Numerical Solution for the Helmholtz Equation with Mixed Boundary Condition

Haibing Wang∗
(School of Mathematics and Computing Science, Hunan University of Science and Technology, Xiangtan 411201, China
E-mail: wanghb845@yahoo.com.cn)

Jijun Liu
(Department of Mathematics, Southeast University, Nanjing 210096, China
E-mail: jjliu@seu.edu.cn)

Received February 20, 2006; Accepted (in revised version) October 29, 2006

Abstract

We consider the numerical solution for the Helmholtz equation in \( \mathbb{R}^2 \) with mixed boundary conditions. The solvability of this mixed boundary value problem is established by the boundary integral equation method. Based on the Green formula, we express the solution in terms of the boundary data. The key to the numerical realization of this method is the computation of weakly singular integrals. Numerical performances show the validity and feasibility of our method. The numerical schemes proposed in this paper have been applied in the realization of probe method for inverse scattering problems.

Keywords: Helmholtz equation; Green formula; potential theory; boundary integral equation; numerics.

Mathematics subject classification: 35J05, 31A10, 65N99

1. Introduction

Boundary value problems for the Helmholtz equation, except for their intrinsic importance, play an important role in obstacle scattering problems, which have been studied widely in recent years. Generally, we must analyze their solvability and give some numerical schemes to solve these problems. Noticing that the direct problems should be solved iteratively in some problems such as inverse scattering, the efficiency and amount of computations should be considered carefully from the numerical point of views. In this paper, we apply the boundary integral method (BIM) to solve the boundary value problem (BVP) for the Helmholtz equation with mixed boundary conditions. Such a problem arises in the probe method for inverse scattering problem by multiple obstacles. Compared with the classical BIM method, the new ingredient in this paper is to take the boundary Dirichlet/Neumann data of the solution as the density function based on the Green formula, rather than to introduce the general density function. Once we determine the Cauchy data

∗Corresponding author.
of the solution on the boundary, the solution to the BVP of the Helmholtz equation can be obtained from the integral expression explicitly. Moreover, the domain for the Helmholtz equation in this paper is non-connected where the mixed boundary conditions are specified for the equation.

Let $D \subset \mathbb{R}^2$ be a bounded domain with three parts $D_j$ ($j = 1, 2, 3$), namely, $D = \bigcup_{j=1}^3 D_j$. We assume that each $D_j$ is a simply connected bounded domain with $C^2$ boundary $\partial D_j$ and $\bar{D}_i \cap \bar{D}_j = \emptyset$ for $i \neq j$. For given $\Omega$ with $C^2$ boundary satisfying $\bar{D} \subset \Omega$, we consider the following mixed boundary value problem for the Helmholtz equation

$$
\begin{cases}
\Delta u(x) + k^2 u(x) = 0 & x \in \Omega \setminus \bar{D}, \\
u(x) = f(x) & x \in \partial \Omega, \\
\partial u / \partial \nu(x) = g_1(x) & x \in \partial D_1, \\
\partial u / \partial \nu(x) + i\lambda u(x) = g_2(x) & x \in \partial D_2, \\
\partial u / \partial \nu(x) + i\lambda u(x) = g_3(x) & x \in \partial D_3,
\end{cases}
$$

(1.1)

with positive wave number $k > 0$, where $\nu$ is the unit normal vector of $\partial D$ directed into the exterior of $D$ and $\lambda > 0$ is the boundary impedance coefficient of $D_3$. This problem arises in the probe method for multiple obstacles applied in an inverse scattering problem, which is to reconstruct the obstacles with different types of boundary from the far-field pattern of the scattered wave.

In [4], as a key step to the test of the probe method for one obstacle, the author considered the following mixed boundary value problem

$$
\begin{cases}
\Delta u(x) + k^2 u(x) = 0 & x \in \Omega \setminus \bar{D}, \\
u(x) = f(x) & x \in \partial \Omega, \\
\partial u / \partial \nu(x) + i\lambda u(x) = g(x) & x \in \partial D,
\end{cases}
$$

(1.2)

and the corresponding Dirichlet-to-Neumann map

$$
\Lambda_{\partial D} : f(x) \mapsto \partial_i u(x)|_{\partial \Omega},
$$

(1.3)

defined by (1.2), which can be viewed as a special case of $D = D_3$ and $D_1 = D_2 = \emptyset$ in (1.1). By applying boundary integral equation method, the solvability of problem (1.2) and an efficient method for solving the Neumann data $\partial_i u(x)$ in $\partial \Omega$ are set up. However, in order to test the validity of the probe method for multiple obstacles with different types of boundary, we should treat the problem (1.1) and Dirichlet-to-Neumann map (1.3) defined by (1.1).

Firstly, the solvability of (1.1) should be clarified for suitable boundary data $f, g_i (i = 1, 2, 3)$ for which the Dirichlet-to-Neumann map (1.3) is well-defined. Secondly, an efficient numerical method to solve (1.1) is necessary. In fact, we can determine the boundary data $\partial_i u|_{\partial \Omega}, \partial_i u|_{\partial D_1}, \partial_i u|_{\partial D_2}$ and $u|_{\partial D_3}$ from a coupled boundary integral equations, therefore $u(x)$ in $\Omega \setminus \bar{D}$ can be calculated by the Green formula. To solve it numerically, the discrete form of integral equations must be given. In this process, the key is how to deal with singular integrals.
This paper is organized as follows. In Section 2, we prove that problem (1.1) is uniquely solvable, and therefore (1.3) is well-defined. This result provides the foundation for our boundary integral method based on the boundary data. Then in Section 3, we discretize the boundary integral equations where the improper integrals are treated by careful singularity analysis. Finally we present some numerical results in Section 4. The validity and feasibility of our method are also checked by comparing the numerical solutions of \( \partial_v u(x) \mid \partial \Omega \) with the exact solutions to model problem of (1.1).

