Stability and Convergence of an Implicit Difference Approximation for the Space Riesz Fractional Reaction-Dispersion Equation

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Abstract

In this paper, we consider a Riesz space-fractional reaction-dispersion equation (RSFRDE). The RSFRDE is obtained from the classical reaction-dispersion equation by replacing the second-order space derivative with a Riesz derivative of order \( \beta \in (1, 2] \). We propose an implicit finite difference approximation for RSFRDE. The stability and convergence of the finite difference approximations are analyzed. Numerical results are found in good agreement with the theoretical analysis.

Keywords: Riesz fractional derivative; fractional reaction-dispersion equation; implicit finite difference approximation; stability; convergence.

Mathematics subject classification: 26A33, 35K57, 65M12

1. Introduction

Recently, a growing number of works have been concerned with dynamical systems described by fractional-order equations which involve derivatives and integrals of non-integer order. It is found and testified that many phenomena in engineering, physics, finance, hydrology, chemistry and other sciences [1,2] can be simulated by fractional-order equations. These models are more adequate than the previously used integer-order models. Fractional order derivatives and integrals provide a powerful instrument for the description of memory and hereditary properties of difference substances. Some partial differential equations of space-time fractional order were successfully used for modeling relevant physical process and financial behavior. For example, it can be used in groundwater hydrology to model the transport of passive traces carried by fluid flow in a porous medium [3, 4]. Furthermore, it can be used in financial markets to model the high-frequency price dynamics [5, 6]. We can also interpret the two-level difference scheme resulting from the Grunwald-Letniko discretization of fractional derivatives as a random walk model discrete in space and time [7].

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The basic analytic theory for the space-fractional diffusion processes was developed in 1952 by Fell [8] via inversion of Riesz potential operators. Mainardi et al. [9] considered the space fractional diffusion equation and presented an explicit representation of the Green function for the equation. Benson et al. [10, 11] considered the space fractional advection-dispersion equation and gave an analytic solution in terms of the α-stable error function. Liu et al. [12] derived the complete solution of the time fractional advection-dispersion equation. However, numerical methods and theoretical analysis of fractional differential equations are still at an early stage of development. Lynch et al. [13] presented two different discretization methods for the fractional diffusion equation, but stability and convergence are not presented. Lin and Liu [14] proposed higher-order approximations of a nonlinear fractional-order ordinary differential equation with initial value and proved consistency, convergence and stability of the fractional higher-order methods. Shen and Liu [15] considered the space-fractional diffusion equation and gave error analysis. Liu et al. [16] presented the numerical solution of a space fractional Fokker-Planck equation. Meerschaert et al. [17] considered the finite difference approximations for two-sided space-fractional partial differential equations and discussed their stability, consistency and convergence of the method. It is also noticed that fractional reaction-diffusion equations can be used to model activator-inhibitor dynamics with anomalous diffusion, which occurs in spatially inhomogeneous media [18].

In this paper we consider a Riesz space-fractional reaction-dispersion equation in a bounded space domain and time domain. We present an implicit difference approximation for the equation and analyze its stability and convergence. Furthermore, to evaluate the efficiency of the obtained difference scheme, a comparison with method of lines (MoL) is provided. The MoL was first introduced to solve space fractional partial differential equations by Liu et al. [16, 19]. Finally, some numerical examples are given to show that the numerical results are in good agreement with our theoretical analysis.

2. An explicit finite approximation for RSFRDE

In this section, we consider RSFRDE in a bounded space domain \([0, L]\) with the following initial and boundary conditions:

\[
\begin{align*}
\frac{\partial u(x, t)}{\partial t} &= -u(x, t) + x D_0^\beta u(x, t), & 0 < x < L, 0 \leq t \leq T, \\
u(x, 0) &= g(x), & 0 \leq x \leq L, \\
\frac{\partial u(0, t)}{\partial x} &= 0, & u(L, t) = 0, & 0 \leq t \leq T,
\end{align*}
\]

(2.1)

where \(1 < \beta \leq 2\), and we assume that both \(u(x, t)\) and \(g(x)\) are real-valued and sufficiently well-behaved functions, the Riesz space-fractional derivative of order \(\beta\), \(x D_0^\beta\), is defined by analytic continuation in the whole range \(1 < \beta \leq 2\) [7]

\[
x D_0^\beta := -I^{-\beta} = -c(I_+^{-\beta} + I_-^{-\beta}),
\]

