq (\det D_1)(\det A \circ B)(\det D_2)$$

$$= (\det D_1 A) \circ (BD_2) = \det \hat{A} \circ \hat{B}$$

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as desired. Since $A = D_1^{-1} \hat{A}, B = D_2^{-1} \hat{B}$ with $D_1^{-1}, D_2^{-1}$ being positive diagonal, the converse part also holds.

**Lemma 2** If $A \in \mathcal{M} \cup \mathcal{M}^{-1}$, then there is a positive diagonal matrix $D$ such that $AD + DA^T \in \mathcal{P}$.

**Proof** When $A \in \mathcal{M}$, the result is well known (see Theorem 2.5.3. of [3]).

If $A \in \mathcal{M}^{-1}$, then $A^{-1} \in \mathcal{M}$ and for some positive diagonal matrix $D$ we have $A^{-1}D + DA^{-T} \in \mathcal{P}$ from which $DA^T + AD \in \mathcal{P}$ follows.

**Lemma 3** For any $n \times n$ real matrix $A$, $H(A) = A + A^T \in \mathcal{P}$ implies $\det A > 0$.

**Proof** Let $F(A) = \{x^*Ax : x \in \mathbb{C}^n, x^*x = 1\}, \sigma(A)$ be the field of values of $A$ (see chapter 1 of [3]) and the spectrum of $A$, respectively. Then $\sigma(A) \subset F(A) \subset \{z \in \mathbb{C} : \Re(z) > 0\}$ by properties 1.2.5 and 1.2.6 of [3] which imply $A$ is positive stable, then $\det A > 0$ by observation 2.1.4 of [3].

**Definition** An $n \times n$ real matrix $A$ is strictly row diagonally dominant if

$$|a_{ii}| \geq \sum_{j \neq i} |a_{ij}|;$$

$A$ is strictly diagonally dominant of its column entries if $|a_{jj}| > |a_{ij}|, \forall i \neq j$.

**Proposition 1** (i) if $A$ is strictly row diagonally dominant, then $\det A > 0$ and $A^{-1}$ is strictly diagonally dominant of its column entries. (ii) if $A \in \mathcal{M}$, then there is a positive diagonal matrix $D$ such that $AD$ is strictly row diagonally dominant. (iii) if $A \in \mathcal{M}^{-1}$, then there exist positive diagonal matrices $D_1, D_2$ such that $D_1AD_2 = (\alpha_{ij})$ satisfy $\alpha_{ii} = 1, \forall i; \alpha_{ij} < 1, \forall i \neq j$.

**Proof** (i) and (ii) are known (see Chapter 2 of [3]); and (iii) can be easily deduced from (i) and (ii).

**Lemma 4** If $A \in \mathcal{P} \cup \mathcal{M}$ and $B \in \mathcal{P} \cup \mathcal{M} \cup \mathcal{M}^{-1}$, then $\det(A \circ B) > 0$.

**Proof** If $A, B \in \mathcal{P}$, then $A \circ B \in \mathcal{P}$ by Schur product theorem (Theorem 7.5.3 of [2]), hence $\det(A \circ B) > 0$ as desired.

If $A \in \mathcal{P}, B \in \mathcal{M} \cup \mathcal{M}^{-1}$, then there is a positive diagonal matrix $D$ such that $BD + DB^T \in \mathcal{P}$ by Lemma 2 and $A \circ (BD) + (A \circ (BD))^T = A \circ (BD + DB^T) \in \mathcal{P}$ by Schur product theorem.

Therefore $\det(A \circ (BD)) > 0$ holds by Lemma 3. Now we have

$$\det(A \circ B) \det D = \det((A \circ B)D) = \det(A \circ (BD)) > 0.$$ 

Since $\det D > 0$, the desired conclusion follows.

If $A \in \mathcal{M}, B \in \mathcal{M} \cup \mathcal{M}^{-1}$, then from Proposition 1 and Lemma 1 we may assume, without loss of generality, that $A$ is strictly row diagonally dominant and $B$ is strictly row diagonally
dominant (when \( B \in \mathcal{M} \)) or satisfies \( b_{ii} = 1, \forall i; b_{ij} < 1, \forall i \neq j \) (when \( B \in \mathcal{M}^{-1} \)). It is easy to see that \( A \circ B \) is strictly row diagonally dominant which implies \( \det(A \circ B) > 0 \).