2. Solvability of problem (1.1)

For given \( \Omega, D_j (j = 1, 2, 3) \), \( \lambda > 0 \), and \( (g_1, g_2, g_3) \in C^{1, \alpha}(\partial D_1) \times C(\partial D_2) \times C(\partial D_3) \), we prove the solvability of (1.1) for \( f \in C^{1, \alpha}(\partial \Omega) \). As a result, the Dirichlet-to-Neumann map (1.3) is well defined.

For a closed curve(surface) \( \Gamma \subset \mathbb{R}^m \), introduce the following operators:

\[
(K_\Gamma \cdot \psi)(x) := 2 \int_{\Gamma} \frac{\partial \Phi(x, y)}{\partial \nu(y)} \psi(y) dy(y), \\
(S_\Gamma \cdot \psi)(x) := 2 \int_{\Gamma} \Phi(x, y) \psi(y) dy(y), \\
(T_\Gamma \cdot \psi)(x) := 2 \int_{\Gamma} \frac{\partial \Phi(x, y)}{\partial \nu(y)} \psi(y) dy(y),
\]

where

\[
\Phi(x, y) = \begin{cases} 
\frac{1}{4\pi |x-y|} H_0^{(1)}(k|x-y|), & m = 2, \\
\frac{\mu(x-y)}{4\pi |x-y|^m}, & m = 3,
\end{cases}
\]

is the fundamental solution to the Helmholtz equation. For \( x \in \Gamma' \) which is a smooth closed curve (surface) satisfying \( \Gamma' \cap \Gamma = \emptyset \), the above integrals are regular. For \( x \in \Gamma \), they are also well-defined for the weakly singular kernels due to the following result.

Lemma 2.1. Let \( \Gamma \in C^2 \). The operators \( S_\Gamma, K_\Gamma, K'_\Gamma \) are bounded from \( C(\Gamma) \) into \( C^{0, \alpha}(\Gamma) \), and \( T \) is bounded from \( C^{1, \alpha}(\Gamma) \) into \( C^{0, \alpha}(\Gamma) \).

Based on this property, we can consider the solvability of (1.1).

Theorem 2.1. For \( D_3 \neq \emptyset \) with \( \lambda > 0 \), there exists a unique solution to (1.1) for given \( (f, g_1, g_2, g_3) \in C^{1, \alpha}(\partial \Omega) \times C^{1, \alpha}(\partial D_1) \times C(\partial D_2) \times C(\partial D_3) \).

Proof. Firstly, we prove the uniqueness. It is enough to show that \( f(x) = g_j(x) = 0 (j = 1, 2, 3) \) means \( u(x) = 0 \) in \( \Omega \setminus \bar{D} \).

Since the conjugate function \( \bar{u} \) satisfies

\[
\begin{cases} 
\Delta \bar{u}(x) + k^2 \bar{u}(x) = 0 & x \in \Omega \setminus \bar{D}, \\
\bar{u}(x) = 0 & x \in \partial \Omega, \\
\bar{u}(x) = 0 & x \in \partial D_1, \\
\frac{\partial \bar{u}}{\partial \nu}(x) = 0 & x \in \partial D_2, \\
\frac{\partial \bar{u}}{\partial \nu}(x) = i\lambda \bar{u}(x) & x \in \partial D_3,
\end{cases}
\]

(2.1)
it follows from the Green formula that

\[ 0 = \int_{\Omega \setminus \bar{D}} (\Delta u + k^2 u) \bar{u} \, dx = \int_{\partial D} \bar{u} \frac{\partial}{\partial v} \, ds - \int_{\Omega \setminus \bar{D}} (\nabla u \cdot \nabla \bar{u} - k^2 u \bar{u}) \, dx, \]

\[ 0 = \int_{\Omega \setminus \bar{D}} (\Delta \bar{u} + k^2 \bar{u}) u \, dx = \int_{\partial D} u \frac{\partial \bar{u}}{\partial v} \, ds - \int_{\Omega \setminus \bar{D}} (\nabla \bar{u} \cdot \nabla u - k^2 \bar{u} u) \, dx. \]

Subtracting these two equalities and noticing the boundary conditions of \( u \) on \( \partial D_1, \partial D_2 \), we can obtain that

\[ \int_{\partial D_3} (\bar{u} \frac{\partial u}{\partial v} - u \frac{\partial \bar{u}}{\partial v}) \, ds = 0 \]

from which and the boundary condition on \( \partial D_3 \) we are led to

\[ \int_{\partial D_3} \lambda |u(x)|^2 \, ds = 0. \]

Therefore \( u = \frac{\bar{u}}{\bar{v}} = 0 \) on \( \partial D_3 \). Due to the uniqueness of the Cauchy problem for the Helmholtz equation outside \( D_3 \), it is obvious that \( u = 0 \) in \( \Omega \setminus \bar{D} \). Clearly, if there exists a solution to (1.1), then it is determined by \( (f, g_1, g_2, g_3) \) uniquely, which says \( \partial_n u|_{\partial \Omega}, \partial_n u|_{\partial D_1}, u|_{\partial D_2}, u|_{\partial D_3} \) are also unique.