(2.2)
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where

\[ c = \frac{1}{2 \cos(\beta \pi/2)} \quad I_\pm^\beta = \frac{d^2 x}{d x^2} l_\pm^{2-\beta}, \]

and the Weyl integrals \( I_\pm^\beta \) are defined as [20]

\[
\begin{cases}
(I_+^\beta \phi)(x) = \frac{1}{\Gamma(\beta)} \int_0^x (x - \xi)^{\beta-1} \phi(\xi) d\xi, & \beta > 0, \\
(I_-^\beta \phi)(x) = \frac{1}{\Gamma(\beta)} \int_x^L (\xi - x)^{\beta-1} \phi(\xi) d\xi, & \beta > 0,
\end{cases}
\]

where \( \phi(x) \in L_1(-\infty, +\infty) \).

We take the symbol of the pseudo-differential operator \( x D_0^\beta \), which is defined through its Fourier presentation, namely \( x \hat{D}_0^\beta := -k^\beta \). In particular, we have \( x D_0^2 = \frac{d^2}{dx^2} \), but \( x D_0^1 \neq \frac{d}{dx} \). We shall discretize the Riesz derivative \( x D_0^\beta \) to derive a numerical solution for RSFRDE.

From (2.2), using the boundary conditions \( \frac{\partial u_0(x,t)}{\partial x} = 0, u(L,t) = 0 \), we have

\[
I_+^\beta = \frac{d^2 x}{d x^2} l_+^{2-\beta} = \frac{d^2 x}{d x^2} \left[ \frac{1}{\Gamma(2-\beta)} \int_0^x (x - \xi)^{1-\beta} u(\xi,t) d\xi \right]
\]

\[
= \frac{u(0,t) x^{-\beta}}{\Gamma(1-\beta)} + \frac{1}{\Gamma(2-\beta)} \int_0^x \frac{\partial^2 u(\xi,t)}{\partial \xi^2} (x - \xi)^{1-\beta} d\xi, \tag{2.4}
\]

\[
I_-^\beta = \frac{d^2 x}{d x^2} l_-^{2-\beta} = \frac{d^2 x}{d x^2} \left[ \frac{1}{\Gamma(2-\beta)} \int_0^L (\xi - x)^{1-\beta} u(\xi,t) d\xi \right]
\]

\[
= \frac{1}{\Gamma(2-\beta)} \left[ -(L - x)^{1-\beta} \frac{\partial u(L,t)}{\partial x} - \int_0^L (\xi - x)^{1-\beta} u''(\xi,t) d\xi \right]. \tag{2.5}
\]

Define \( x_i = lh \) (\( l = 0, 1, \cdots, M \)) and \( t_n = n \tau \) (\( n = 0, 1, \cdots, N \)), where \( M \) and \( N \) are positive integers, \( h = L/M, \tau = T/N \) are space step and time step respectively. Assume \( u_n^\beta \) be the numerical approximation to \( u(x_i, t_n) \). Using a second-order difference approximation, the
resulting discretization on $I_{1+}^{-\beta}, I_{1-}^{-\beta}$ takes the following form

$$
I_{1+}^{-\beta} u(x_i, t_n) \approx h^{-\beta} I_{1+}^{-\beta} u(x_i, t_n) = \frac{h^{-\beta}}{\Gamma(3-\beta)} \left[ \frac{u_0^n(1-\beta)(2-\beta)}{\Gamma(1-\beta)} u_0^n 
+ \sum_{j=0}^{l-1} c_j(u_{l-j+1}^n - 2u_{l-j}^n + u_{l-j-1}^n) \right],
$$

(2.6)

$$
I_{1-}^{-\beta} u(x_i, t_n) \approx h^{-\beta} I_{1-}^{-\beta} u(x_i, t_n) = \frac{h^{-\beta}}{\Gamma(3-\beta)} \left[ -\frac{(2-\beta)(u_M^n - u_{M-1}^n)}{(M-1)^{\beta-1}} 
+ \sum_{j=0}^{M-l-1} c_j(u_{l+j-1}^n - 2u_{l+j}^n + u_{l+j+1}^n) \right],
$$

(2.7)

where $c_j = (j + 1)^{2-\beta} - j^{2-\beta}$, and $I_{1+}^{-\beta}$ denote the approximating operators $I_{1+}^{-\beta}$. Hence we can introduce the difference operator $h D_0^\beta$ to approximate the Riesz derivative $D_0^\beta$ which is defined in (2.2)

$$
h D_0^\beta u(x_i, t_n) = -c[h I_{1+}^{-\beta} u(x_i, t_n) + h I_{1-}^{-\beta} u(x_i, t_n)] \approx D_0^\beta u(x_i, t_n),
$$

(2.8)

for $l = 1, \ldots, M-1, n = 0, 1, \ldots, N$, where $c$ is defined in (2.2).

