The Generalized Arrow-Hurwicz Method with Applications to Fluid Computation

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Received 26 November 2017; Accepted (in revised version) 15 January 2018

Abstract. In this paper, we first discuss the existence and uniqueness of a class of nonlinear saddle-point problems, which are frequently encountered in physical models. Then, a generalized Arrow-Hurwicz method is introduced to solve such problems. For the method, the convergence rate analysis is established under some reasonable conditions. It is also applied to solve three typical discrete methods in fluid computation, with the computational efficiency demonstrated by a series of numerical experiments.

AMS subject classifications: 65N30, 65N22, 75D05

Key words: Nonlinear saddle-point problems, the generalized Arrow-Hurwicz method, convergence rate analysis, fluid computation.

1 Introduction

This paper is intended to study existence and uniqueness of the solution to the following abstract problem and then design and analyze a generalized Arrow-Hurwicz method iteratively solving it. Afterwards, we will apply the algorithm to solve several discrete methods in fluid mechanics.

Problem P. Find \((u, p) \in V \times Q\) such that

\[
\begin{align*}
    a_0(u, v) + N(u; u, v) - b(v, p) &= \langle f, v \rangle & \forall v \in V, \quad (1.1) \\
    b(u, q) &= 0 & \forall q \in Q. \quad (1.2)
\end{align*}
\]

Here, \(V\) and \(Q\) are two finite or infinite dimensional Hilbert spaces, \(f \in V'\) and \(\langle \cdot, \cdot \rangle\) denotes the bilinear form between the dual pair \(V'\) and \(V\). In addition, denote by \(a_0(\cdot, \cdot)\) (resp. \(N(\cdot; \cdot, \cdot)\)) a bounded and coercive bilinear (resp. a bounded trilinear) form over \(V\), i.e., there exist two positive numbers \(a_0\) and \(a_1 (a_0 \leq a_1)\) such that

\[
a_0 \|v\|_V^2 \leq a_0(v, v) \quad \forall v \in V, \quad a_0(u, v) \leq a_1 \|u\|_V \|v\|_V \quad \forall u, v \in V, \quad (1.3)
\]

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and there exists a positive number \( N \) such that
\[
|N(u;v,w)| \leq N \|u\|_V \|v\|_V \|w\|_V \quad \forall u,v,w \in V. \tag{1.4}
\]

We also assume \( N(\cdot;\cdot) \) is anti-symmetric with respect to the second and third components, that is, \( N(\cdot;v,w) = -N(\cdot;w,v) \) for any \( v \) and \( w \) in \( V \). Meanwhile, let \( b(\cdot,\cdot) \) be a bounded bilinear form over \( V \times Q \), i.e., there exists a positive constant \( \alpha_2 \) such that
\[
b(v,q) \leq \alpha_2 \|v\|_V \|q\|_Q \quad \forall v \in V, \ q \in Q. \tag{1.5}
\]

Throughout this paper, we use \( (\cdot,\cdot)_V \) and \( \|\cdot\|_V \) (resp. \( (\cdot,\cdot)_Q \) and \( \|\cdot\|_Q \)) to stand for the inner product and the induced norm over \( V \) (resp. \( Q \)).

To the best of our knowledge, many important mathematical-physical models and their numerical methods can be described in the setting of problem \( P \)—an abstract framework. The typical examples include steady incompressible Navier-Stokes equations (cf. [15, 19]), steady incompressible magnetohydrodynamics (MHD) equations (cf. [12]), and many numerical methods for solving the previous problems (cf. [15, 19]).

As a matter of fact, consider the following steady incompressible MHD model:
\[
\begin{cases}
-R_e^{-1} \Delta u + (u \cdot \nabla) u - S_c B \times \text{curl} B + \nabla p = f & \text{in } \Omega, \\
\text{div} u = 0 & \text{in } \Omega, \\
S_m R_m^{-1} \text{curl}(\text{curl} B) - S_c \text{curl}(u \times B) = g & \text{in } \Omega, \\
\text{div} B = 0 & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega, \\
B \cdot n = 0, \ n \times \text{curl} B = 0 & \text{on } \partial \Omega,
\end{cases} \tag{1.6}
\]

where \( \Omega \subset \mathbb{R}^d \) (\( d = 2,3 \)), \( n \) is the unit outward normal to the boundary \( \partial \Omega \). Here \( u \) denotes the velocity field, \( B \) the magnetic field, \( f \) and \( g \) the external force terms, and \( p \) the pressure field. There are three physical parameters \( R_e, R_m \) and \( S_c \) in the equations, called the hydrodynamic Reynolds number, the magnetic Reynolds number and the coupling number, respectively.

Using some notations and symbols (see Section 4.1 for details), the variational formulation of MHD model can be described as follows.

**Problem** \( P_1 \). Find \((\tilde{u},p) \in W_{0n} \times M \) such that
\[
\begin{cases}
A_0(\tilde{u},\tilde{v}) + A_1(\tilde{u};\tilde{u},\tilde{v}) - d(\tilde{v},p) = (F,\tilde{v}) & \forall \tilde{v} \in W_{0n}, \\
d(\tilde{u},q) = 0 & \forall q \in M. \tag{1.7}
\end{cases}
\]

If we let \( V=W_{0n}, Q=M, \) and for \( u=\tilde{u} \) and \( v=\tilde{v} \), write \( a_0(u,v)=A_0(\tilde{u},\tilde{v}), b(v,q)=d(\tilde{v},q) \) and \( N(u;u,v)=A_1(\tilde{u};\tilde{u},\tilde{v}) \), then problem \( P_1 \) can be viewed as a specific case of problem \( P \). It is evident that the variational form of steady incompressible Navier-Stokes equation can be written in the form of problem \( P \) as well (cf. [9, 10, 18, 20]). Moreover, many
important discrete methods for solving the Navier-Stokes equations can be formulated in the form of problem $P$ (see Section 4 for details). Hence, the mathematical analysis and algorithm design for problem $P$ is of great importance in theory and applications. Even so, it deserves to point out that, there are no systematic studies about problem $P$ or similar abstract framework in the literature.

In this paper, we will first show problem $P$ has a unique solution under some reasonable conditions. Concretely speaking, following some ideas in [18], we first reformulate it as a primal problem of $u$, and then show the latter problem has a unique solution by means of the Banach contraction mapping theorem. Next, we will propose and analyze an Arrow-Hurwicz method for solving problem $P$. Historically, the Arrow-Hurwicz method was first mentioned in [2] for constrained optimization problems. Later on, this method was mentioned in [20] for solving steady incompressible Navier-Stokes equations. More recently, it was proved in [6] that when the method was used for solving incompressible Navier-Stokes equations discretized by mixed element methods, it converges with a contraction number independent of the finite element mesh size $h$, even for regular triangulations. We remark that the modified Uzawa-type iterative method devised in [5] can be viewed as a specific case of the Arrow-Hurwicz discussed in [6]. Here, motivated by some ideas in [6], we will design a generalized Arrow-Hurwicz method for solving problem $P$ and then establish the corresponding convergence rate analysis. As shown in [6] that one of the main advantages of the method is that it requires to solve no linear saddle-point systems at each iteration step (except for the initial step). To show the power of our abstract framework, we will apply it to study three specific nonlinear saddle-point problems, including mathematical model and its numerical method for the steady incompressible magnetohydrodynamic equations, the variational multi-scale (VMS) method and the defect-correction method for numerically solving the steady incompressible Navier-Stokes equations. For all the previous discrete methods, we will propose the related generalized Arrow-Hurwicz methods to solve these problems and develop their convergence rate analysis. Finally, a series of numerical results are reported to show computational performance of the generalized Arrow-Hurwicz method.

The rest of this paper is organized as follows. In Section 2, we study the existence and uniqueness of solutions to problem $P$. The generalized Arrow-Hurwicz method and its convergence rate analysis is presented in Section 3. In Section 4, three typical problems from fluid computation are discussed in detail based on the abstract results obtained in the last two sections. In Section 5, we present a series of numerical experiments to show the computational performance of our generalized Arrow-Hurwicz method. A brief conclusion is given in the last section.

2 Existence and uniqueness of the solution to problem $P$

In this section, we are concerned with existence and uniqueness of the solution to problem $P$. 
First of all, observing the bilinear form \( b(\cdot, \cdot) \) is bounded over \( V \times Q \), we know there is a bounded linear operator \( B \in L(V, Q) \) such that
\[
b(v, q) = (Bv, q)_Q = (v, B^Tq)_V .
\] (2.1)

Next, assume that the bilinear form \( b(\cdot, \cdot) \) satisfies the following inf-sup condition: there exists a positive constant \( \beta \) such that
\[
\inf_{q \in Q} \sup_{v \in V} \frac{b(v, q)}{\|v\|_V \|q\|_Q} \geq \beta .
\] (2.2)

Letting \( V_{\ker} = \ker(B) := \{v \in V; Bv = 0\} \), it is evident to see that if \((u, p) \in V \times Q\) is a solution of problem \( \bar{P} \), then \( u \) is a solution of the following problem.

**Problem \( \bar{P} \).** Find \( u \in V_{\ker} \) such that
\[
a_0(u, v) + N(u; u, v) = \langle f, v \rangle \quad \forall v \in V_{\ker} .
\] (2.3)

Moreover, the converse is also true in the sense described by the following result.

**Lemma 2.1.** Let \( u \) be a solution to problem (2.3). Then there exists a unique \( p \in Q \) such that \((u, p)\) is the solution of problem (1.1)-(1.2).