Secondly, we prove the existence by boundary integral equation method.

For \( x \in \Omega \setminus \bar{D} \), using the Green formula ([2], Theorem 2.1) in \( \Omega \setminus \bar{D} \), we get

\[ u(x) = \int_{\partial (\Omega \setminus \bar{D})} \left\{ \frac{\partial u}{\partial v}(y) \Phi(x, y) - u(y) \frac{\partial \Phi(x, y)}{\partial v(y)} \right\} \, ds(y) \]

\[ = \int_{\partial \Omega} \frac{\partial u}{\partial v}(y) \Phi(x, y) \, ds(y) - \int_{\partial \Omega} f(y) \frac{\partial \Phi(x, y)}{\partial v(y)} \, ds(y) \]

\[ - \int_{\partial D_1} \frac{\partial u}{\partial v}(y) \Phi(x, y) \, ds(y) + \int_{\partial D_1} g_1(y) \frac{\partial \Phi(x, y)}{\partial v(y)} \, ds(y) \]

\[ - \int_{\partial D_2} g_2(y) \Phi(x, y) \, ds(y) + \int_{\partial D_2} \frac{\partial \Phi(x, y)}{\partial v(y)} u(y) \, ds(y) \]

\[ - \int_{\partial D_3} g_3(y) \Phi(x, y) \, ds(y) + \int_{\partial D_3} \left( \frac{\partial \Phi(x, y)}{\partial v(y)} + i \lambda \Phi(x, y) \right) u(y) \, ds(y). \quad (2.2) \]

Introduce a parallel surface of \( \partial \Omega \) in \( \Omega \setminus \bar{D} \) as

\[ \partial \Omega_h := \{ x = z + hv(z), z \in \partial \Omega, h < 0 \} \subset \Omega \setminus \bar{D}. \]

Then we can consider the normal derivative of \( u(x) \) on \( \partial \Omega_h \) along the direction of \( v(x) = v(z) \).
Let \( x \in \partial \Omega_h \) and \( h \to 0 - \). It follows from the jump relation of single- and double-layer potential theory ([2], Theorem 3.1) that

\[
\frac{\partial u}{\partial \nu}(z) = \int_{\partial \Omega} \frac{\partial u}{\partial \nu}(y) \frac{\Phi(z,y)}{\partial \nu(z)} ds(y) + \frac{1}{2} \frac{\partial u}{\partial \nu}(z) - \frac{\partial}{\partial \nu(z)} \int_{\partial \Omega} f(y) \frac{\partial \Phi(z,y)}{\partial \nu(y)} ds(y)
- \frac{\partial}{\partial \nu(z)} \int_{\partial D_1} \frac{\partial u}{\partial \nu}(y) \Phi(z,y) ds(y) + \frac{\partial}{\partial \nu(z)} \int_{\partial D_1} g_1(y) \frac{\partial \Phi(z,y)}{\partial \nu(y)} ds(y)
- \frac{\partial}{\partial \nu(z)} \int_{\partial D_2} g_2(y) \Phi(z,y) ds(y) + \frac{\partial}{\partial \nu(z)} \int_{\partial D_2} g_3(y) \frac{\partial \Phi(z,y)}{\partial \nu(y)} u(y) ds(y)
+ \frac{\partial}{\partial \nu(z)} \int_{\partial D_3} (\frac{\partial \Phi(z,y)}{\partial \nu(y)} + i \lambda \Phi(z,y)) u(y) ds(y),
\]

which may be rewritten as

\[
(I_{\partial \Omega} - K_{\partial \Omega}) \frac{\partial u}{\partial \nu} + K'_{\partial D_1} \frac{\partial u}{\partial \nu} - T_{\partial D_2} u - (T_{\partial D_3} + i \lambda K_{\partial D_3}) u = -T_{\partial \Omega} f + T_{\partial D_1} g_1 - K'_{\partial D_2} g_2 - K'_{\partial D_3} g_3. \tag{2.4}
\]

Similarly, we can define the parallel surface of \( \partial D_j(j = 1, 2, 3) \) in \( \Omega \setminus \bar{D} \) by

\[
\partial (D_j)_h := \{ x = z + hv(z), z \in \partial D_i, h > 0 \} \subset \Omega \setminus \bar{D}.
\]

For \( z \in \partial D_1 \), by the same procedure as that for \( \partial \Omega_h \), we can get

\[
-I_{\partial \Omega} f + I_{\partial D_1} g_1 - K'_{\partial D_2} g_2 - K'_{\partial D_3} g_3.
\]