Adopting an implicit (Euler) method in time at level $t = t_n$ (and $x = x_i$) and substituting the above the expressions into Eq. (2.1), we finally get the implicit difference approximation

$$
\frac{u_i^n - u_{i-1}^{n-1}}{\tau} = -u_i^n - \frac{h^{-\beta}}{2\cos \beta \pi} \frac{(1-\beta)(2-\beta)}{\Gamma(3-\beta)} u_0^n 
+ \sum_{j=0}^{l-1} c_j(u_{l-j+1}^n - 2u_{l-j}^n + u_{l-j-1}^n) 
+ \sum_{j=0}^{M-l-1} c_j(u_{l+j-1}^n - 2u_{l+j}^n + u_{l+j+1}^n) 
-(2-\beta)(M-1)^{1-\beta}(u_M^n - u_{M-1}^n),
$$

(2.9)

with the boundary conditions $u_0^n = u_0^n$ and $u_M^n = 0$.

The above equation with the boundary conditions can be written as the following implicit finite difference approximation (IFDA)

$$
(1 + \tau)u_i^n - d_1(I)u_{i+1}^n - k \sum_{j=0}^{l-1} c_j(u_{l-j+1}^n - 2u_{l-j}^n + u_{l-j-1}^n) 
-k \sum_{j=0}^{M-l-1} c_j(u_{l+j-1}^n - 2u_{l+j}^n + u_{l+j+1}^n) - d_2(I)u_{M-1}^n = u_i^{n-1},
$$

(2.10)
Lemma 3.2. Let $c \in \mathbb{R}^+$ positive numbers $c_1, c_2, d_1(l), d_2(l)$ are given by the following expressions

\[
\begin{cases}
  k = -\frac{\tau h^{-\beta}}{2 \cos \frac{\beta \pi}{2} \Gamma(3 - \beta)} > 0, \\
  d_1(l) = k(1 - \beta)(2 - \beta)(2l - 1) > 0, \\
  d_2(l) = k(2 - \beta)(M - 1)^{1-\beta} > 0.
\end{cases}
\]  

(2.11)

Eq. (2.10) results in a linear system of equations

\[BU^n = U^{n-1},\]  

(2.12)

where $U^n = (u_1^n, u_2^n, \ldots, u_{M-1}^n)^T$, and $B = (b_{ij})_{(M-1) \times (M-1)}$ is a matrix of coefficients.

3. Analysis of stability and convergence

To prove the stability and the convergence we need the following Lemmas.

Lemma 3.1. Let $A \in \mathbb{C}^{m \times n}$ and $\rho(A)$ be the spectral radius of the matrix $A$. Then for any given positive numbers $\epsilon$, there exists a norm $\| \cdot \|_m$ of the matrix $A$, such that $\|A\|_m \leq \rho(A) + \epsilon$.

Proof. See [21].

Lemma 3.2. Let $c_l = (l + 1)^{2-\beta} - l^{2-\beta}$, and $k, d_1(l), d_2(l)$ be defined in (2.11). Then

1. $c_{l-1} > c_l > 0, c_{l-1} - 2c_l + c_{l+1} > 0, \quad l \geq 1,$
2. $d_2(l) + k(-2c_{M-l-1} + c_{M-l-2}) > 0, \quad 0 \leq l \leq M - 1,$
3. $d_2(l) - k c_{M-l-1} < 0, \quad 0 \leq l \leq M - 1,$
4. $d_1(l) + k(c_{l-2} - c_{l-1}) > 0, \quad l \geq 2.$

Proof.

