Superconvergence for Triangular Linear Edge Elements

Chao Wu¹,*, Yunqing Huang², Wenying Lu¹, Zhiguang Xiong¹ and Jinyun Yuan³

¹ School of Mathematics and Computational Science, Hunan University of Science and Technology, China.
² Hunan Key Laboratory for Computation and Simulation in Science and Engineering and School of Mathematics and Computational Science, Xiangtan University, Xiangtan, China.
³ Departamento de Matemática, Universidade Federal do Paraná, Centro Politécnico, Curitiba, Brazil.

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Abstract. Superconvergence for the lowest-order edge finite elements on strongly regular triangulation is studied. By the averaging technique, superconvergence of order $O(h^2)$ is established at the midpoint of the interior edge for both the finite element solution and the curl of the finite element solution. Numerical results justifying our theoretical analysis are presented.

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1 Introduction

Superconvergence of finite element methods (FEMs) has been an active research topic due to its strong relevance with a posteriori error estimations for the adaptive finite element method and most of the interest was devoted to elliptic and parabolic equations, see for example the surveys [2,4,7,15,25,26] and the monographs [5,9]. In regards to edge elements and their applications to Maxwell’s equations, the superconvergence are limited. The first superconvergence result is due to Monk’s 1994 work [23] for time-dependent Maxwell system. The integral identity technique was applied by Lin and Yan [20] to deal with the same problem once more. One order of superconvergent factor was obtained by them for $k$-th Nédélec elements on cubic meshes, which improved the result in [23].
For 2-D Maxwell’s equations, it was demonstrated that similar result remains true, see- 
ing Lin [19] for the lowest order rectangular edge elements, Lin [17] for the second order 
rectangular edge elements and Brandts [3] for $k$-th triangular edge elements. Brandts’ 
result was refined by Lin [18] in 2003: if the domain is rectangular, two order of super-
convergent factor was derived for the $k$-th ($k \geq 1$) Nédélec elements on the rectangular 
meshes. The Maxwell’s equations in the aforementioned surveys were investigated in 
vacuum and the case of complicated medium can be also found in the literature, see for 
instance [16] where three popular dispersive media were considered. Note that the above 
mentioned results are all global superconvergence. Recently, superconvergence results at 
some special points are established by Huang and Li: the cubic center [14], the rectangu-
lar center [11], the midpoint of interior edge for uniform triangular mesh [13] (in these 
papers, the mixed FEM is used) and [10] (where the obtained superconvergence by FEM 
is utilized to construct an adaptive FEM method for cloaking simulation) and the mid-
point of interior edge for uniform tetrahedral mesh [12]. For recent progress, we refer 
to the work of Chen [6], Chung [8] and Qiao [24]. A new Hybridizable Discontinuous 
Galerkin (HDG) method, the Staggered Discontinuous Galerkin (SDG) method and the 
nonuniform mixed FEM were exploited, respectively.

The superconvergence analysis on strongly regular mesh is known to be much more 
complicated than that on uniform mesh. It has been shown by Chen [4] that there ex-
stists superconvergence for elliptic equations on strongly regular mesh. The main goal of 
this paper is to transfer the superconvergence result in [4] to Maxwell’s equations. Pi-
ola transformation [21], which is the covariant transformation for vector-fields, plays an 
essential role in our analysis.

We focus our analysis on time harmonic Maxwell’s equations [10, 12, 22]:
\[
\begin{align*}
\nabla \times \nabla \times u - \kappa_0^2 u &= f \quad \text{in} \quad \Omega, \\
n \times u &= 0 \quad \text{on} \quad \partial \Omega.
\end{align*}
\]