For \( x \in \partial (D_j)_h(i = 2, 3) \), we deduce that

\[
-S_{\partial \Omega} \frac{\partial u}{\partial \nu} + S_{\partial D_1} \frac{\partial u}{\partial \nu} + (I_{\partial D_2} - K_{\partial D_2}) u - (K_{\partial D_3} + i \lambda S_{\partial D_3} u = -K_{\partial \Omega} f + K_{\partial D_1} g_1 - S_{\partial D_2} g_2 - S_{\partial D_3} g_3, \quad z \in \partial D_2, \tag{2.6}
\]

\[
-S_{\partial \Omega} \frac{\partial u}{\partial \nu} + S_{\partial D_1} \frac{\partial u}{\partial \nu} - K_{\partial D_2} u + (I_{\partial D_3} - K_{\partial D_3} - i \lambda S_{\partial D_3}) u = -K_{\partial \Omega} f + K_{\partial D_1} g_1 - S_{\partial D_2} g_2 - S_{\partial D_3} g_3, \quad z \in \partial D_3. \tag{2.7}
\]

Obviously, (2.4)-(2.7) constitute an integral equations with respect to the unknowns.
\[ \partial_v u|_{\partial \Omega}, \partial_v u|_{\partial D_1}, u|_{\partial D_2} \text{ and } u|_{\partial D_3}. \text{ Its matrix form is} \]
\[
I - \begin{pmatrix}
K_{2\Omega}' & -K_{2D_1}' & T_{2D_2} & T_{2D_3} + i\lambda K_{2D_3}' \\
K_{2\Omega}' & -K_{2D_1}' & T_{2D_2} & T_{2D_3} + i\lambda K_{2D_3}' \\
S_{2\Omega} - S_{2D_1} & K_{2D_1} & K_{2D_3} + i\lambda S_{2D_3} \\
S_{2\Omega} - S_{2D_1} & K_{2D_1} & K_{2D_3} + i\lambda S_{2D_3}
\end{pmatrix}
\begin{pmatrix}
\partial_v u|_{\partial \Omega} \\
\partial_v u|_{\partial D_1} \\
u|_{\partial D_2} \\
u|_{\partial D_3}
\end{pmatrix}
\]
\[
= \begin{pmatrix}
-T_{2\Omega} f + T_{2D_1} g_1 - K_{2D_2}' g_2 - K_{2D_3}' g_3(x)|_{\partial \Omega} \\
-T_{2\Omega} f + T_{2D_1} g_1 - K_{2D_2}' g_2 - K_{2D_3}' g_3(x)|_{\partial D_1} \\
-K_{2\Omega} f + K_{2D_1} g_1 - S_{2D_2} g_2 - S_{2D_3} g_3(x)|_{\partial D_2} \\
-K_{2\Omega} f + K_{2D_1} g_1 - S_{2D_2} g_2 - S_{2D_3} g_3(x)|_{\partial D_3}
\end{pmatrix},
\]
(2.8)

where \( I \) is a \( 4 \times 4 \) unit matrix.

From Lemma 2.1, \((K_{2\Omega}', K_{2D_1}', K_{2D_2}, K_{2D_3} + i\lambda S_{2D_3})\) is compact from \((C(\partial \Omega), C(\partial D_1), C(\partial D_2), C(\partial D_3))\) into \((C(\partial \Omega), C(\partial D_1), C(\partial D_2), C(\partial D_3))\), while other operators in the left-hand side of (2.8) are regular. Therefore this operator equation \((I - \mathcal{A})\psi = \Phi\) is the Fredholm type of the second kind with a compact operator \(\mathcal{A}\) in \((C(\partial \Omega) \times C(\partial D_1) \times C(\partial D_2) \times C(\partial D_3)).\) Due to the Riesz theorem, the existence of solution to (2.8) can be obtained from the uniqueness of \((\partial_v u|_{\partial \Omega}, \partial_v u|_{\partial D_1}, u|_{\partial D_2}, u|_{\partial D_3})\), which has been established. Finally the existence of solution to (1.1) comes from the integral expression (2.2).

**Remark 2.1.** The numerical importance of Theorem 2.1 is that it provides an alternative method for the direct determination of \(\partial_v u|_{\partial \Omega}\). That is, we can determine the boundary data from the couple integral equations (2.8) directly, without the necessity of solving (1.1) in the whole domain.

### 3. Discrete of the integral system (2.8)

In this section, we will solve the linear integral equations (2.8) numerically. The main task is to treat the singular integrals carefully. More precisely, it can be checked that

- \((K_{2\Omega}' u)(x)|_{\partial \Omega}, (K_{2D_1}' \partial_v u)(x)|_{\partial D_1}, K_{2D_1} u(x)|_{\partial D_2}(K_{2D_3} + i\lambda S_{2D_3}) u(x)|_{\partial D_3},\)
- \((S_{2D_2} g_2(x)|_{\partial D_2} \text{ and } (S_{2D_3} g_3(x)|_{\partial D_3} \text{ are weakly singular;} \)
- \((T_{2\Omega} f(x)|_{\partial \Omega} \text{ and } (T_{2D_1} g_1(x)|_{\partial D_1} \text{ are hyper-singular.} \)

- The other integrals in (2.8) are regular.

In the sequel, we consider the case \(D \subset \mathbb{R}^2\), since the parameterization of integration and \(\Phi(x, y)\) depends on the dimension of the space. The solution to (1.1) in the three-dimensional case can be treated analogously.