**Proof.** Fixed \( u \in V \), let
\[
F(v) := a_0(u, v) + N(u; u, v) - \langle f, v \rangle \quad \forall v \in V .
\] (2.4)

It is easy to check that \( F(v) \in V' \). Hence, according to the Riesz representation theorem, there exists an element \( \sigma(F) \in V \) such that
\[
F(v) = (v, \sigma(F))_V .
\] (2.5)

In addition, since the bilinear form \( b(\cdot, \cdot) \) satisfies the condition (2.2), or equivalently,
\[
\beta \|q\|_Q \leq \|B^Tq\|_V \quad \forall q \in Q ,
\] (2.6)

we easily know the image of \( B^T \), i.e. \( \text{Im}(B^T) \), is closed. Hence, it follows from the equation (4.1.55) in [3] that the following identity holds:
\[
\text{Im}(B^T) = (\ker(B))^\perp .
\] (2.7)

On the other hand, since \( u \in \ker(B) \) is the solution of Problem \( \bar{P} \), we have by (2.3)-(2.5) that \( \sigma(F) \perp \ker(B) \), and with the identity (2.7) we know \( \sigma(F) \in \text{Im}(B^T) \). Consequently, there is a certain \( p \in Q \) such that \( \sigma(F) = -B^Tp \), which combined with (2.5) gives \( F(v) = -(Bv, p)_Q = -b(v, p) \) for all \( v \in V \). In other words, \((u, p) \in V \times Q\) satisfies Eq. (1.1). Eq. (1.2) for \( u \) naturally holds since \( u \in \ker(B) \). The uniqueness of \( p \) is due to the estimate (2.6). The proof is complete. \( \square \)
Now, let us focus on existence and uniqueness of the solution to problem (2.3). Define
\[
\Lambda = a_0^{-2} N', \quad \|f\|_{V'} := \sup_{v \in V} \frac{\langle f, v \rangle}{\|v\|_V}.
\] (2.8)

**Theorem 2.1.** If the function \(f\) in problem (1.1)-(1.2) satisfies that \(\Lambda < 1\), with \(\Lambda\) defined by (2.8), then problem (2.3) has a unique solution \(u \in V_{\ker}\).

**Proof.** The proof is based on some ideas in [18]. For each \(w \in V_{\ker}\), define a bilinear form by
\[
a_w (z, v) = a_0 (z, v) + N (w; z, v) \quad \forall z, v \in V_{\ker},
\] (2.9)
It is evident that the bilinear form is bounded. Moreover, recalling that \(N (\cdot; v, v) = 0\) for all \(v \in V\) and \(a_0 (\cdot, \cdot)\) is coercive (cf. (1.3)), we easily know \(a_w (\cdot, \cdot)\) is coercive:
\[
a_0 \|v\|^2_V \leq a_0 (v, v) = a_w (v, v) \quad \forall v \in V.
\] (2.10)
Therefore, according to the Lax-Milgram lemma, for each \(w \in V_{\ker}\) there exists a unique solution \(z = z (w) \in V_{\ker}\) to the problem
\[
a_w (z, v) = \langle f, v \rangle \quad \forall v \in V_{\ker}.
\] (2.11)
Moreover, it follows readily from (2.10) and (2.11) that
\[
a_0 \|z\|^2_V \leq \langle f, v \rangle \leq \|f\|_{V'} \|z\|_V,
\]
i.e.,
\[
\|z\|_V \leq a_0^{-1} \|f\|_{V'}.
\] (2.12)
To sum up, for each \(w \in V_{\ker}\), we have defined a nonlinear mapping \(z = z (w)\) in terms of Eq. (2.11), which admits the above estimate.

Next, introduce a closed subset \(K \subset V_{\ker}\) by
\[
K := \left\{ v \in V_{\ker}; \|v\|_V \leq a_0^{-1} \|f\|_{V'} \right\}.
\] (2.13)
It is easy to see from the estimate (2.12) that the relation \(z = z (w)\) defined by (2.11) is a mapping from \(K\) into \(K\). Now, let us further show the mapping is contractive. For this, we first derive the following estimate:
\[
\|z_1 - z_2\|_V = \|z (w_1) - z (w_2)\|_V \leq \Lambda \|w_1 - w_2\|_V \quad \forall w_1, w_2 \in K,
\] (2.14)
where the constant \(\Lambda\) is given by (2.8). In fact, subtracting Eq. (2.11) related to \(w = w_1\) and \(w = w_2\) gives
\[
a_0 (z_1 - z_2, v) + N (w_1; z_1, v) - N (w_2; z_2, v) = 0 \quad \forall v \in V_{\ker},
\]
or equivalently,
\[
a_0 (z_1 - z_2, v) + N (w_1; z_1 - z_2, v) + N (w_1 - w_2; z_2, v) = 0 \quad \forall v \in V_{\ker}.
\]
Choosing \( v = z_1 - z_2 \) in the last equation and using the fact that \( N(v;v,v) = 0 \) for all \( v \in V \), we find
\[
a_0(z_1 - z_2, z_1 - z_2) + N(w_1 - w_2; z_1, z_1 - z_2) = 0,
\]
from which, the estimates (1.3) and (2.12) we are led to
\[
a_0(z_1 - z_2, z_1 - z_2) + N(w_1 - w_2; z_2, z_1 - z_2) = 0,
\]
i.e.,
\[
\|z_1 - z_2\|_V \leq N\|w_1 - w_2\|_V \|z_2\|_V \|z_1 - z_2\|_V,
\]
from (2.8). We then obtain the inequality (2.14). Hence, under the condition that \( \Lambda < 1 \), we have from the Banach contraction mapping theorem that the mapping \( z = z(w) \) has a unique fixed point \( w = w_* \) in \( K \), which is exactly a solution to problem (2.3). On the other hand, according to (2.3), any solution to problem (2.3) must belong to \( K \) and is a fixed point of the mapping \( z = z(w) \), so the uniqueness of solution is also obtained.

The combination of Lemma 2.1 and Theorem 2.1 immediately leads to the following result.

**Theorem 2.2.** If the function \( f \) in problem (1.1)-(1.2) satisfies that \( \Lambda < 1 \), with \( \Lambda \) defined by (2.8), then the problem \( P \) has a unique solution \( (u, p) \in V \times Q \).

**Algorithm 1** The generalized Arrow-Hurwicz method for problem \( P \)

Let \( \rho \) and \( \alpha \) be two positive parameters.

Step 1. Choose two functions \( u^0 \in V \) and \( p^0 \in Q \) satisfying the following discrete linear equations:
\[
\begin{align*}
& a_0(u^0, v) - b(v, p^0) = \langle f, v \rangle \quad \forall v \in V, \quad \text{(2.15)} \\
& b(u^0, q) = 0 \quad \forall q \in Q. \quad \text{(2.16)}
\end{align*}
\]

Step 2. For \( n = 0, 1, \cdots \), if \( u^n \in V \) and \( p^n \in Q \) are available, determine \( u^{n+1} \in V \) and \( p^{n+1} \in Q \) by solving the following two equations sequentially:
\[
\begin{align*}
& \rho^{-1} a_0(u^{n+1} - u^n, v) + a_0(u^n, v) + N(u^n; u^{n+1}, v) = b(v, p^n) + \langle f, v \rangle \quad \forall v \in V, \quad \text{(2.17)} \\
& \alpha (p^{n+1} - p^n, q) + \rho b(u^{n+1}, q) = 0 \quad \forall q \in Q. \quad \text{(2.18)}
\end{align*}
\]

3 The generalized Arrow-Hurwicz method and convergence rate analysis

In this section, we want to extend the Arrow-Hurwicz method in [20] to solve the abstract problem \( P \) (see Algorithm 1 in the next subsection), and then study convergence rate analysis of the method following some ideas in [6].
3.1 The generalized Arrow-Hurwicz method

Motivated by the Arrow-Hurwicz method mentioned in [20], we extend it to solve problem $P$, described as Algorithm 1.

3.2 Some basic estimates

In order to develop convergence rate analysis of Algorithm 1, we require to establish some basic estimates in advance. We emphasize that throughout this paper, all the assumptions given in the last section are always fulfilled. Moreover, we always assume that $\Lambda < 1$ so that problem $P$ has a unique solution.

**Lemma 3.1.** Let $(u, p) \in V \times Q$ be a solution of problem $P$. Then there holds

$$
\|u\|_V \leq \alpha_0^{-1} \|f\|_{V'}, \quad \beta \|p\|_Q \leq \left(2 + \frac{\alpha_1}{\alpha_0}\right) \|f\|_{V'}.
$$

**Proof.** Since $N(\cdot; v) = 0$ for all $v \in V$, the first estimate follows readily from the coercive condition (1.3). For the second estimate, the combination of (1.1) and (1.4) implies

$$
b(v, p) = a_0(u, v) + N(u; u, v) - \langle f, v \rangle 
\leq \mathcal{N}\|u\|_V^2 \|v\|_V + \alpha_1 \|u\|_V \|v\|_V + \|f\|_{V'} \|v\|_V.
$$

This together with (2.2), (2.8) and the first estimate just obtained leads to

$$
\beta \|p\|_Q \leq \sup_{v \in V} \frac{b(v, p)}{\|v\|_V} \leq \mathcal{N}\|u\|_V^2 + \alpha_1 \|u\|_V + \|f\|_{V'} 
\leq \left(1 + \Lambda + \frac{\alpha_1}{\alpha_0}\right) \|f\|_{V'} \leq \left(2 + \frac{\alpha_1}{\alpha_0}\right) \|f\|_{V'},
$$

as required. \qed

**Lemma 3.2.** Let $(u^0, p^0)$ be the solution to problem (2.15)-(2.16) and let $(u, p)$ be the solution to problem $P$. Then there holds

$$
\|u - u^0\|_V \leq \Lambda \alpha_0^{-1} \|f\|_{V'}, \quad \|p - p^0\|_Q \leq \left(1 + \frac{\alpha_1}{\alpha_0}\right) \beta^{-1} \|f\|_{V'}.
$$

**Proof.** Set

$$
E^0 := u - u^0, \quad e^0 := p - p^0.
$$

Subtracting (2.15) from (1.1) gives

$$
a_0(E^0, v) - b(v, e^0) = -N(u; u, v).
$$

Taking $v = e^0$ in (2.12) and (2.16) yields

$$
b(E^0, e^0) = b(u, e^0) - b(u^0, e^0) = 0.
$$
Hence, taking $v = E^0$ in (3.2) we find
\[ a_0(E^0, E^0) = -N(u; u, E^0), \]
which, in conjunction with the estimates (1.3) and (1.4), leads to
\[ a_0\|E^0\|_V^2 \leq a_0(E^0, E^0) = -N\|u\|_V^2 \|E^0\|_V, \]
i.e.,
\[ \|E^0\|_V \leq a_0^{-1}N\|u\|_V^2 \leq a_0^{-1}N(a_0^{-1}\|f\|_{V'}^2)^2 = \Lambda a_0^{-1}\|f\|_{V'}. \]
We then derive the first estimate in the lemma. For the second one, using (2.2), (3.2), (1.4) and (3.1), we get
\[ \beta\|e^0\|_Q \leq \sup_{v \in V} \frac{b(v, e^0)}{\|v\|_V} \]
\[ = \sup_{v \in V} \frac{1}{\|v\|_V} (a_0(E^0, v) + N(u; u, v)) \]
\[ \leq a_1\|E^0\|_V + N\|u\|_V^2 \leq a_1\Lambda a_0^{-1}\|f\|_{V'} + N(a_0^{-1}\|f\|_{V'})^2 \]
\[ = (1 + \alpha_1 a_0^{-1}\Lambda\|f\|_{V'}) \leq (1 + \alpha_1 a_0^{-1}\Lambda\|f\|_{V'}). \]
The second estimate is also obtained.