1. Let $f(l) = c_l = (l + 1)^{2-\beta} - l^{2-\beta}$. Then for any $l \geq 0$, we have

\[
\begin{align*}
  f'(l) &= (2 - \beta)(l + 1)^{1-\beta} - l^{1-\beta} = (2 - \beta)(1 - \beta)\xi^{-\beta} < 0, \\
  f''(l) &= (2 - \beta)(1 - \beta)[(l + 1)^{-\beta} - l^{-\beta}] \\
  &= (2 - \beta)(1 - \beta)(-\beta)\eta^{-\beta - 1} > 0,
\end{align*}
\]

(3.1)

(3.2)

where $\xi, \eta \in (l, l+1)$. It follows from (13) and (14) that $c_l > c_{l+1} > 0$. Since $f''(l) > 0$, we get $\frac{2}{\beta}(f(l+1) + f(l-1)) > f(l)$, i.e., $c_{l-1} - 2c_l + c_{l+1} > 0$.

2. Because $c_l = (l + 1)^{2-\beta} - l^{2-\beta} = (2 - \beta)\xi^{1-\beta}, \xi \in (l, l+1)$, it follows that

\[(2 - \beta)(l + 1)^{1-\beta} < c_l < (2 - \beta)l^{1-\beta}.
\]

(3.3)
Consequently,
\[
d_2(l) + k(-2c_{M-l-1} + c_{M-l-2}) \\
= k(2 - \beta)(M - 1)^{1-\beta} + k(-2c_{M-l-1} + c_{M-l-2}) \\
= k[(2 - \beta)(M - l)^{1-\beta} - c_{M-l}] + k(c_{M-l} - 2c_{M-l-1} + c_{M-l-2}) > 0. \tag{3.4}
\]

3. It follows from (3.3) that
\[
d_2(l) - kc_{M-l-1} = k[(2 - \beta)(M - l)^{1-\beta} - c_{M-l}] < 0.
\]

4. Because
\[
c_{l-2} - c_{l-1} = f(l - 2) - f(l - 1) = -(2 - \beta)[(\xi + 1)^{1-\beta} - \xi^{1-\beta}] \\
= -(2 - \beta)(1 - \beta)\eta^{\beta - 1} \\
> -(2 - \beta)(1 - \beta)l^{-\beta}, \tag{3.5}
\]

where \( \xi \in (l - 2, l - 1) \), \( \eta \in (\xi, \xi + 1) \), we find that
\[
(1 - \beta)(2 - \beta)l^{-\beta} + (c_{l-2} - c_{l-1}) > 0.
\]

Noticing \( k > 0, d_1 = k(1 - \beta)(2 - \beta)l^{-\beta} \), we have
\[
d_1 + k(c_{l-2} - c_{l-1}) = k[(1 - \beta)(2 - \beta)l^{-\beta} + (c_{l-2} - c_{l-1})] > 0. \]

Lemma 3.3. Let \( u(x, t) \) is smooth function. Then
\[
\frac{h}{x} D_0^\beta u(x_1, t_n) = x D_0^\beta u(x_1, t_n) + \mathcal{O}(h^{2-\beta}), \tag{3.6}
\]

where \( \frac{h}{x} D_0^\beta u(x_1, t_n) \) is defined in (2.8), \( x D_0^\beta u(x_1, t_n) \) is defined in (2.2).

Proof. See [22].

Theorem 3.1. The implicit finite difference method (2.10) for Eq. (2.1) is unconditionally stable.

Proof. Notice that \( k > 0, d_1(l) < 0, d_2(l) > 0, l = 1, \cdots, M - 1 \). Moreover,
\[
\sum_{j=1, j \neq i}^{M+1} |b_{ij}| = |d_1(i) - k(c_{i-2} - c_{i-1})| + |\sum_{j=1}^{i-3} (c_j - 2c_{j+1} + c_{j+2})| \\
+ |k - k(c_0 - 2c_1 + c_2)| + |k - k(c_0 - 2c_1 + c_2)| \\
+ \sum_{j=1}^{M+1} (c_j - 2c_{j+1} + c_{j+2})| \\
+ |d_2(i) - k(-2c_{M-l-1} + c_{M-l-2})|. \tag{3.7}
\]
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and

\[ b_{ii} = 1 + \tau + 4k - 2k c_1, \]  

(3.8)

where \( 2 \leq i \leq M - 2 \), and \( \sum_{j=p}^{q} = 0 \) for \( q < p \). Applying Lemma 3.2, we conclude that

\[ b_{ij} < 0 \ (j \neq i), \quad b_{ii} > 1, \]  