Assume that the closed curve \(\Gamma \subset \mathbb{R}^2\) has the parametric representation
\[
\Gamma := \{ y := \gamma(\tau) = (\gamma_1(\tau), \gamma_2(\tau)) \in \mathbb{R}^2 : \tau \in [0, 2\pi]\}
\]
(3.1)

with 2\(\pi\)-periodic function \(\gamma_i(\tau)\) for \(i = 1, 2\), we define \(\psi(y) = \psi(\gamma(\tau)) := \tilde{\psi}(\tau)\) for \(y \in \Gamma\) and
\[
|\gamma'(\tau)| = \sqrt{|\gamma'_1(\tau)|^2 + |\gamma'_2(\tau)|^2} > 0.
\]
Numerical Solution for the Helmholtz Equation with Mixed Boundary Condition

Firstly, we consider the singular integrals in (2.8). For $K'_\Gamma, K_\Gamma$ and $S_\Gamma$, it is not difficult to get that

\[
(K'_\Gamma \cdot \psi)(x) = \frac{1}{|\gamma'(t)|} \int_0^{2\pi} H(t, \tau) \psi'(\tau)|\gamma'(\tau)|d\tau, \tag{3.2}
\]

\[
(S_\Gamma \cdot \psi)(x) = \int_0^{2\pi} M(t, \tau) \psi'(\tau)|\gamma'(\tau)|d\tau, \tag{3.3}
\]

\[
(K_\Gamma \cdot \psi)(x) = \int_0^{2\pi} H(\tau, t) \psi'(\tau)d\tau, \tag{3.4}
\]

with the kernel functions

\[
H(t, \tau) := \frac{ik}{2} n(t) \cdot [\gamma(\tau) - \gamma(t)] \frac{H_1^{(1)}(k|\gamma(\tau) - \gamma(t)|)}{|\gamma(\tau) - \gamma(t)|},
\]

\[
M(t, \tau) := \frac{i}{2} H_0^{(1)}(k|\gamma(\tau) - \gamma(t)|),
\]

where $n(t) = (\gamma'_2(t), -\gamma'_1(t)) = |\gamma'(t)|\nu(\gamma(t))$. For operator $T_\Gamma$, it can be shown that [7]

\[
(T_\Gamma \cdot \psi)(x) = \frac{1}{|\gamma'(t)|} \int_0^{2\pi} \left( \frac{1}{2\pi} \text{cot} \frac{\tau - t}{\pi} \frac{d\psi}{d\tau} - N(t, \tau) \psi'(\tau) \right) d\tau + \frac{k^2}{|\gamma'(t)|} \int_0^{2\pi} M(t, \tau) n(t) \cdot n(\tau) \psi'(\tau)d\tau, \tag{3.5}
\]

where the function $N(t, \tau)$ has the expression

\[
N(t, \tau) = \frac{\partial^2}{\partial \tau \partial \tau} \left[ \frac{i}{2} H_0^{(1)}(k|\gamma(\tau) - \gamma(t)|) + \frac{1}{2\pi} \ln \left( 4\sin^2 \frac{t - \tau}{2} \right) \right]. \tag{3.6}
\]

Decomposing the singularity of functions $H(t, \tau), M(t, \tau)$ and $N(t, \tau)$ at $t = \tau$, we have

\[
A(t, \tau) = A_1(t, \tau) \ln \left( 4\sin^2 \frac{t - \tau}{2} \right) + A_2(t, \tau)
\]

with the vector

\[
A(t, \tau) = (H(t, \tau), M(t, \tau), N(t, \tau)),
\]

\[
A_j(t, \tau) = (H_j(t, \tau), M_j(t, \tau), N_j(t, \tau)), \quad j = 1, 2,
\]

where the analytic functions $H_1, M_1, N_1$ have the expressions

\[
H_1(t, \tau) = \begin{cases} 
-\frac{k}{2\pi} n(t) \cdot [\gamma(\tau) - \gamma(t)] \frac{J_0(k|\gamma(\tau) - \gamma(t)|)}{|\gamma(\tau) - \gamma(t)|}, & t \neq \tau, \\
0, & t = \tau,
\end{cases} \tag{3.7}
\]

\[
M_1(t, \tau) = \begin{cases} 
-\frac{1}{2\pi} J_0(k|\gamma(\tau) - \gamma(t)|), & t \neq \tau, \\
-\frac{k}{2\pi}, & t = \tau,
\end{cases} \tag{3.8}
\]

\[
N_1(t, \tau) = \begin{cases} 
-\frac{1}{2\pi} \frac{\partial^2}{\partial \tau \partial \tau} J_0(k|\gamma(\tau) - \gamma(t)|), & t \neq \tau, \\
-\frac{k}{4\pi} |\gamma'(t)|^2, & t = \tau.
\end{cases} \tag{3.9}
\]
For $t \neq \tau$, $H_2, M_2$ and $N_2$ can be represented by

$$A_2(t, \tau) = A(t, \tau) - A_1(t, \tau) \ln \left( \frac{4 \sin^2 \frac{t - \tau}{2}}{2} \right).$$