3.3 Convergence rate analysis

In the last part of this section, we analyze convergence rate of the generalized Arrow-Hurwicz method. Define
\[ E^n = u^n - u, \quad e^n = p^n - p. \] (3.3)
For the positive parameters $\rho$ and $\alpha$ used in Algorithm 1, we make the following assumption:
\[ |1 - \rho|a_1 + \rho a_0\Lambda + \frac{\rho^2 a_0^2}{2\alpha} < a_0. \] (3.4)

**Lemma 3.3.** Let $(u, p) \in V \times Q$ be the solution of problem $P$, and let $\{ (u^n, p^n) \}$ be the function sequence generated by Algorithm 1. Suppose that $\Lambda < 1$ and condition (3.4) holds. Then the sequences $\{\|u^n\|_V\}$ and $\{\|p^n\|_Q\}$ are bounded from above, with an upper bound depending only on the generic constants related to problem $P$ (including $\alpha_i$, $0 \leq i \leq 2$, $N$, $\beta$ and $\|f\|_{V'}$) and the parameters $\rho$ and $\alpha$ used in the algorithm.

**Proof.** Because of (3.1), it suffices to discuss boundedness of the sequences $\{\|E^n\|_V\}$ and $\{\|e^n\|_Q\}$. In fact, according to the definitions of $E^n$ and $e^n$ (cf. (3.3)), subtracting (1.1) from (2.17) gives
\[ \rho^{-1}a_0(E^{n+1}, v) + a_0(E^n, v) - b(v, e^n) = -N(E^n; u, v) - N(u^n; E^{n+1}, v). \] (3.5)
Furthermore, we take $v = E^{n+1}$ in the above equation to find

$$a_0(E^{n+1},E^{n+1}) - \rho b(E^{n+1},E^{n+1}) = (1 - \rho)a_0(E^n,E^{n+1}) - \rho N(E^n,u,E^{n+1}). \quad (3.6)$$

On the other hand, it follows from (1.2) and (2.18) that

$$-\rho b(E^{n+1},E^{n}) = -\rho b(u^{n+1},e^{n}) = \alpha(e^{n+1} - e^{n}),$$

$$= \frac{\alpha}{2}(\|e^{n+1}\|_Q^2 - \|e^n\|_Q^2 - \|e^{n+1} - e^n\|_Q^2). \quad (3.7)$$

Now, we require to derive the estimate

$$\|e^{n+1} - e^n\|_Q \leq \rho \alpha^{-1}\alpha_2\|E^{n+1}\|_V. \quad (3.8)$$

For this purpose, we have by (1.2) and (2.18) that

$$(e^{n+1} - e^n,q) = -\rho \alpha^{-1}b(E^{n+1},q) \quad \forall q \in Q.$$ 

The estimate (3.8) then follows by letting $q = e^{n+1} - e^n$ in the above equation and using the estimate (1.5).

Furthermore, inserting (3.7) in (3.6), we have by the estimate (3.8) and Young’s inequality that

$$2a_0(E^{n+1},E^{n+1}) + \alpha\|e^{n+1}\|_Q^2 = \alpha\|e^n\|_Q^2 + \alpha\|e^{n+1} - e^n\|_Q^2 + 2(1-\rho)a_0(E^n,E^{n+1})$$

$$-2\rho N(E^n,u,E^{n+1}) \leq \alpha\|e^n\|_Q^2 + \rho^2 \alpha^{-1}\alpha_2\|E^{n+1}\|_V^2 + 2(1-\rho)\alpha_1\|E^n\|_V\|E^{n+1}\|_V$$

$$+2\rho\alpha_0\Lambda\|E^n\|_V\|E^{n+1}\|_V$$

$$\leq \alpha\|e^n\|_Q^2 + \rho^2 \alpha^{-1}\alpha_2\|E^{n+1}\|_V^2$$

$$+ ((1-\rho)\alpha_1 + \rho\alpha_0\Lambda)(\varepsilon\|E^{n+1}\|_V^2 + \varepsilon^{-1}\|E^n\|_V^2),$$

where $\varepsilon > 0$ is a parameter to be determined later on.

By setting $r := |1-\rho|\alpha_1 + \rho\alpha_0\Lambda$ and using the estimate (1.3), we can rewrite the above estimate as

$$(2\alpha_0 - \rho^2 \alpha^{-1}\alpha_2^2 - r\varepsilon)\|E^{n+1}\|_V^2 + \alpha\|e^{n+1}\|_Q^2 \leq r\varepsilon^{-1}\|E^n\|_V^2 + \alpha\|e^n\|_Q^2. \quad (3.9)$$

Now, let us look for a parameter $\varepsilon$ such that it satisfies

$$2\alpha_0 - \rho^2 \alpha^{-1}\alpha_2^2 - r\varepsilon = r\varepsilon^{-1}. \quad (3.10)$$

In other words, $\varepsilon$ is the positive root of the quadratic equation

$$re^2 - (2\alpha_0 - \rho^2 \alpha^{-1}\alpha_2^2)e + r = 0.$$
Note that the discriminant of the equation is
\[
G = (2\alpha_0 - \rho^2 \alpha^{-1} \alpha_2)^2 - 4r^2 = 4 \left( \alpha_0 - \frac{\rho^2 \alpha_2}{2\alpha} \right)^2 - 4r^2,
\]
which is positive under the assumption (3.4). Hence, we choose
\[
\varepsilon = \epsilon^* := \frac{2\alpha_0 - \rho^2 \alpha^{-1} \alpha_2 - \sqrt{G}}{2r}.
\]

Based on the above choice, write
\[
D = 2\alpha_0 - \rho^2 \alpha^{-1} \alpha_2 - r\varepsilon = r/\varepsilon = \alpha_0 - \frac{\rho^2 \alpha_2}{2\alpha} + \frac{1}{2} \sqrt{G}.
\]

It is easy to show that \(D > 0\) and the estimate (3.9) can be expressed as
\[
D\|E^{n+1}\|_V^2 + \alpha \|e^{n+1}\|_Q^2 \leq \mathcal{D}\|E^n\|_V^2 + \alpha \|e^n\|_Q^2,
\]
which combined with Lemma 3.2 implies the required estimate readily.

Now we are ready to discuss the convergence rate analysis of Algorithm 1.

**Theorem 3.1.** Let \((u, p) \in V \times Q\) be the solution of problem \(P\). Let \(\{u^n, p^n\}\) be the function sequence generated by Algorithm 1. If \(\Lambda < 1\) and condition (3.4) holds, then we have the following estimate
\[
\mathcal{F}\|E^{n+1}\|_V^2 + \alpha \|e^{n+1}\|_Q^2 \leq \gamma (\mathcal{F}\|E^n\|_V^2 + \alpha \|e^n\|_Q^2),
\]
where \(\mathcal{F} \in (0,1)\) and \(\gamma \in (0,1)\) are two constants depending only on the generic constants as for Lemma 3.3.

**Proof.** From Lemma 3.3, we know there is a positive number \(\mathcal{F}_1\) such that \(\|u^n\|_V \leq \mathcal{F}_1\). Thus, it follows from (2.2), (2.8), (1.4), (3.5) and Lemma 3.1 that
\[
\beta \|e^n\|_Q \leq \sup_{v \in V} \frac{b(v, e^n)}{\|v\|_V} \leq \alpha_1 \|E^n\|_V + \rho^{-1} \alpha_1 \|E^{n+1}\|_V + \mathcal{N}\|E^n\|_V \|u\|_V + \mathcal{N}\|u^n\|_V \|E^{n+1}\|_V
\]
\[
\leq \alpha_1 \|E^n\|_V + \rho^{-1} \alpha_1 \|E^{n+1}\|_V + \rho^{-1} \alpha_1 \|E^n\|_V + \alpha_0 \Lambda \|E^n\|_V + \mathcal{N}\|\mathcal{F}_1 \|E^{n+1}\|_V
\]
\[
= (\alpha_1 + \alpha_0 \Lambda + \rho^{-1} \alpha_1) \|E^n\|_V + (\mathcal{N}\mathcal{F}_1 + \rho^{-1} \alpha_1) \|E^{n+1}\|_V,
\]
which immediately yields
\[
\beta^2 \|e^n\|_Q^2 \leq 2(\alpha_1 + \alpha_0 \Lambda + \rho^{-1} \alpha_1)^2 \|E^n\|_V^2 + 2(\mathcal{N}\mathcal{F}_1 + \rho^{-1} \alpha_1)^2 \|E^{n+1}\|_V^2.
\]
Therefore,
\[
\|E^{n+1}\|_V^2 \geq \mathcal{F}_2 \|e^n\|_Q^2 - \mathcal{F}_3 \|E^n\|_V^2,
\]
\[\text{(3.11)}\]
where
\[ \mathcal{F}_2 := \frac{1}{2} \left( \frac{\beta}{\mathcal{N} \mathcal{F}_1 + \rho^{-1} a_1} \right)^2, \quad \mathcal{F}_3 := \left( \frac{a_1 + a_0 \Lambda + \rho^{-1} a_1}{\mathcal{N} \mathcal{F}_1 + \rho^{-1} a_1} \right)^2. \]

Rewrite (3.9) in the form
\[ \delta \| E^{n+1} \|^2_{\bar{V}} + (2a_0 - \rho^2 a^{-1} a_2^2 - re - \delta) \| E^{n+1} \|^2_{\bar{V}} + \alpha \| e^{n+1} \|_Q^2 \leq \epsilon^{-1} \| E^n \|^2_{\bar{V}} + \epsilon \| e^n \|_Q^2, \]

where \( \delta > 0 \) is a parameter to be determined. We then insert (3.11) into the above inequality to arrive at
\[ (2a_0 - \rho^2 a^{-1} a_2^2 - re - \delta) \| E^{n+1} \|^2_{\bar{V}} + \alpha \| e^{n+1} \|_Q^2 \leq \epsilon^{-1} \| E^n \|^2_{\bar{V}} + \epsilon \| e^n \|_Q^2. \] (3.12)

To derive the required estimate, we have to find \( \epsilon \) and \( \delta \) satisfy
\[ (2a_0 - \rho^2 a^{-1} a_2^2 - re - \delta) \bar{\alpha} = (\epsilon^{-1} + \delta \mathcal{F}_3) / (\alpha - \delta \mathcal{F}_2). \] (3.13)