It is well known that Hadamard’s inequality and Fischer’s inequality hold in the three classes of matrices: \( \mathcal{P}, \mathcal{M} \) and \( \mathcal{M}^{-1} \). In sequent for a square matrix \( A \) we denote the number \( \det(A)/\det(A(n)) \) by \( \alpha(A) \). For \( A \in \mathcal{P} \cup \mathcal{M} \cup \mathcal{M}^{-1} \) we have \( \det(A) \leq a_{nn}(\det(A(n))) \) by Fischer’s inequality which implies \( a_{nn} \geq \alpha(A) \).

**Lemma 5**  
(i) If \( A \in \mathcal{P} \), then \( A_t = A - t \text{diag}(0, \cdots, 0, 1) \in \mathcal{P}, \forall t < \alpha(A) \);  
(ii) If \( A \in \mathcal{M} \), then \( A_t = A - t \text{diag}(0, \cdots, 0, 1) \in \mathcal{M}, \forall t < \alpha(A) \).

**Proof**  
By Theorem 7.2.5 of [2] and Theorem 2.5.3 of [3] it suffices to prove all the leading principal minors of \( A_t \) are positive.

If \( A \in \mathcal{P} \cup \mathcal{M} \), then \( A_t(n) = A(n) \in \mathcal{P} \cup \mathcal{M} \). Therefore the leading principal minors of \( A_t(n) \) are all positive by Theorem 7.2.5 of [2] or Theorem 2.5.3 of [3]. Now we have \( \det(A_t) = \det(A_t)\det(A(n)) > \det(A) - \alpha(A) \det(A(n)) = 0 \), by the definition of \( \alpha(A) \). Finally the leading principal minors of \( A_t(n) \) are all positive and the proof is completed.

**Theorem 1**  
If \( A \in \mathcal{P} \cup \mathcal{M} \), and \( B \in \mathcal{P} \cup \mathcal{M} \cup \mathcal{M}^{-1} \), then

\[
\det(A \circ B) \geq (\det(A)) \prod_{i=1}^{n} b_{ii} \geq (\det(A))(\det(B)).
\]  

**Proof**  
We prove the first inequality of (2) by induction on \( n \). When \( n = 2 \), \( A \in \mathcal{P} \cup \mathcal{M} \) implies \( a_{11}a_{22} > a_{12}a_{21} \) and \( B \in \mathcal{P} \cup \mathcal{M} \cup \mathcal{M}^{-1} \) implies \( b_{11}b_{22} > 0; (b_{12}b_{21}/b_{11}b_{22}) < 1 \). Therefore the conclusion holds, because

\[
\det(A \circ B) = (a_{11}a_{22} - a_{12}a_{21}(b_{12}b_{21}/b_{11}b_{22}))b_{11}b_{22} \\
\geq (a_{11}a_{22} - a_{12}a_{21})b_{11}b_{22} = (\det(A)) \prod_{i=1}^{2} b_{ii}.
\]

Suppose that \( n > 2 \) and the conclusion holds for \( n - 1 \). Since any principal submatrix of a symmetric positive matrix (\( M \)-matrix) is also a symmetric positive matrix (\( M \)-matrix), the induction hypothesis implies

\[
\det(A(n) \circ B(n)) \geq (\det(A(n))) \prod_{i=1}^{n-1} b_{ii}.
\]  

Now \( A \in \mathcal{P} \cup \mathcal{M} \) implies \( A_t \in \mathcal{P} \cup \mathcal{M}, \forall t < \alpha(A) \) by Lemma 5 and \( A \in \mathcal{P} \cup \mathcal{M}, B \in \mathcal{P} \cup \mathcal{M} \cup \mathcal{M}^{-1} \) implies \( 0 < \det(A_t \circ B) = \det(A_t \circ B - tb_{nn}E_{nn}) \) by Lemma 4, where \( E_{nn} = \text{diag}(0, \cdots, 0, 1) \). Since \( \det(A_t \circ B - tb_{nn}E_{nn}) = \det(A_t \circ B) - tb_{nn}\det(A(n)) \), we have \( \det(A_t \circ B) > tb_{nn}\det(A(n) \circ B(n)) \). As \( t \) approaches \( \alpha(A) \) from the left side taking the limit gives

\[
\det(A \circ B) \geq (\det(A)/\det(A(n)))(b_{nn})(\det(A(n))) \prod_{i=1}^{n-1} b_{ii} = (\det(A)) \prod_{i=1}^{n} b_{ii}.
\]
This completes the proof of the first inequality of (2), and the second inequality of (2) follows immediately from Hadamard’s inequality of matrix $B \in \mathcal{P} \cup \mathcal{M}^{-1} B : \prod_{i=1}^{n} b_{ii} \geq \det B$.