For $t = \tau$, we have

$$\begin{cases}
H_2(t, t) = \frac{1}{2\pi} \left( \frac{\gamma''(t)\nu(t)}{\nu(t)} \right)^2, \\
M_2(t, t) = \frac{1}{2} - \frac{C}{\pi} - \frac{1}{\pi} \frac{\ln \nu(t)}{\nu(t)}, \\
N_2(t, t) = \left( C_1 - 2\ln \frac{|\gamma''(t)|^2}{2\pi} \right) + \frac{1}{12\pi} + \frac{1}{2\pi\nu(t)^4} - \frac{3|\nu''(t)|^2 + 2\nu'(t)\nu''(t)}{12\pi
\nu(t)^4},
\end{cases}$$

with constant $C_1 = \pi i - 1 - 2C$, $C$ is the Euler constant. For $\partial \Omega$ and $\partial D_j$ ($j = 1, 2, 3$) with regular analytic and $2\pi$-periodic parametric representation

$$\partial \Omega = \{ x := \gamma_\Omega(t) | t \in [0, 2\pi] \}, \partial D_i = \{ x := \gamma_{D_i}(t) | t \in [0, 2\pi] \},$$

we define

$$\tilde{\omega}_1(t) = \omega_1(\gamma_\Omega(t)) := \partial_\nu u(\gamma_\Omega(t)), \quad \tilde{\omega}_2(t) = \omega_2(\gamma_{D_1}(t)) := \partial_\nu u(\gamma_{D_1}(t)), \quad \tilde{\omega}_3(t) = \omega_3(\gamma_{D_2}(t)) := u(\gamma_{D_2}(t)), \quad \tilde{\omega}_4(t) = \omega_4(\gamma_{D_3}(t)) := u(\gamma_{D_3}(t)). \tag{3.10}$$

Based on the expressions (3.2)-(3.4), we can transform the weakly singular integrals in (2.8) into the parametric forms

$$\begin{align*}
(K'_{\partial \Omega} \omega_1)(\gamma_\Omega(t)) &= \frac{1}{|\gamma_\Omega(t)|} \int_0^{2\pi} H_\Omega(t, \tau) \tilde{\omega}_1(\tau)|\gamma_\Omega'(\tau)| d\tau, \tag{3.11} \\
(K'_{\partial D_1} \omega_2)(\gamma_{D_1}(t)) &= \frac{1}{|\gamma_{D_1}(t)|} \int_0^{2\pi} H_{D_1}(t, \tau) \tilde{\omega}_2(\tau)|\gamma_{D_1}'(\tau)| d\tau, \tag{3.12} \\
K_{\partial D_2} \omega_3(\gamma_{D_2}(t)) &= \int_0^{2\pi} H_{D_2}(\tau, t) \tilde{\omega}_3(\tau) d\tau, \tag{3.13} \\
(K_{\partial D_3} + i\lambda S_{\partial D_3}) \omega_4(\gamma_{D_3}(t)) &= \int_0^{2\pi} (H_{D_3}(\tau, t) + i\lambda M_{D_3}(t, \tau)|\gamma_{D_3}'(\tau)|)|\tilde{\omega}_4(\tau)| d\tau, \tag{3.14} \\
(S_{\partial D_2} g_2)(\gamma_{D_2}(t)) &= \int_0^{2\pi} M_{D_2}(t, \tau) \tilde{g}_2(\tau)|\gamma_{D_2}'(\tau)| d\tau, \tag{3.15} \\
(S_{\partial D_3} g_3)(\gamma_{D_3}(t)) &= \int_0^{2\pi} M_{D_3}(t, \tau) \tilde{g}_3(\tau)|\gamma_{D_3}'(\tau)| d\tau, \tag{3.16}
\end{align*}$$

where $H_\Omega(t, \tau)$ means we take $(\gamma(\tau), \gamma(t)) = (\gamma_\Omega(\tau), \gamma_\Omega(t))$ in the expression of $H(t, \tau)$, while $H_{D_i}$ and $M_{D_i}$ mean we take $(\gamma(\tau), \gamma(t)) = (\gamma_{D_i}(\tau), \gamma_{D_i}(t))$ ($i = 1, 2, 3$) in $H(\tau, t)$, $M(t, \tau)$. 
In terms of both (3.5) and (3.6), the hyper-singular integrals $(T_{\partial \Omega} f)(x)|_{\partial \Omega}$ and $(T_{\partial D_1} g_1)(x)|_{\partial D_1}$ become that

$$
(T_{\partial \Omega} f)(\gamma_\Omega(t)) = \frac{1}{|\gamma_\Omega'(t)|} \int_0^{2\pi} \frac{1}{2\pi} \frac{\cot \frac{\tau - t}{2} d\tilde{f}(\tau)}{d\tau} d\tau \\
- \frac{1}{|\gamma_\Omega'(t)|} \int_0^{2\pi} \left[ N_\Omega(t, \tau) - k^2 M_\Omega(t, \tau) n_\Omega(t) \cdot n_\Omega(\tau) \right] \tilde{f}(\tau) d\tau, \quad (3.17)
$$

$$
(T_{\partial D_1} g_1)(\gamma_{D_1}(t)) = \frac{1}{|\gamma_{D_1}'(t)|} \int_0^{2\pi} \frac{1}{2\pi} \frac{\cot \frac{\tau - t}{2} d\tilde{g}_1(\tau)}{d\tau} d\tau \\
- \frac{1}{|\gamma_{D_1}'(t)|} \int_0^{2\pi} N_{D_1}(t, \tau) \tilde{g}_1(\tau) d\tau \\
- \frac{1}{|\gamma_{D_1}'(t)|} \int_0^{2\pi} k^2 M_{D_1}(t, \tau) n_{D_1}(t) \cdot n_{D_1}(\tau) \tilde{g}_1(\tau) d\tau. \quad (3.18)
$$

From the decompositions (3.6)-(3.9) , we can obtain the explicit expressions of (3.11)-(3.14) and (3.17)-(3.18), in which only the integrals with kernels $\ln \left( \frac{4 \sin^2 \frac{t - \tau}{2}}{2} \right)$ or $\cot \frac{t - \tau}{2}$ are improper, the others are regular.