For this purpose, we first consider how to choose \( \epsilon \). It is obvious that (3.13) is equivalent to
\[ \mathcal{F}_2 \delta^2 - (\alpha + \alpha \mathcal{F}_3 + \mathcal{F}_2(2a_0 - \rho^2 a^{-1} a_2^2 - re)) \delta + \alpha (2a_0 - \rho^2 a^{-1} a_2^2 - re - re^{-1}) = 0. \] (3.14)

If we suppose \( \alpha - \delta \mathcal{F}_2 > 0 \) and \( 2a_0 - \rho^2 a^{-1} a_2^2 - re > \delta > 0 \), then we find from (3.14) that
\[ \alpha (2a_0 - \rho^2 a^{-1} a_2^2 - re - re^{-1}) > 0. \] (3.15)

Define \( G = (2a_0 - \rho^2 a^{-1} a_2^2)^2 - 4r^2 \), where \( G > 0 \) under the condition (3.4). Therefore, due to (3.15), the required \( \epsilon \) must satisfy
\[ \frac{2a_0 - \rho^2 a^{-1} a_2^2 - \sqrt{G}}{2r} < \epsilon < \frac{2a_0 - \rho^2 a^{-1} a_2^2 + \sqrt{G}}{2r}. \]

Based on the above analysis, we choose
\[ \epsilon = \epsilon^* := \frac{2a_0 - \rho^2 a^{-1} a_2^2}{2r} = \frac{a_0 - \rho^2 a_2^2}{2a} \frac{r}{r}. \]

In this case, the quadratic equation (3.14) can be expressed as
\[ a \delta^2 - b \delta + c = 0, \]
where
\[ s := a_0 - \rho^2 a_2^2 \frac{s}{2a}, \quad a := \mathcal{F}_2, \quad b := \alpha + \alpha \mathcal{F}_3 + \mathcal{F}_2 s, \quad c := \alpha \left( s - \frac{r^2}{s} \right). \]

Note that \( a > 0, b > 0, c > 0 \) and
\[ b^2 - 4ac > (\alpha + \mathcal{F}_2 s)^2 - 4a \mathcal{F}_2 s \geq 0, \]
so Eq. (3.14) has two real roots and we select 

$$\delta = \delta^* := \frac{b - \sqrt{b^2 - 4ac}}{2a}. $$

With the parameters $\varepsilon$ and $\delta$ chosen as above, it follows from (3.12) that

$$ F \| E^{n+1} \|_V^2 + \alpha \| e^{n+1} \|_Q^2 \leq \gamma (F \| E^n \|_V^2 + \alpha \| e^n \|_Q^2), $$

where $F := s - \delta^* = a_0 - \rho^2 a_2^2/(2a) - \delta^*$ and $\gamma := 1 - \alpha^{-1} \delta^* F_2$.

It is apparent that $F < a_0$ and $\gamma < 1$. Now, let us prove that $F > 0$ and $\gamma > 0$. Consider the quadric function $f(x) = ax^2 - bx + c$. Since $s > 0$, $b > a + F_2 s$ and $c < a s$, we have $\lim_{x \to -\infty} f(x) = \infty$ and

$$ f(s) = as^2 - bs + c < F_2 s^2 - (a + F_2 s)s + a s = 0, $$

so the smaller root $\delta^*$ of $f(x)$ belongs to $(-\infty, s)$, i.e., $F > 0$. Furthermore, we know by (3.13) that $\gamma = (re^{*+1} + \delta^* F_3)/F > 0$. The proof is complete.

**Remark 3.1.** If $\Lambda < 1$, we have by a direct manipulation that condition (3.4) is equivalent to $\rho < (a_0 + a_1)/(a_1 + a_0 \Lambda)$ and

$$ a > \begin{cases} 
\frac{\rho^2 a_2^2}{2(a_0(1-\rho \Lambda) + a_1(\rho - 1))}, & \rho \leq 1, \\
\frac{\rho^2 a_2^2}{2(a_0(1-\rho \Lambda) + a_1(1-\rho))}, & 1 < \rho < \frac{a_0 + a_1}{a_1 + a_0 \Lambda}.
\end{cases} $$

Hence, whenever $\Lambda < 1$, we can always find $\rho$ and $a$ such that Algorithm 1 is convergent.

**Remark 3.2.** By checking the derivation given above, it is easy to see that if the initial functions vanish, Algorithm 1 still converges geometrically with a contraction number depending only on the generic constants as for Lemma 3.3.

### 4 Some applications

In this section, we will apply the mathematical theory and the generalized Arrow-Hurwicz method to study three specific nonlinear saddle-point problems, including mathematical model and its numerical method for the steady incompressible magnetohydrodynamic equations, the variational multi-scale (VMS) method and the defect-correction method for numerically solving the steady incompressible Navier-Stokes equations. For all the previous discrete methods, we will propose the related Arrow-Hurwicz methods to solve these problems and develop their convergence rate analysis.
4.1 The steady incompressible MHD equations

For simplicity, consider the following steady incompressible MHD equations (cf. [12]):

\[
\begin{align*}
-R_\nu^{-1} \Delta u + (u \cdot \nabla) u - S_c B \times \text{curl} B + \nabla p &= f & \text{in } \Omega, \\
\text{div} u &= 0 & \text{in } \Omega, \\
S_c R_m^{-1} \text{curl}(\text{curl} B) - S_c \text{curl}(u \times B) &= g & \text{in } \Omega, \\
\text{div} B &= 0 & \text{in } \Omega, \\
u &= 0 & \text{on } \partial \Omega, \\
B \cdot n &= 0, \quad n \times \text{curl} B &= 0 & \text{on } \partial \Omega,
\end{align*}
\]

(4.1)

where \( \Omega \subset \mathbb{R}^d \) \((d = 2, 3)\) and \( n \) is the unit outward normal to the boundary \( \partial \Omega \). Here \( u \) denotes the velocity field, \( B \) the magnetic field, \( f \) and \( g \) the external force terms, \( p \) the pressure. Obviously, the MHD equations are characterized by three parameters: \( R_\nu, R_m \) and \( S_c \), which are the hydrodynamic Reynolds number, the magnetic Reynolds number and the coupling number, respectively.

In order to derive the variational formulation to the above problem, we require to use some notations and symbols in Sobolev spaces (cf. [1]). For instance, for all non-negative integers \( k \) and \( r \), \( W^{k,r}(\Omega) \) consists of all functions in \( L^r(\Omega) \) whose weak derivatives with the degree no more than \( k \) are also \( L^r(\Omega) \)-integrable. The related norm is defined by \( \|v\|_{k,r} = (\sum_{|\alpha|=0}^k \|D^\alpha v\|_{0,r}^r)^{1/r} \). And \( W_0^{k,r}(\Omega) \) is the closure of \( C_0^\infty(\Omega) \) with respect to the norm \( \|\cdot\|_{k,r} \). In particular, \( W^{k,2}(\Omega) \) (resp. \( W_0^{k,2}(\Omega) \)) is simply written as \( H^k(\Omega) \) (resp. \( H_0^k(\Omega) \)) and the norm \( \|\|_{k,2} \) as \( \|\|_k \). If \( v = (v_1, \cdots, v_d)^T \), then \( v \in (W^{k,r}(\Omega))^d \) if \( v_1 \in W^{k,r}(\Omega) \) for \( i = 1, \cdots, d \); \( \|v\|_{k,r} := (\sum_{i=1}^d \|v_i\|_{k,r}^r)^{1/r} \). The convention for \( r = 2 \) also applies in vector cases. Furthermore, introduce the following function spaces:

\[
\begin{align*}
X &=: (H^1_0(\Omega))^d = \{ w \in (H^1(\Omega))^d; w|_{\partial \Omega} = 0 \}, \\
W &=: (H^1_0(\Omega))^d = \{ w \in (H^1(\Omega))^d; w \cdot n|_{\partial \Omega} = 0 \}, \\
M &=: L^2_0(\Omega) = \{ q \in L^2(\Omega); \int_{\Omega} q \, dx = 0 \}.
\end{align*}
\]

With the above notations in mind, the variational formulation of the problem (4.1) reads (cf. [12,25]):

**Problem P1.** Find \((\bar{u}, p) \in W_{0n} \times M\) such that

\[
\begin{align*}
A_0(\bar{u}, \bar{\theta}) + A_1(\bar{u}; \bar{\theta}) - d(\bar{\theta}, p) &= \langle F, \bar{\theta} \rangle & \forall \bar{\theta} \in W_{0n}, \\
\langle d(\bar{u}, q) = 0 & \forall q \in M,
\end{align*}
\]

(4.2)

(4.3)
where
\[ W_{0n} := X \times W, \tilde{u} = (u,B), \tilde{\sigma} = (v,\Psi), \]
\[ A_0(\tilde{u},\tilde{\sigma}) = b_0(u,v) + c_0(B,\Psi), \]
\[ b_0(u,v) = R_c^{-1}(\nabla u, \nabla v), c_0(B,\Psi) = S_c R_m^{-1}(\text{curl}B,\text{curl}\Psi) + S_c R_m^{-1}(\text{div}B,\text{div}\Psi), \]
\[ d(\tilde{\sigma},p) = (\text{div}v,p), \]
\[ A_1(\tilde{u};\tilde{\sigma}) = b_1(u;u,v) - c_1(B;B,v) + c_1(B;\Psi,u), \]
\[ b_1(u;u,v) = \frac{1}{2}((u \cdot \nabla)v,u) - \frac{1}{2}((u \cdot \nabla)v,u), \]
\[ c_1(B;B,v) = S_c(\text{curl}B \times B,v), \]
\[ \langle f,\tilde{\sigma} \rangle = \langle f,v \rangle + \langle g,\Psi \rangle, \]
where and in what follows, \((\cdot,\cdot)\) denotes the usual \(L^2\)-inner product and \(\langle \cdot,\cdot \rangle\) is the bilinear form between the dual pair \(H^{-1}(\Omega)\) and \(H_0^1(\Omega)\). These symbols also applies to vector functions similarly.