(3.9)

\[ b_{ii} - \sum_{j=1, j \neq i}^{M-1} |b_{ij}| = 1 + \tau - d_1(i) - (d_2(i) - k c_{M-i-1}) > 1, \]  

(3.10)

for \( 2 \leq i \leq M - 2 \). Similarly, we easily get

\[ b_{1j} < 0 \ (j \neq 1), \quad b_{M-1j} < 0 \ (j \neq M - 1), \quad b_{11} > 1, \quad b_{M-1,M-1} > 1, \]  

(3.11)

and

\[ b_{11} - \sum_{j=2}^{M-1} |b_{1j}| = 1 + \tau - d_1(1) - (d_2(1) - k c_{M-2}) > 1, \]  

(3.12)

\[ b_{M-1,M-1} - \sum_{j=1}^{M-2} |b_{M-1,j}| = 1 + \tau - d_1(M - 1) + k \]  

\[ -(d_2(M - 1) - k c_0) > 1. \]  

(3.13)

Combining (3.9)-(3.13), we conclude that

\[ b_{ij} < 0 \ (j \neq i), \quad b_{ii} > 1, \quad b_{ii} - \sum_{j=1, j \neq i}^{M-1} |b_{ij}| > 1, \quad i = 1, \ldots, M - 1. \]  

(3.14)

Let \( \lambda \) be an eigenvalues of the matrix \( B \) to linear system of Eq. (2.12). According to the Greschgorin Theorem [21], the eigenvalues of the matrix \( B \) lie in the union of the circles centered at \( b_{ii} \) with radius \( r_i = \sum_{j=1, j \neq i}^{M-1} b_{ij} \), i.e.

\[ |\lambda - b_{ii}| < \sum_{j=1, j \neq i}^{M-1} |b_{ij}|. \]  

(3.15)

Consequently,

\[ \lambda > b_{ii} - \sum_{j=1, j \neq i}^{M-1} |b_{ij}| > 1, \quad i = 1, 2, \ldots, M - 1. \]

Hence \( B \) is invertible and \( \rho(B^{-1}) < 1 \). Using Lemma 3.1, we obtain that IFDA (2.10) is unconditionally stable. ■
Remark 3.1. We can conclude that the coefficients matrix $B$ is strictly diagonally dominant and irreducible. Hence the difference scheme (2.12) is uniquely solvable.

Theorem 3.2. The implicit difference method (2.10) for RSFRDE (2.1) is unconditionally convergent.

Proof. Let $U^n_l$ be the exact solution of the Eq. (2.1), and $u^n_l$ be the exact solution of the difference equation (2.9), error $e^n_l = U^n_l - u^n_l$, $u^n_0 = U^n_l - e^n_l$ at the mesh points $(x_l, t_n)$. Substitution into the difference equation (2.9) and using Taylor theorem and Lemma 3.3 we have

$$
e^n_l - e^{n-1}_l = -e^n_l - \frac{h^{-\beta}}{2\cos \frac{\beta \pi}{2} \Gamma(3 - \beta)} \left[ (1 - \beta)(2 - \beta)l^{-\beta}e^n_0 \\
+ \sum_{j=0}^{l-1} c_j (e^n_{l+j+1} - 2e^n_{l+j} + e^n_{l+j-1}) - (2 - \beta)(M - 1)^{1-\beta}(e^n_M - e^n_{M-1}) \\
+ \sum_{j=0}^{M-l-1} c_j (e^n_{l+j-1} - 2e^n_{l+j} + e^n_{l+j+1}) \right] + O(h^{2-\beta}) + O(\tau).$$

Using the initial and boundary conditions

$$e^n_0 = 0, \ (l = 0, \cdots, M), \quad e^n_0 = e^n_1 + O(h^2), \quad e^n_M = 0, \quad n \in N,$$

Eq. (3.16) can be rewritten in the matrix form as

$$BE^n = E^{n-1} + R, E^0 = 0, \text{ i.e., } \sum_{i=1}^{M-1} b_{ij} e^n_i = e^{n-1}_i + \tau (O(h^{2-\beta} + \tau)), \quad (j = 1, \cdots, M - 1),$$

where $E^n = (e^n_1, e^n_2, \cdots, e^n_{M-1})^T, R = \tau (O(h^{2-\beta}) + O(\tau))(1, \cdots, 1)^T, (j = 1, \cdots, M - 1)$, and

$B = (b_{ij})_{(M-1) \times (M-1)}$ is defined in (2.12).