**Remark 1** In the case of $A \in \mathcal{M}$ and $B \in \mathcal{M}^{-1}$ the result is known (see Problem 13 in page 128 of [3]).

**Theorem 2** If $A, B \in \mathcal{P} \cup \mathcal{M}$, then

$$\det A \circ B \geq \max \left\{ (\det A) \prod_{i=1}^{n} b_{ii}, (\det B) \prod_{i=1}^{n} a_{ii} \right\} \geq (\det A)(\det B). \quad (4)$$

**Proof** By virtue of Hadamard’s inequality we only need to prove the first inequality. By Theorem 1 we have

$$\det A \circ B \geq (\det) \prod_{i=1}^{n} b_{ii}. \quad (5)$$

Now since $\det A \circ B = \det B \circ A$, we may change the roles of $A, B$ and use Theorem 1 to deduce

$$\det A \circ B = \det B \circ A \geq (\det B) \prod_{i=1}^{n} a_{ii}. \quad (6)$$

Finally the first inequality of (4) follows immediately from (5) and (6) and the proof is completed.

**Definition 2** Matrix $A$ is called an inverse $M_0$-matrix if there is a sequence $\{A_k\} \subset \mathcal{M}^{-1}$ such that $A = \lim_{k \to \infty} A_k$. Meanwhile the following formulas clearly hold:

$$\lim_{k \to \infty} \det A_k \circ B = \det A \circ B; \lim_{k \to \infty} \det A_k = \det A; \lim_{k \to \infty} \prod_{i=1}^{n} (a_{ii} + 1/k) = \prod_{i=1}^{n} a_{ii}.$$

In addition, the next lemma can be easily proved by using Theorem 7.2.5 of [2], Theorem 2.5.3 of [3] and some other properties of the matrices involved.

**Lemma 6** If $A_k = A + (1/k)I \in \mathcal{P}(\mathcal{M}), \forall k$, then $A = \lim_{k \to \infty} A_k \in \mathcal{SP}(\mathcal{M})$.

Conversely, if $A \in \mathcal{SP}(\mathcal{M})$, then $A_k = A + (1/k)I \in \mathcal{P}(\mathcal{M}), \forall k$ and $\lim_{k \to \infty} A_k = A$.

Now Theorem 1 and Theorem 2 can be generalized as follows:

**Theorem 3** Inequality (2) holds if $A \in \mathcal{SP} \cup \mathcal{M}$, and $B \in \mathcal{SP} \cup \mathcal{M} \cup \mathcal{M}^{-1}$.

**Proof** By Lemma 6 $A_k = A + (1/k)I \in \mathcal{P} \cup \mathcal{M}; B_k = B + (1/k)I \in \mathcal{P} \cup \mathcal{M} \cup \mathcal{M}^{-1}$ for $k = 1, 2, \cdot \cdot \cdot$. And by Theorem 1

$$\det A_k \circ B_k \geq (\det A_k) \prod_{i=1}^{n} (b_{ii} + 1/k) \geq (\det A_k)(\det B_k). \quad (7)$$

Letting let $k$ approaches to infinity and taking the limits in (7) the inequality (2) follows.
Theorem 4  If $A, B \in S \mathcal{P} \cup \widetilde{\mathcal{M}}$, then
\[
\det A \circ B \geq \max \left\{ (\det A)^n \prod_{i=1}^{n} b_{ii}, (\det B)^n \prod_{i=1}^{n} a_{ii} \right\} \geq (\det A)(\det B).
\]

Proof  The proof is similar to that of Theorem 3.

 Remark 2  In the case of $A, B \in S \mathcal{P}$, the inequality in Theorem 4 is sharper than the original Oppenheim’s inequality (1).

References


Yang Shangjun  Ministry of Education Key Lab. of IC & SP of Anhui University, Hefei 230039, PRC.

Cai Qian  Department of Applied Mathematics, Nanjing Audit University, Nanjing 210029, PRC.