Here, $u$ represents the electric field in $\Omega$, a Lipschitz polyhedron in $\mathbb{R}^2$. $f$ stands for a 
given function related to the imposed current sources, which is assumed to be smooth 
ENO. $\kappa_0$ indicates the wavenumber assumed to be real and positive and $n$ denotes the 
outward unit normal vector field. Eq. (1.2) specifies a standard perfectly conducting 
boundary condition on the boundary of $\Omega$. Let $s \geq 0$. To obtain the weak formulation of 
(1.1)-(1.2), we introduce the following Sobolev spaces
\[
\begin{align*}
H(curl; \Omega) &= \{ v \in (L^2(\Omega))^2; \nabla \times v \in L^2(\Omega) \}, \\
H_0(curl; \Omega) &= \{ v \in H(curl; \Omega); n \times v = 0 \quad \text{on} \quad \partial \Omega \}, \\
H^s(curl; \Omega) &= \{ v \in (H^s(\Omega))^2; \nabla \times v \in H^s(\Omega) \}
\end{align*}
\]
equipped with the norms
\[
\begin{align*}
||v||_{H(curl; \Omega)} &= (||v||_{0, \Omega}^2 + ||\nabla \times v||_{0, \Omega}^2)^{1/2}, \\
||v||_{H^s(curl; \Omega)} &= (||v||_{H^s(\Omega)}^2 + ||\nabla \times v||_{H^s(\Omega)}^2)^{1/2},
\end{align*}
\]
Let $(\cdot , \cdot )$ be the standard $L^2(\Omega )$ inner product. Multiplying (1.1) by a function $v \in H_0(curl; \Omega )$ and integrating the curl term by parts, we obtain the variational formulation of Maxwell’s equations: find $u \in H_0(curl; \Omega )$ such that

$$a(u, v) := (\nabla \times u, \nabla \times v) - \kappa_0^2(u, v) = (f, v), \quad \forall v \in H_0(curl; \Omega ).$$

(1.3)

It is well-known that Eq. (1.3) is well-posed [21], meaning that the problem has a unique solution unless $\kappa_0$ is an interior Maxwell eigenvalue for $\Omega$.

To solve (1.3) by finite element method (FEM), we assume that $\Omega$ is partitioned by a family of strongly regular triangles $\Gamma_h$ with mesh size $h$. Its definition [4, 5] is as follows:

**Definition 1.1.** A triangulation $\Gamma_h$ is strongly regular if any two adjacent triangles of $\Gamma_h$ form an $O(h^2)$ approximate parallelogram, meaning that the lengths of any two opposite edges differ by $O(h^2)$.

The finite element space for solving (1.3) are built by using piecewise polynomial functions. Let $P_k$ denote the standard space of polynomials of total degree less than or equal to $k$, and let $P_k$ denote homogeneous polynomials of total degree exactly $k$. Now we define $S^1 \subset (P_1)^2$ and $R^1 \subset (P_1)^2$ by

$$S^1 := \{ p \in (P_1)^2 : p \cdot x = 0 \}, \quad R^1 := (P_0)^2 \oplus S^1. $$

Following [21], we introduce the following Nédélec’s first type elements $V^1_h$ and Nédélec’s second type elements $V^2_h$:

$$V^1_h := \{ v_h \in H(curl; \Omega ) : v_h |_K \in R^1 \text{ for all } K \in \Gamma_h \},$$

$$V^2_h := \{ v_h \in H(curl; \Omega ) : v_h |_K \in (P_1)^2 \text{ for all } K \in \Gamma_h \}. $$

Note that the basis functions in $V^1_h$ consist of

$$\phi^1_k = \lambda_{k+1} \nabla \lambda_{k-1} - \lambda_{k-1} \nabla \lambda_{k+1}, \quad k = 1, 2, 3,$$

(1.4)

where $\lambda_k$ is the barycentric coordinate at the $k$-th vertex of element $K$ and $k \pm 1$ permuted cyclically. It is known that the basis functions in $V^2_h$ are formed by $\phi^1_k$ and

$$\phi^2_k = \lambda_{k+1} \nabla \lambda_{k-1} + \lambda_{k-1} \nabla \lambda_{k+1}. $$

(1.5)