From (3.14), we know that

$$b_{ij} < 0 \ (j \neq i), \quad b_{ii} > 1,$$

$$\sum_{j=1}^{M-1} b_{ij} = b_{ii} - \sum_{j \neq i} |b_{ij}| > 1 \ (i = 1, \cdots, M - 1).$$
Let \( \| E^n \|_{\infty} = \max_{1 \leq j \leq M-1} |e^n_j| \leq |e^m_n| \) (1 \( \leq m \leq M - 1 \)). Then
\[
\| E^n \|_{\infty} = |e^n_m| \leq \sum_{j=1}^{M-1} b_{mj} |e^m_n| = b_{mn} |e^n_m| \leq \sum_{j=1,j \neq m}^{M-1} b_{mj} |e^m_n| \leq \sum_{j=1,j \neq m}^{M-1} b_{mj} |e^n_j| \leq |e^{n-1}_m| + \tau(\Theta(h^{2-\beta} + \tau)) \leq \| E^{n-1} \|_{\infty} + \tau(\Theta(h^{2-\beta} + \tau)) \leq \| E^0 \|_{\infty} + n\tau(\Theta(h^{2-\beta} + \tau)) \leq C(h^{2-\beta} + \tau). \tag{3.19}
\]
So the implicit difference method (2.10) is unconditionally convergent.

4. Method of Line for RSFRDE

It is very difficult to obtain the exact solutions of fractional partial differential equation. In order to demonstrate the efficiency of the numerical approximation, a comparison with Method of Lines (MoL) is used. The MoL was firstly introduced by Liu et al. [16,19] to solve the space fractional Fokker-Planck equation successfully. The MoL only discretized space variable, then the problem can be transformed a system of ordinary differential equation. The tests show that our numerical results are agreement with the convergence analysis and close to the results of MoL.

The MoL for RSFRDE can be written in the following form: for \( 1 < \beta < 2, (l = 1, \ldots, M - 1) \)
\[
\frac{du_l}{dt} = -u_l^\beta - \frac{h^{-\beta}}{2\cos \frac{\pi\beta}{2}}[u_0(1 - \beta)(2 - \beta)l^{-\beta}
+ \sum_{j=0}^{l-1} c_j(u_{l-j+1} - 2u_{l-j} + u_{l-j-1})
- (2 - \beta)(M - 1)^{1-\beta}(u_M - u_{M-1})
+ \sum_{j=0}^{M-l-1} c_j(u_{l+j+1} - 2u_{l+j} + u_{l+j+1})], \tag{4.1}
\]
with \( u_0 = u_1, u_M = 0, u_l = u(x_l, t) \).

5. Numerical results

To test the numerical schemes, it is important to use simple analytical model. In this section we present an example in a bounded domain to demonstrate that RSFRDE can be
applied to simulate the behavior of the solution of a fractional reaction-dispersion equation. We consider the system

$$\begin{aligned}
    \frac{\partial u(x,t)}{\partial t} &= -u(x,t)+x D_0^\beta u(x,t), \\
    u(x,0) &= g(x) = x^2 \sin x, \\
    \frac{\partial u(0,t)}{\partial x} &= 0, \quad u(\pi,t) = 0,
\end{aligned} \quad 0 \leq x \leq \pi, \quad 0 \leq t \leq T.
$$

Fig. 5.1 shows that the numerical solutions using MoL and the implicit difference approximation (IFDA) with $h = \pi/100$, $\tau = 0.003$ for $\beta = 1.7$, $T = 0.3$. From Fig. 5.1, it can be seen that the numerical results using IFDA is in good agreement with MoL.

Fig. 5.2 shows the evolution result using IFDA with $h = \pi/100$, $\tau = 0.003$, $\beta = 1.7 (0 \leq T \leq 1, 0 \leq x \leq \pi)$, it is apparent that the order $\beta = 1.7$ exhibits diffusive behavior for different times.

Figs. 5.3 and 5.4 compare the response of the RSFRDE for different real numbers $1.2 \leq \beta \leq 1.8$, $T = 0.4$ and $T = 0.8$. 

6. Conclusions

In this paper, we present an implicit finite difference approximation for a Riesz space-fractional reaction-dispersion equation in a bounded domain. The implicit difference approximation is proved to be unconditionally stable and convergent. Furthermore numerical examples are presented to show that the numerical results are in good agreement with our theoretical analysis.

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