Now, if we write \(V := W_{0n}, Q := M\), and for \(u = \tilde{u}\) and \(v = \tilde{\sigma}\), define \(a_0(u,v) = A_0(\tilde{u},\tilde{\sigma}), b(v,q) = d(\tilde{\sigma},q)\) and \(N(u;u,v) = A_1(\tilde{u};\tilde{\sigma})\), then problem \(P_1\) can be viewed a specific case of problem \(P\). Thus, all the results developed in the last section can be applied to problem \(P_1\). In addition, the bilinear forms \(A_0(\cdot,\cdot)\) and the trilinear form \(A_1(\cdot;\cdot;\cdot)\) satisfy the following conditions (cf. [12]): for all \(\tilde{u},\tilde{\sigma},\tilde{w} \in W_{0n},\)
\[
\begin{align*}
A_0(\tilde{\sigma},\tilde{w}) &\leq \max\{R_c^{-1},A_c R_m^{-1}\} \|\tilde{\sigma}\|_1 \|\tilde{w}\|_1, \\
A_0(\tilde{\sigma},\tilde{\sigma}) &\geq \min\{R_c^{-1},C_1 S_c R_m^{-1}\} \|\tilde{\sigma}\|_1^2, \\
A_1(\tilde{u};\tilde{\sigma};\tilde{w}) &\leq \sqrt{2} C_0^2 \max\{1,\sqrt{2} S_c\} \|\tilde{u}\|_1 \|\tilde{w}\|_1 \|\tilde{w}\|_1,
\end{align*}
\]
where \(C_0\) and \(C_1\) are two positive constants such that
\[
\|v\|_{0,4} \leq C_0 \|v\|_1 \quad \forall v \in X; \quad C_1 \|B\|_1 \leq \|\text{curl}B\|_0 + \|\text{div}B\|_0 \quad \forall B \in (H^1_0(\Omega))^d.
\]

On the other hand, it is evident from [10,18,20] that there exists a constant \(\beta > 0\) such that for any \(q \in M\), there holds
\[
\sup_{v \in W_{0n}} d(\tilde{\sigma},q) / \|\tilde{\sigma}\|_1 \geq \sup_{v \in X} (\text{div}v,q) / \|v\|_1 \geq \beta \|q\|_0.
\]

Therefore, all the conditions given for problem \(P_1\) are satisfied in this case. According to Theorem 2.2, if \(\Lambda = a_0^{-2}N\|F\|_{-1} < 1\), where \(a_0 = \min\{R_c^{-1},C_1 S_c R_m^{-1}\},\)
\(N = \sqrt{2} C_0^2 \max\{1,\sqrt{2} S_c\}\), then problem \(P_1\) has a unique solution \((\tilde{u},p) \in W_{0n} \times M\). This condition coincides with the one given in [12].

Now, let us consider the finite element method for problem \(P_1\). For simplicity, we assume that \(\Omega\) is a convex polygonal/polyhedral domain. Let \(T_h = \{K\}_{K \in T_h}\) be a regular family of triangulations of \(\Omega\), and \(h\) denotes the mesh size of \(T_h\) (cf. [4]). With each triangulation \(T_h\), we associate a triple of the conforming finite element spaces \((X_h,W_h,M_h)\)
such that $X_h \subset X$, $W_h \subset W$ and $M_h \subset M$. Recalling the derivation of (4.5), we know if the pairs of finite element spaces $(X_h, M_h)$ are uniformly stable for solving steady incompressible Navier-Stokes equations, so are the triples of finite element spaces $(X_h, W_h, M_h)$. That means, there exists a generic constant $\beta_1 > 0$ independent of $h$ such that

$$\inf_{\tilde{v} \in M_h} \sup_{\tilde{\vartheta} \in W_h} \frac{d(\tilde{\vartheta}, \tilde{v})}{\|\tilde{\vartheta}\|_1 \|\tilde{v}\|_0} \geq \beta_1,$$

where $W_h^0 := X_h \times W_h$. Hence, we can choose the stable finite element spaces $(X_h, M_h)$ used traditionally for steady incompressible Navier-Stokes equations to approximate the velocity field and pressure field (e.g. the Taylor-Hood element), and any appropriate finite element subspace $W_h$ of $W$ to approximate the magnetic field. Thus, the mixed element method for (4.2)-(4.3) can be expressed as follows.

**Problem $P_h^1$.** Find $(\tilde{u}_h, p_h) \in W_h^0 \times M_h$ such that

$$\begin{cases} A_0(\tilde{u}_h, \tilde{\vartheta}) + A_1(\tilde{u}_h; \tilde{u}_h, \tilde{\vartheta}) - d(\tilde{\vartheta}, p_h) = \langle F, \tilde{\vartheta} \rangle, & \forall \tilde{\vartheta} \in W_h^0, \\ d(\tilde{u}_h, q) = 0 & \forall q \in M_h. \end{cases}$$

The Arrow-Hurwicz algorithm can be used naturally to solve problem $P_h^1$, leading to Algorithm 2 given below.

**Algorithm 2** The generalized Arrow-Hurwicz algorithm for problem $P_h^1$.

Let $\rho$ and $\alpha$ be two positive parameters.

Step 1. Choose two initial functions $\tilde{u}_h^0 \in W_h^0$ and $p_h^0 \in M_h$ such that

$$\begin{cases} A_0(\tilde{u}_h^0, \tilde{\vartheta}) - d(\tilde{\vartheta}, p_h^0) = \langle F, \tilde{\vartheta} \rangle, & \forall \tilde{\vartheta} \in W_h^0, \\ d(\tilde{u}_h^0, q) = 0 & \forall q \in M_h. \end{cases}$$

Step 2. For $n = 0, 1, \ldots$, if $\tilde{u}_h^n \in W_h^0$ and $p_h^n \in M_h$ are available, determine $\tilde{u}_h^{n+1} \in W_h^0$ and $p_h^{n+1} \in M_h$ by solving the following two equations sequentially:

$$\begin{cases} A_0((\tilde{u}_h^{n+1} - \tilde{u}_h^n), \tilde{\vartheta}) + \rho A_1(\tilde{u}_h^n; \tilde{u}_h^{n+1}, \tilde{\vartheta}) + \rho A_0(\tilde{u}_h^n, \tilde{\vartheta}) - \rho d(\tilde{\vartheta}, p_h^n) = \rho \langle F, \tilde{\vartheta} \rangle, & \forall \tilde{\vartheta} \in W_h^0, \\ \alpha(p_h^{n+1} - p_h^n, q) + \rho d(\tilde{u}_h^{n+1}, q) = 0 & \forall q \in M_h. \end{cases}$$

The following theorem is a direct consequence of Theorem 3.1 combined with the above analysis.

**Theorem 4.1.** Let $(\tilde{u}_h, p_h) \in W_h^0 \times M_h$ be the solution of problem $P_h^1$. Let $(\tilde{u}_h^n, p_h^n)$ be the function sequence generated by Algorithm 2. Set $a_0 = \min\{R_e^{-1}, C_1 S_c R_m^{-1}\}$, $a_1 = \max\{R_e^{-1}, 4 S_c R_m^{-1}\}$,
\( \alpha_2 = 1 \) and \( \beta \) is the \( \beta_1 \) given in (4.6). Then, if \( \Lambda = \alpha_0^{-2} \mathcal{N}(F) \| F \|_{-1} < 1 \) and condition (3.4) holds, we have

\[
\mathcal{F}(\bar{E}_h^{n+1})_1^{\frac{1}{2}} \| \mathcal{E}_h^{n+1} \|_{0,0}^2 \leq \gamma (\mathcal{F}(\bar{E}_h^n)_1^{\frac{1}{2}} + \alpha \| \mathcal{E}_h^n \|),
\]

where \( \bar{E}_h^n := \bar{u}_h^n - \bar{u}_h, \mathcal{E}_h^n := p_h^n - p_h, \mathcal{F} \in (0,1) \) and \( \gamma \in (0,1) \) are two constants independent of \( h \).

### 4.2 The steady incompressible Navier-Stokes equations using the VMS method

In this and the next subsections, we focus on numerical solution of the following steady incompressible Navier-Stokes equations:

\[
\begin{aligned}
-\nu \nabla u + (u \cdot \nabla)u + \nabla p &= f \quad \text{in } \Omega, \\
\text{div} u &= 0 \quad \text{in } \Omega, \\
u = 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

(4.12)

where \( \Omega \) represents a bounded polygonal/polyhedral domain of \( \mathbb{R}^d \) (\( d = 2,3 \)) with the boundary \( \partial \Omega \), \( u \) the velocity, \( p \) the pressure, \( f \) the prescribed body force, and \( \nu > 0 \) the kinematic viscosity, which is inversely proportional to the Reynolds number \( R_e \).

The mixed formulation for the above problem can be described as follows (cf. \[9, 10, 18, 20\]).

**Problem** \( P_2 \). Find \((u, p) \in X \times M\) such that

\[
\begin{aligned}
va(u, v) + b(u; u, v) - d(v, p) &= \langle f, v \rangle \quad \forall v \in X, \\
d(u, q) &= 0 \quad \forall q \in M,
\end{aligned}
\]

(4.13)

(4.14)

where

\[
\begin{aligned}
X := (H_0^1(\Omega))^d &= \{ w \in (H^1(\Omega))^d; w|_{\partial \Omega} = 0 \}, \\
M := L_0^2(\Omega) &= \{ q \in L^2(\Omega); \int_\Omega q dx = 0 \}, \\
a(u, v) := (\nabla u, \nabla v) \quad \forall u, v \in X, \\
b(u; u, v) := \frac{1}{2}((u \cdot \nabla)v, w) - \frac{1}{2}(u \cdot \nabla)w, v) \quad \forall u, v, w \in X.
\end{aligned}
\]

If the Reynolds number \( R_e \) is not very large, the standard mixed element method based on (4.14) works well (cf. \[3, 10, 18\]). Concretely speaking, for a regular family of triangulation \( T_h \), we choose a pair of conforming finite element spaces \((X_h, M_h) \subset (X, M)\) satisfying the following discrete inf-sup condition

\[
\inf_{q \in M_h} \sup_{v \in X_h} \frac{d(v, q)}{\| v \|_1 \| q \|_0} \geq \beta_2,
\]

(4.15)
where $\beta_2 > 0$ is a constant independent of $h$. Then, the mixed element method for solving (4.2) is obtained by replacing $(X,M)$ with $(X_h,M_h)$ in (4.14). The typical stable pairs of $(X_h,M_h)$ include the MINI element, the Girault-Raviart element, and the $P_k - P_{k-1}$ element (cf. [3]). The $P_2 - P_1$ element is also called the Taylor-Hood element.