To solve (2.8) numerically, we should approximate the integrals by quadrature formula.

For regular integrals, noting the integrands are periodic smooth functions, the simple rectangular rule is superior to any other quadrature formula [6]. However, for improper integrals, we should approximate them by

$$
\left\{ \begin{array}{l}
\frac{1}{2\pi} \int_0^{2\pi} \cot \frac{t - \tau}{2} \psi'(\tau) d\tau \approx \sum_{j=0}^{2n-1} T_j^{(n)}(t) \psi(t_j), \\
\frac{1}{2\pi} \int_0^{2\pi} \ln \left( 4 \sin^2 \frac{t - \tau}{2} \right) \psi(\tau) d\tau \approx \sum_{j=0}^{2n-1} R_j^{(n)}(t) \psi(t_j), 
\end{array} \right. \quad (3.19)
$$

with the weights

$$
\left\{ \begin{array}{l}
T_j^{(n)}(t) = -\frac{1}{n} \left( \sum_{m=1}^{n-1} m \cos m(t - t_j) + \frac{1}{2} n \cos n(t - t_j) \right), \\
R_j^{(n)}(t) = -\frac{1}{n} \left( \sum_{m=1}^{n-1} \frac{1}{m} \cos m(t - t_j) + \frac{1}{2n} \cos n(t - t_j) \right). 
\end{array} \right. \quad (3.20)
$$

The regular integrals in (2.8) can be computed by the standard method, we omit the lengthy expressions.

Based on the above treatments, by straightforward calculations, the integrals equations (2.8) may be discretized which leads to a linear equations with respect to

$$
\partial_t u(x)|_{\partial \Omega} = \omega_1(\gamma_\Omega(t)) = \tilde{\omega}_1(t), \quad \partial_t u(x)|_{\partial D_1} = \omega_2(\gamma_{D_1}(t)) = \tilde{\omega}_2(t), \\
u(x)|_{\partial D_2} = \omega_3(\gamma_{D_2}(t)) = \tilde{\omega}_3(t), \quad u(x)|_{\partial D_3} = \omega_4(\gamma_{D_3}(t)) = \tilde{\omega}_4(t),
$$
Example 4.2. \( \Omega \) should be taken as non-convex so that the probe needle outside \( D \) can approach to \( \partial D_j \) \((j = 1, 2, 3) \) as follows:

\[
\begin{align*}
\partial \Omega &= \{ x := \gamma_{\Omega} = 3(\cos t + 0.65 \cos 2t - 0.65, 1.5 \sin t) : t \in [0, 2\pi] \}, \\
\partial D_1 &= \{ x := \gamma_{D_1} = (-2 + 0.5 \cos t, 2 + 0.5 \sin t) : t \in [0, 2\pi] \}, \\
\partial D_2 &= \{ x := \gamma_{D_2} = (\cos t, 0.5 \sin t) : t \in [0, 2\pi] \}, \\
\partial D_3 &= \{ x := \gamma_{D_3} = (-2 + 0.5 \cos t, -2 + 0.5 \sin t) : t \in [0, 2\pi] \},
\end{align*}
\]

4. Numerical performance of our method

To simply test the validity of our method proposed in this paper, we will only compare the numerical solution of \( \partial u |_{\partial \Omega} \) with the exact explicit solution to some model problems of (1.1) constructed specially.

Example 4.1. In this model, we take

\[
\begin{align*}
\partial \Omega &= \{ x := \gamma_{\Omega} = (2 \cos t, 2 \sin t) : t \in [0, 2\pi] \}, \\
\partial D_1 &= \{ x := \gamma_{D_1} = (1 + 0.5 \cos t, 0.5 \sin t) : t \in [0, 2\pi] \}, \\
\partial D_2 &= \{ x := \gamma_{D_2} = (0.5 \cos t, 1 + 0.5 \sin t) : t \in [0, 2\pi] \}, \\
\partial D_3 &= \{ x := \gamma_{D_2} = (0.5 \cos t, -1 + 0.5 \sin t) : t \in [0, 2\pi] \}.
\end{align*}
\]

For \( x_0 = (3, 0) \notin \Omega \), \( u(x) = H_0^{(1)}(k|x - x_0|) \) solves the Helmholtz equation in \( \Omega \). So the Cauchy data on the boundary of \( \Omega \) in (1.1) can be given as:

\[
\begin{align*}
f(x) &= H_0^{(1)}(k|x - x_0|), \quad x \in \partial \Omega, \\
g_1(x) &= H_0^{(1)}(k|x - x_0|), \quad x \in \partial D_1, \\
g_2(x) &= -H_0^{(1)}(k|\gamma_{D_2}(t) - x_0|)k\left(\frac{(\gamma_{D_2}(t) - x_0) \cdot n_{D_2}(t)}{|\gamma_{D_2}(t) - x_0| \cdot |\gamma_{D_2}'(t)|}\right), \quad x \in \partial D_2, \\
g_3(x) &= -H_0^{(1)}(k|\gamma_{D_3}(t) - x_0|)k\left(\frac{(\gamma_{D_3}(t) - x_0) \cdot n_{D_3}(t)}{|\gamma_{D_3}(t) - x_0| \cdot |\gamma_{D_3}'(t)|}\right) + i\lambda H_0^{(1)}(k|\gamma_{D_3}(t) - x_0|), \quad x \in \partial D_3,
\end{align*}
\]