However, if the Reynolds number $Re$ is very large, the above methods will fail to produce the numerical solution with desired accuracy. This is due to the fact that in this case, owing to the domination of the convection, spurious oscillations may occur in numerical simulations if the underlying mesh used for the discretization is not fine enough. The other point is that the iterative method used to solve the nonlinear discrete system may fail to converge and can not yield a solution. Therefore, we require to develop more sophisticated methods to overcome the difficulty, among which the variational multiscale (VMS) method (cf. [11, 14, 15]) and the defect-correction method (cf. [19]) are two typical ones. For later uses, we briefly review some existing results for the two methods. In [14], Hughes proposed a variational method capable of representing multiscale phenomena. Later on, Guermond introduced in [11] the VMS method by imposing the stabilization effect with subgrid modeling for a Galerkin approximation solving transport equations. In [15], Layton established the connection between the subgrid-scale eddy viscosity and mixed methods. The defect-correction method is an iterative improvement technique well established for solving nonlinear steady-state problems; for details, see Stetter [19]. In [16], Layton initially investigated the defect-correction method for the incompressible viscous flow with high Reynolds number. Subsequently, Layton et al. furthered their studies in [17] along this line. For solving nonlinear steady-state problems, the defect-correction method or the VMS method can be viewed as an iterative improvement technique (cf. [19]). we notice that both methods mentioned above involve solving a saddle-point system at each iteration step. Here, we hope to combine the two methods with our method proposed here to produce some new methods for solving steady incompressible Navier-Stokes equations with high Reynolds number, without solving any saddle-point systems at each iteration step (except for the initial step).

In this subsection, we are devoted to the above study for the VMS method. To fix the main ideas clearly, let us consider the method corresponding to the Taylor-Hood element (cf. [24]). To this end, we fist introduce a piecewise constant space given by

$$R_0 = \{ v_h \in L^2(\Omega); v_h|_K \in P_0(K) \ \forall K \in T_h \},$$

where $P_0(K)$ consists of all constants on $K$. Throughout this paper, we use $C$ to denote a generic constant independent of $h$, which may take different values at different occurrences.

Since the usual finite element discretization is unstable in the case of high Reynolds number, the use of stabilization becomes necessary. We then define two spaces $L = L^2(\Omega)^{d \times d}$ and $L_h = R_0(\Omega)^{d \times d}$, the latter is defined on the same grid as $X_h$ for the velocity deformation tensor. We define the orthogonal projection operator $\Pi_h : L \rightarrow L_h$ with
the following properties:

\[
\begin{align*}
((I - \Pi_h)l, g_h) &= 0 \quad \forall l \in L, \ g_h \in L_h, \\
\|\Pi_h l\|_{0,K} &\leq C\|l\|_{0,K} \quad \forall l \in L, \ K \in \mathcal{T}_h, \\
\|(I - \Pi_h)l\|_{0,K} &\leq Ch\|l\|_{1,K} \quad \forall l \in L \cap H^1(\Omega)^{d \times d}, \ K \in \mathcal{T}_h,
\end{align*}
\]  

(4.16)

where \( I \) is the identity operator. For all \((u_h, v) \in X_h \times X_h\), with the help of the orthogonal projection operator \(\Pi_h\), define

\[
G_h(u_h, v) = \eta((I - \Pi_h) \nabla u_h, (I - \Pi_h) \nabla v).
\]

According the estimates in (4.16) and the local inverse inequality for finite elements, we have

\[
G_h(u_h, v) \leq \eta\|(I - \Pi_h) \nabla u_h\|_0\|(I - \Pi_h) \nabla v\|_0 \leq C\eta \|u_h\|_1\|v\|_1,
\]

(4.17)

where \( \eta \) is an artificial viscosity parameter with the scale of \( \mathcal{O}(h) \) in order to stabilize the convective term appropriately.

Then the VMS method can be written as follows (cf. [24]).

**Problem** \( P_h^2 \). Find \((u_h, p_h) \in V_h \times P_h\) such that

\[
\begin{cases}
G_h(u_h, v) + \nu a(u_h, v) + b(u_h, v) - d(v, p_h) = (f, v) &\forall v \in X_h, \\
d(u_h, q) = 0 &\forall q \in M_h.
\end{cases}
\]

(4.18)

If we let \( V = X_h, \ Q = M_h \), and for \( u = u_h \) and \( v = v_h \), write \( a_0(u, v) = G_h(u_h, v) + \nu a(u_h, v), \ b(v, p) = d(v, p_h) \) and \( N(u; u, v) = b(u_h; u_h, v) \), then problem \( P_h^2 \) can be viewed as a specific case of problem \( P \). Furthermore, for all \( u_h, v \in X_h \), there hold

\[
\begin{align*}
G_h(u_h, v) + \nu a(u_h, v) &\leq (C\eta + \nu) |u_h|_1 |v|_1, \\
G_h(v, v) + \nu a(v, v) &\geq \nu |v|_1^2, \\
b(u_h; u_h, v) &\leq N |u_h|_1 |u_h|_1 |v|_1.
\end{align*}
\]

(4.19)

Now, the abstract Arrow-Hurwicz method (cf. Algorithm 1) can be used to solve the discrete problem \( P_h^2 \) directly, leading to Algorithm 3 below.

By means of the estimates in (4.19), we can derive the following result from Theorem 3.1 readily.

**Theorem 4.2.** Let \((u_h, p_h) \in X_h \times M_h\) be the solution of problem \( P_h^2 \). Let \(\{u^n_h, p^n_h\}\) be the function sequence generated by Algorithm 3. Set \( a_0 = \nu, \ a_1 = \nu, \ a_2 = 1 \) and \( \beta = \beta_2 \) given in (4.15). Then, if \( \Lambda := \nu^{-2}N\|f\|_{-1} < 1 \) and the condition \( |1 - \nu\Lambda + \rho\nu\Lambda + \frac{\rho^2}{2\alpha} < 1 \) holds, we have

\[
\mathcal{F}\|E_{h}^{n+1}\|_1^2 + \alpha\|e_{h}^{n+1}\|_1^2 \leq \gamma(\mathcal{F}\|E_{h}^{n}\|_1^2 + \alpha\|e_{h}^{n}\|_1^2),
\]

where \( E_{h}^{n} := u_{h}^{n} - u_{h}, \ e_{h}^{n} := p_{h}^{n} - p_{h} \) and \( \mathcal{F} \in (0, 1) \) and \( \gamma \in (0, 1) \) are two generic constants independent of \( n \) and \( h \).
First of all, let us recall the following defect-correction method to be used later on (cf. Algorithm 2). We consider the linear problem for the correction
\[ \begin{align*}
G_h(u^0_h, v) + \nu (\nabla u^0_h, \nabla v) - (p^0_h, \text{div } v) &= \langle f, v \rangle \quad \forall v \in X_h, \\
(\text{div } u^0_h, q) &= 0 \quad \forall q \in M_h.
\end{align*} \tag{4.20} \]

Step 2. For \( n = 0, 1, \ldots \), if \( u^n_h \in X_h \) and \( p^n_h \in M_h \) are available, determine \( u^{n+1}_h \in X_h \) and \( p^{n+1}_h \in M_h \) by solving the following two equations sequentially:
\[ \begin{align*}
(\nabla (u^{n+1}_h - u^n_h), \nabla v) + \rho G_h(u^{n+1}_h, v) + \rho (\nabla u^{n+1}_h, \nabla v) + \rho b(u^n_h; u^{n+1}_h, v) \\
- \rho (p^n_h, \text{div } v) &= \rho \langle f, v \rangle \quad \forall v \in V_h, \tag{4.21} \\
\alpha (p^{n+1}_h - p^n_h, q) + \rho (\text{div } u^{n+1}_h, q) &= 0 \quad \forall q \in P_h. \tag{4.22}
\end{align*} \]

### 4.3 The steady incompressible Navier-Stokes equations using the defect-correction method

In this subsection, we intend to develop a new method for solving steady incompressible Navier-Stokes equations by combining our method with the defect-correction method. First of all, let us recall the following defect-correction method to be used later on (cf. [17]).

**Algorithm 3** The generalized Arrow-Hurwicz algorithm for problem \( P^1_2 \).

Let \( \rho \) and \( \alpha \) be two positive parameters.

**Step 1.** Choose two initial functions \( u^0_h \in X_h \) and \( p^0_h \in M_h \) such that:
\[ \begin{align*}
G_h(u^0_h, v) + \nu (\nabla u^0_h, \nabla v) - (p^0_h, \text{div } v) &= \langle f, v \rangle \quad \forall v \in X_h, \\
(\text{div } u^0_h, q) &= 0 \quad \forall q \in M_h.
\end{align*} \]

**Step 2.** For \( n = 0, 1, \ldots \), if \( u^n_h \in X_h \) and \( p^n_h \in M_h \) are available, determine \( u^{n+1}_h \in X_h \) and \( p^{n+1}_h \in M_h \) by solving the following two equations sequentially:
\[ \begin{align*}
(\nabla (u^{n+1}_h - u^n_h), \nabla v) + \rho G_h(u^{n+1}_h, v) + \rho (\nabla u^{n+1}_h, \nabla v) + \rho b(u^n_h; u^{n+1}_h, v) \\
- \rho (p^n_h, \text{div } v) &= \rho \langle f, v \rangle \quad \forall v \in V_h, \\
\alpha (p^{n+1}_h - p^n_h, q) + \rho (\text{div } u^{n+1}_h, q) &= 0 \quad \forall q \in P_h.
\end{align*} \]

**Algorithm 4** The defect-correction method.

**Step 1.** Find \( (u^0_h, p^0_h) \in X_h \times M_h \) such that
\[ \begin{align*}
\delta h a(u^0_h, v) + v a(u^0_h, v) + b(u^0_h; u^0_h, v) - d(v, p^0_h) &= \langle f, v \rangle \quad \forall v \in X_h, \\
d(u^0_h, q) &= 0 \quad \forall q \in M_h.
\end{align*} \tag{4.23} \]

where \( \delta > 0 \) is a prescribed stabilizing factor, and \( (X_h, M_h) \) is a stable pair of finite element spaces.

**Step 2.** If \( u^0_h \in X_h \) and \( p^0_h \in M_h \) are available, determine \( u^1_h \in X_h \) and \( p^1_h \in M_h \) by solving the linear problem for the correction
\[ \begin{align*}
\delta h a(u^1_h, v) + v a(u^1_h, v) + b(u^1_h; u^1_h, v) + b(u^0_h; u^0_h, v) - d(v, p^1_h) \\
= \langle f, v \rangle + \delta h a(u^0_h, v) + b(u^0_h; u^0_h, v) \quad \forall v \in X_h, \\
d(u^1_h, q) &= 0 \quad \forall q \in M_h.
\end{align*} \tag{4.24} \]

We find the main cost of the above method involves implementing Step 1, or equiv-
alently, solving a nonlinear saddle-point problem. For simplicity of presentation, we reformulate it as follows.

**Problem** $P^h_3$. Find $(u_h, p_h) \in X_h \times M_h$ such that

$$
\begin{align*}
\delta h(u_h, v) + v(a(u_h, v) + b(u_h; u_h, v) - d(v, p_h) = \langle f, v \rangle & \quad \forall v \in X_h, \\
\delta d(u_h, q) = 0 & \quad \forall q \in M_h. \tag{4.26}
\end{align*}
$$

If we let $V = X_h$, $Q = M_h$, and for $u = u_h$ and $v = v_h$, write $a_0(u, v) = \delta h(a(u, v) + v(a(u, v))$, $b(v, p) = d(v, p_h)$ and $N(u; u, v) = b(u_h; u_h, v)$, then problem $P^h_3$ can also be viewed as a specific case of problem $P$. Furthermore, $\forall u_h, v \in X_h$, there hold

$$
\begin{align*}
\delta h(u_h, v) + v(a(u_h, v) & \leq (\delta h + v)|u_h|v|1, \\
\delta h(v, v) + v(a(v, v) & \geq (\delta h + v)|v|2, \\
b(u_h, v) & \leq N|u_h|v|1|v|1. \tag{4.27}
\end{align*}
$$

Then, as shown in the last two subsections, we can design an Arrow-Hurwicz method for solving problem $P^h_3$, described as Algorithm 5, with its convergence rate analysis given in Theorem 4.3 which follows readily from (4.27) and Theorem 3.1.

**Algorithm 5** The generalized Arrow-Hurwicz algorithm for problem $P^h_3$.

Let $\rho$ and $\alpha$ be two positive parameters.

Step 1. Choose two initial functions $u^0_h \in X_h$ and $p^0_h \in M_h$ such that:

$$
\begin{align*}
\delta h(\nabla u^0_h, \nabla v) + v(\nabla u^0_h, \nabla v) - (p^0_h, \text{div} v) = \langle f, v \rangle & \quad \forall v \in X_h, \\
\text{(div} u^0_h, q) = 0 & \quad \forall q \in M_h. \tag{4.28}
\end{align*}
$$

Step 2. For $n = 0, 1, \cdots$, if $u^n_h \in X_h$ and $p^n_h \in M_h$ are available, determine $u^{n+1}_h \in X_h$ and $p^{n+1}_h \in M_h$ by solving the following two defect equations sequentially:

$$
\begin{align*}
\begin{cases}
(\nabla u^{n+1}_h - u^n_h, \nabla v) + \rho \delta h(\nabla u^{n+1}_h, \nabla v) + \rho v(\nabla u^n_h, \nabla v) + \rho b(u^n_h; u^{n+1}_h, v) \\
- \rho (p^n_h, \text{div} v) = \rho \langle f, v \rangle & \quad \forall v \in X_h, \tag{4.29}
\end{cases}
\end{align*}
$$

$$
\begin{align*}
\begin{cases}
\alpha (p^{n+1}_h - p^n_h, q) + \rho (\text{div} u^{n+1}_h, q) = 0 & \quad \forall q \in M_h. \tag{4.30}
\end{cases}
\end{align*}
$$

**Theorem 4.3.** Let $(u_h, p_h) \in X_h \times M_h$ be the solution of problem $P^h_3$. Let $\{u_{nh}, p_{nh}\}$ be the function sequence generated by Algorithm 5. Set $\alpha_0 = \delta h + v, \alpha_1 = \delta h + v, \alpha_2 = 1$ and $\beta$ is the $\beta_2$ given in (4.15). Then, if $\Lambda := (\delta h + v)^{-2} |f||1 < 1$ and the condition $|1 - \rho v| + \rho (v + \delta h) \Lambda + \frac{\rho \delta h}{2a} < 1$ holds, we have

$$
\mathcal{E}(E^{n+1}_h) |1^2 + \alpha e^{n+1}_h |1^2 \leq \Lambda (\mathcal{E}(E^n_h) |1^2 + \alpha e^n_h |1^2),
$$

where $E^n_h := u^n_h - u_h, e^n_h := p^n_h - p_h$ and $\mathcal{E} \in (0, 1)$ and $\lambda \in (0, 1)$ are two generic constants independent of $n$ and $h$. 
The combination of the defect-correction method with the Arrow-Hurwicz algorithm for problem $P_2$, we finally obtain the following algorithm for solving problem $P_2$. In this case, we can not use the abstract theory (cf. Theorem 3.1) directly to develop related error analysis. It is our forthcoming issue to be studied.

Algorithm 6 The Arrow-Hurwicz algorithm for problem $P_2$.

Let $\rho$ and $\alpha$ be two positive parameters.

Step 1. Choose two initial functions $u^n_0 \in X_h$ and $p^n_0 \in M_h$ such that:

$$
\begin{align*}
\delta h(\nabla u^n_h, \nabla v) + \nu(\nabla u^n_h, \nabla v) - (p^n_h, \text{div } v) &= \langle f, v \rangle \quad \forall v \in X_h, \\
(\text{div } u^n_h, q) &= 0 \quad \forall q \in M_h.
\end{align*}
$$

(4.31)

Step 2. For $n = 0, 1, \cdots$, if $u^n_h \in X_h$ and $p^n_h \in M_h$ are available, determine $u^{n+1}_h \in X_h$ and $p^{n+1}_h \in M_h$ by solving the following two defect equations sequentially:

$$
\begin{align*}
\begin{cases}
\delta h(\nabla u^{n+1}_h, \nabla v) + \nu(\nabla u^{n+1}_h, \nabla v) + \rho \delta h(\nabla u^n_h, \nabla v) + \rho \nu(\nabla u^n_h, \nabla v) + \rho b(u^n_h, u^{n+1}_h, v) \\
- \rho (p^n_h, \text{div } v) = \rho \langle f, v \rangle \\
\alpha (p^{n+1}_h - p^n_h, q) + \rho (\text{div } u^{n+1}_h, q) = 0
\end{cases}
\forall v \in X_h. \quad (4.32)
\end{align*}
$$

Step 3. If $u^n_h \in X_h$ and $p^n_h \in M_h$ are obtained by the above iteration, determine $u^n_h \in X_h$ and $p^n_h \in M_h$ by solving the following two correction equations sequentially:

$$
\begin{align*}
\begin{cases}
\delta h(\nabla u^n_h, \nabla v) + \nu(\nabla u^n_h, \nabla v) + \rho \delta h(\nabla u^n_h, \nabla v) + \rho \nu(\nabla u^n_h, \nabla v) + \rho b(u^n_h, u^n_h, v) \\
- \rho (p^n_h, \text{div } v) = \rho \langle f, v \rangle + \rho \delta h(\nabla u^n_h, \nabla v) \\
\alpha (p^n_h - p^n_h, q) + \rho (\text{div } u^n_h, q) = 0
\end{cases}
\forall v \in X_h, \quad (4.34)
\end{align*}
$$

$$
\begin{align*}
\begin{cases}
\delta h(\nabla u^n_h, \nabla v) + \nu(\nabla u^n_h, \nabla v) + \rho \delta h(\nabla u^n_h, \nabla v) + \rho \nu(\nabla u^n_h, \nabla v) + \rho b(u^n_h, u^n_h, v) \\
- \rho (p^n_h, \text{div } v) = \rho \langle f, v \rangle + \rho \delta h(\nabla u^n_h, \nabla v) \\
\alpha (p^n_h - p^n_h, q) + \rho (\text{div } u^n_h, q) = 0
\end{cases}
\forall q \in M_h. \quad (4.35)
\end{align*}
$$

5 Numerical experiments

In order to illustrate computational performance of the generalized Arrow-Hurwicz method, we apply it to solve several nonlinear saddle-point problems from fluid computation, including the steady incompressible MHD equations, the steady incompressible Navier-Stokes equations discretized by the VMS method and the defect-correction method, discussed in the previous section. In our numerical simulation, the solution domain is always taken as $\Omega = (0, 1)^2$, which is triangulated by a family of uniform triangulations, and the pair of finite element spaces $(X_h, M_h)$ are taken as the Taylor-Hood element. Moreover, $W_h$ consists of continuous piecewise quadratic vector functions whose normal components vanish on $\partial \Omega$. Similar to [6], the stopping criterion during the iteration is that the relative error of consecutive approximate solutions for each physical variable in the maximum norm is no more than $10^{-6}$. The choice of parameters is very technical, shown in detail in each numerical example.
5.1 The MHD model with closed-form solutions

The MHD model has been given in Section 4.1, used to describe the interaction between a viscous, incompressible, electrically conducting fluid and an external magnetic field. We impose the body forces $f$ and $g$, so that the exact solutions of (4.1) are

\begin{align*}
  u_1(x,y) &= \pi \cos(\pi y) \sin(\pi x)^2 \sin(\pi y), \\
  u_2(x,y) &= -\pi \cos(\pi x) \sin(\pi y)^2, \\
  p(x,y) &= \cos(\pi x) \cos(\pi y), \\
  B_1(x,y) &= \cos(\pi y) \sin(\pi x), \\
  B_2(x,y) &= -\cos(\pi x) \sin(\pi y).
\end{align*}

Here, we set $R_e = 1$, $S_c = 1$ and $R_m = 1$.

We use Algorithm 2 to carry out numerical simulation. The parameters are chosen as $\rho = 1/2\nu$, $\alpha = 1/3\nu^2$ (cf. [6]). We display the CPU time, relative errors and the convergence orders of the generalized Arrow-Hurwicz method in Table 1 and Fig. 1, respectively. Here, $r$ stands for convergence order. From these numerical results, we may observe that the convergence order of solutions of the Arrow-Hurwicz method in $L^2$ and $H^1$ norms are in accordance with error estimates in theory. When the mesh size is taken as $h = \frac{1}{64}$, the log absolute errors of the numerical solutions in $H^1$-seminorm, $H^1$-seminorm and $L^2$ norm at different iteration steps are shown in Fig. 2, from which we can see that the convergence rates are almost the same for $u^*_h, B^*_h$ and $p^*_h$, but $u^*_h$ and $B^*_h$ achieve the desired results with much few iteration steps.