with \( n_{D_j}(t) = (\gamma_{D_2}'(t), -\gamma_{D_1}'(t)) \) \((j = 2, 3) \). In this case, we know the exact Neumann data

\[
\begin{align*}
\partial_n u(x) &= \frac{\partial}{\partial v(x)} H_0^{(1)}(k|x - x_0|) \\
&= -H_0^{(1)}(k|\gamma_{\Omega}(t) - x_0|)k\left(\frac{(\gamma_{\Omega}(t) - x_0) \cdot n_{\Omega}(t)}{|\gamma_{\Omega}(t) - x_0| \cdot |\gamma_{\Omega}'(t)|}\right), \quad x \in \partial \Omega.
\end{align*}
\]

Let \( k = 2.0, \lambda = 1.5 \). Obviously, the numerical results for \( n = 16 \) are satisfactory, see Figs. 4.1 and 4.2.

In the numerical realization of probe method for multiple obstacles, the domain \( \Omega \) should be taken as non-convex so that the probe needle outside \( \Omega \) can approach to \( \partial D_j \) \((j = 1, 2, 3) \). Therefore, we have to test the efficiency of our method for non-convex domain \( \Omega \).

Example 4.2. Take \( \Omega, D_j \) \((j = 1, 2, 3) \) as follows:

\[
\begin{align*}
\partial \Omega &= \{ x := \gamma_{\Omega} = 3(\cos t + 0.65 \cos 2t - 0.65, 1.5 \sin t) : t \in [0, 2\pi] \}, \\
\partial D_1 &= \{ x := \gamma_{D_1} = (-2 + 0.5 \cos t, 2 + 0.5 \sin t) : t \in [0, 2\pi] \}, \\
\partial D_2 &= \{ x := \gamma_{D_2} = (\cos t, 0.5 \sin t) : t \in [0, 2\pi] \}, \\
\partial D_3 &= \{ x := \gamma_{D_3} = (-2 + 0.5 \cos t, -2 + 0.5 \sin t) : t \in [0, 2\pi] \},
\end{align*}
\]

at nodal \( t_i \) with \( i = 0, 1, \cdots 2n - 1 \). Based on these boundary data of solution, \( u(x) \) for each \( x \in \Omega \setminus \bar{D} \) can be computed by the expression (2.2).
Let $x_0 = 4(\cos \frac{\pi}{4}, \sin \frac{\pi}{4}) \not\in \Omega, k = 2.0, \lambda = 1.5$. Similarly, we give the boundary data on $\partial \Omega$ and $\partial D_j$ ($j = 1, 2, 3$) by (4.1). Then the exact Neumann data $\partial_n u |_{\partial \Omega}$ can also be determined by (4.2).
Table 4.1: Real part of $\partial_\nu u|_{\partial\Omega}$ for different $n$ in Example 4.2.

<table>
<thead>
<tr>
<th>$t$</th>
<th>Exact</th>
<th>$n=32$</th>
<th>$n=64$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>3.9697509E-002</td>
<td>3.9709472E-002</td>
<td>3.9698366E-002</td>
</tr>
<tr>
<td>$\pi/2$</td>
<td>-7.3205722E-002</td>
<td>-7.3208948E-002</td>
<td>-7.3206033E-002</td>
</tr>
<tr>
<td>$\pi$</td>
<td>0.142882387</td>
<td>0.142887882</td>
<td>0.142882443</td>
</tr>
<tr>
<td>$3\pi/2$</td>
<td>-5.3274408E-002</td>
<td>-5.3282921E-002</td>
<td>-5.3274450E-002</td>
</tr>
</tbody>
</table>

Table 4.2: Imaginary part of $\partial_\nu u|_{\partial\Omega}$ for different $n$ in Example 4.2.

<table>
<thead>
<tr>
<th>$t$</th>
<th>Exact</th>
<th>$n=32$</th>
<th>$n=64$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>9.4956045E-003</td>
<td>9.3670063E-003</td>
<td>9.4984262E-003</td>
</tr>
<tr>
<td>$\pi/2$</td>
<td>0.253276951</td>
<td>0.253281226</td>
<td>0.253277117</td>
</tr>
<tr>
<td>$\pi$</td>
<td>0.372841380</td>
<td>0.372855458</td>
<td>0.372841499</td>
</tr>
<tr>
<td>$3\pi/2$</td>
<td>8.8648127E-002</td>
<td>8.8686804E-002</td>
<td>8.8647969E-002</td>
</tr>
</tbody>
</table>

In the case of $n = 16$ (Figs. 4.3 and 4.4), the accuracy of numerical solution is unsatisfactory at some points, which show that small division number $n$ can not reveal the oscillation property of true solution.

If we take $n = 32$, then the whole numerical performances are shown in Figs. 4.5 and 4.6, from which we see that the difference between exact value and numerical solution is hardly distinguished. The numerical results at four special points for different $n$ are also given in Tables 4.1 and 4.2.

We can conclude from these numerical facts that the proposed method in this paper is efficient and stable for determining the boundary Neumann data of the solution.

Acknowledgment This work is supported by NSFC (No. 10371018).

References