![Figure 1: $L^2$ errors vs $h$ in log-log scale for Algorithm 2.](image-url)
Furthermore, we can also demonstrate that the number of iterations for deriving desired solutions are uniformly bounded, independent of $h$, coinciding with the theoretical result in Theorem 4.1. Moreover, we choose $h = \frac{1}{25}, \frac{1}{50}, \frac{1}{100}$ and $\frac{1}{25}$ for numerical simulation, respectively. Comparisons of CPU times and iteration numbers with these meshes are given in Table 2, from which we may find that the method converges geometrically with a contraction number independent of the finite element mesh size $h$.

### 5.2 The Navier-Stokes equations with closed-form solutions

In this subsection, we discuss the numerical performance of Algorithm 3 and Algorithm 6 for solving steady incompressible Navier-Stokes equations. In this case, we choose $f$ in

---

**Table 1: CPU times, relative errors and convergence orders of Algorithm 2 with $R_c = 1$.**

| $h$  | CPU(s) | $\frac{|u-u_h|_1}{|u|_1}$ | $r$ | $\frac{|B-B_h|_1}{|B|_1}$ | $r$ | $\frac{\|p-p_h\|_0}{\|p\|_0}$ | $r$ |
|------|--------|--------------------------|-----|--------------------------|-----|--------------------------|-----|
| $\frac{1}{5}$ | 2.292  | 3.533e-02                | /   | 1.183e-02                | /   | 2.102e-02                | /   |
| $\frac{1}{10}$ | 8.341  | 1.138e-02                | 1.964 | 3.776e-03                | 1.985 | 3.944e-03                | 2.908 |
| $\frac{1}{25}$ | 29.087 | 4.688e-03                | 1.986 | 1.551e-03                | 1.993 | 1.395e-03                | 2.329 |
| $\frac{1}{50}$ | 77.260 | 2.266e-03                | 1.994 | 7.491e-04                | 1.996 | 6.472e-04                | 2.106 |
| $\frac{1}{100}$ | 164.671 | 1.244e-03              | 1.997 | 4.047e-04                | 1.998 | 3.453e-04                | 2.038 |
| $\frac{1}{200}$ | 450.575 | 7.179e-04              | 1.998 | 2.373e-04                | 1.998 | 2.015e-04                | 2.016 |
Table 2: Errors, CPU times and iteration numbers of Algorithm 2 with different meshes with $R_e = 1$.

| Mesh Size  | $|u - u_h|_{L^1}$ | $|B - B_h|_{L^1}$ | $\|p - p_h\|_{L^0}$ | CPU(s) | Iter. |
|------------|------------------|------------------|------------------|-------|-------|
| Level 2 1/15 | 0.00468771 | 1.551e-03 | 1.395e-03 | 34.963 | 41 |
| Level 3 1/30 | 0.00226569 | 7.491e-04 | 6.472e-04 | 89.593 | 42 |
| Level 4 1/45 | 0.00122415 | 4.047e-04 | 3.453e-04 | 168.646 | 37 |
| Level 5 1/60 | 0.00071791 | 2.373e-04 | 2.015e-04 | 495.775 | 44 |

Figure 3: Error analysis for Algorithms 3 and 6. (a): $H^1$ errors vs $h$ in log-log scale for the velocity; (b): $L^2$ errors vs $h$ in log-log scale for the pressure.

(4.2) such that the exact solution for $u = (u_1, u_2)^T$ and the pressure $p$ are given by

$$
\begin{cases}
    p(x,y) = 10(2x-1)(2y-1), \\
    u_1(x,y) = 10x^2(x-1)^2y(y-1)(2y-1), \\
    u_2(x,y) = -10x(x-1)(2x-1)y^2(y-1)^2.
\end{cases}
$$

(5.1)

For implementing Algorithms 3 and 6, we choose $\nu = 1$, $\rho = 1/2\nu$, $\alpha = 1/3\nu^2$, while the mesh size $h$ is taken as 1/5, 1/10, 1/15, 1/20, 1/25 and 1/30, respectively. The numerical results are given in Fig. 3, Tables 3 and 4, respectively.

From Fig. 3, we may see our two methods predict a convergence rate of order $O(h^2)$ for velocity in the $H^1$-norm and $O(h^2)$ for pressure in the $L^2$-norm, which agree with the theoretical analysis in Section 3.

Moreover, we study the effect of meshes on the two methods under discussion. Choose $h = 1/15$, $1/20$ and $1/25$, respectively. The numerical results are shown in Tables 3 and 4, which coincide with our theoretical results in Theorems 4.2 and 4.3.
Table 3: Errors, CPU times and iteration numbers of Algorithm 3 with different meshes with $\nu = 1$.

| Mesh Size | $h$ | $|u - u_h|_{1}$ | $\|p - p_h\|_0$ | CPU(s) | Iter |
|-----------|-----|----------------|-----------------|--------|------|
| Level 1   | $\frac{1}{16}$ | 1.142e-02 | 3.043e-02 | 1.263  | 32   |
| Level 2   | $\frac{1}{64}$ | 8.196e-04 | 3.698e-04 | 19.712 | 29   |
| Level 3   | $\frac{1}{128}$ | 2.951e-04 | 3.202e-04 | 177.048| 31   |

Table 4: Errors, CPU times and iteration numbers of Algorithm 6 with different meshes with $\nu = 1$.

| Mesh Size | $h$ | $|u - u_h|_{1}$ | $\|p - p_h\|_0$ | CPU(s) | Iter |
|-----------|-----|----------------|-----------------|--------|------|
| Level 1   | $\frac{1}{16}$ | 1.145e-02 | 3.043e-02 | 1.334  | 33   |
| Level 2   | $\frac{1}{64}$ | 7.798e-04 | 3.716e-04 | 21.041 | 30   |
| Level 3   | $\frac{1}{128}$ | 2.448e-04 | 3.191e-04 | 212.572| 35   |

5.3 The lid-driven cavity problem

Finally, we test a popular benchmark problem—the lid-driven problem, which was studied in [13]. This problem is chosen because some benchmark data is available for comparison. In this case, the computation is carried out in the domain $\Omega = (0,1) \times (0,1)$ as before, while the boundary conditions for the velocity on the top edge is $u = (1,0)^T$ and no-slip condition is applied on the other edges. The body force is taken as $f = 0$. In numerical simulation, we always take $\eta = 0.1h$ and $\delta = 0.01$, but the parameters $\alpha$, $\rho$ and mesh sizes are changing for different viscosity coefficients.

We use Algorithm 3 and Algorithm 6 to achieve the numerical results, with the parameters given by $\nu = 10^{-3}$ (or equivalently, $Re = 10^3$), $\alpha = 0.8$ and $\rho = \frac{7}{15}$, while $h = \frac{1}{30}$. The computed results are shown in Fig. 4 and Fig. 5, respectively. Concretely speaking, we present the horizontal velocity distribution at $x = 0.5$ and the vertical distribution at $y = 0.5$ of the two algorithms in Fig. 4. The isobars and stream lines are shown in Fig. 5. Compared with the numerical results obtained by Ghia et al. (cf. [13]), we find that they are in excellent agreement. Moreover, we can observe the same physical phenomenon from Fig. 5 as that given in [13], that means, the position of the main vortex moves towards the center of cavity when the viscosity decreases, and additionally second vortex may appear in the right bottom corner of the cavity and a third vortex appears at the left bottom corner. All these results demonstrate that the methods are very stable and effective for solving this lid-driven cavity problem with a smaller viscosity coefficient.

Specially, we choose $\nu = 0.01$, $\alpha = 1.2$ and $\rho = \frac{7}{10}$ for solving the lid-driven cavity problem with three different uniform meshes. Comparisons of CPU times and iteration numbers are given in Tables 5 and 6, respectively. These numerical results indicate that the iteration numbers to reach the required accuracy are uniformly bounded from above with respect to the mesh sizes, coinciding with our theoretical analysis.
Figure 4: $u_1$-vertical velocity (a) and $u_2$-horizontal velocity (b) with $\nu = 10^{-3}$ by using Algorithms 3 and 6 for the lid-driven cavity problem.

Figure 5: Isobars (top) and streamlines (bottom) obtained by Algorithms 3 and 6 for the lid-driven problem, respectively.
Table 5: CPU times and iteration numbers of Algorithm 3 with $\nu = 0.01$

<table>
<thead>
<tr>
<th>Mesh Size $h$</th>
<th>CPU(s)</th>
<th>Iter.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Level 1 1/16</td>
<td>72.683</td>
<td>1189</td>
</tr>
<tr>
<td>Level 2 1/32</td>
<td>493.374</td>
<td>1108</td>
</tr>
<tr>
<td>Level 3 1/64</td>
<td>13030.6</td>
<td>1326</td>
</tr>
</tbody>
</table>

Table 6: CPU times and iteration numbers of Algorithm 6 with $\nu = 0.01$

<table>
<thead>
<tr>
<th>Mesh Size $h$</th>
<th>CPU(s)</th>
<th>Iter.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Level 1 1/16</td>
<td>80.375</td>
<td>1108</td>
</tr>
<tr>
<td>Level 2 1/32</td>
<td>475.467</td>
<td>1015</td>
</tr>
<tr>
<td>Level 3 1/64</td>
<td>13296.9</td>
<td>1323</td>
</tr>
</tbody>
</table>

6 Conclusions

This paper aims to systematically study a class of nonlinear saddle-point problems, which are formulated as problem $P$, an abstract frame, in the introduction part. Under some reasonable conditions, we first show problem $P$ has a unique solution. The key idea is to reformulate it as a primal problem of $u$, and then show the latter problem has a unique solution by means of the Banach contraction mapping theorem. Next, we propose and analyze an Arrow-Hurwicz method for solving problem $P$, following some ideas in [6]. To show the power of this abstract framework, we apply it to study three specific nonlinear saddle-point problems, including mathematical model and its numerical method for the steady incompressible magnetohydrodynamic equations, the variational multi-scale (VMS) method and the defect-correction method for numerically solving the steady incompressible Navier-Stokes equations. For all the previous discrete methods, we devise the related Arrow-Hurwicz methods to solve these problems and develop their convergence rate analysis. A series of numerical results are reported to show computational performance of the generalized Arrow-Hurwicz method.

On the other hand, in the past two decades many researchers have also developed discontinuous Galerkin methods and weak Galerkin methods for approximating incompressible Navier-Stokes equations with high Reynolds numbers effectively (cf. [7, 8, 21–23]). Therefore, it is very valuable for us to propose and analyze the related generalized Arrow-Hurwicz methods for the preceding discrete problems. This is our forthcoming issue under investigation.

Acknowledgments

This work was supported by NSFC (Grant no. 11571237). The authors thank the referees for valuable suggestions, which greatly improved the presentation of an early version of the paper.
References