To approximate the functions in $H_0(curl; \Omega )$, we define

$$V^1_{h,0} := \{ v_h \in V^1_h \mid v_h \times n = 0 \text{ on } \partial \Omega \}. $$

(1.6)

With the above preparations, the FEM [21] for solving (1.3) is: Find $u_h \in V^1_{h,0}$ such that

$$a(u_h, v_h) = (f, v_h), \quad \forall v_h \in V^1_{h,0}. $$

(1.7)

The optimal convergence rate and some fundamental error estimates have been obtained by Monk:
Theorem 1.1. [21, Theorem 7.1] Let \( u \) be the weak solution of (1.3) and \( u_h \in V_h^1 \) the FEM solution of (1.7). Denote \( e_h = u - u_h \). Assume that the wavenumber \( \kappa_0 \) is not an eigenvalue of Maxwell’s equations (1.1). Then there exists a constant \( C > 0 \) (independent of \( h \)) such that

\[
\| e_h \|_{H(\text{curl}; \Omega)} \leq C h \| u \|_{H(\text{curl}; \Omega)}.
\]

Lemma 1.1. [21, Lemma 7.7] Let \( e_h \) be as in Theorem 1.1. For small enough \( h \), there exists a constant \( \delta \in (0, \frac{1}{2}) \) (independent of \( h \)) such that

\[
\sup_{\nu_h \in V_h} \left( \frac{\| (e_h, \nu_h) \|_{H(\text{curl}; \Omega)}}{\| \nu_h \|_{H(\text{curl}; \Omega)}} \right) \leq C h^{\delta + \frac{1}{2}} \| e_h \|_{H(\text{curl}; \Omega)}.
\]

The rest of the paper is organized as follows. In Section 2, some preliminaries are presented. And then the superconvergence at the middle point of interior edge for both Nédélec’s interpolation and its curl are deduced. In Section 3, we use the superconvergence at the middle point of interior edge to derive the superconvergence for both the solution of (1.7) and its curl. The illustrative numerical examples justifying our analysis are presented in Section 4.

2 Analysis in elements

In this section, we derive some basic knowledge in the reference element \( \hat{K} \) shown in Fig. 1 (left), the general element \( K \) with vertices \( A_k \ (k=1,2,3) \) and a patch \( \omega = K \cup K' \) shown in Fig. 1 (right). And we use these basic analysis to prove that the Nédélec’s interpolation has one order higher convergence rates at the midpoint of interior edge in \( L^\infty \) norm for triangular linear edge elements (1.4) when strongly regular mesh is involved.

2.1 Preliminary lemmas in \( \hat{K} \)

In this subsection, we present some preliminary lemmas in \( \hat{K} \). As shown in Fig. 1 (left), let \( \hat{K} \) have vertices \( \hat{A}_1(1,0), \hat{A}_2(0,1), \hat{A}_3(0,0) \) oriented counterclockwise and edges \( \{ \hat{e}_k \}_{k=1}^3 \). Let \( \hat{\lambda} \) be the corresponding barycentric coordinate related to \( \hat{A}_k \). Let us introduce some geometric parameters \( \{ \hat{\xi}_k \}_{k=1}^3 \) and \( \{ \hat{\eta}_k \}_{k=1}^3 \):

\[
\hat{\xi}_k = \hat{x}_{k+1} - \hat{x}_{k-1}, \quad \hat{\eta}_k = \hat{y}_{k+1} - \hat{y}_{k-1},
\]

where \( 1 \leq k \leq 3 \) and \( k \pm 1 \) permuted cyclically. Denote

\[
\Lambda^\hat{\xi}_k = \hat{\lambda}_{k+1} \hat{\xi}_{k-1} + \hat{\lambda}_{k-1} \hat{\xi}_{k+1}, \quad \Lambda^\hat{\eta}_k = \hat{\lambda}_{k+1} \hat{\eta}_{k-1} + \hat{\lambda}_{k-1} \hat{\eta}_{k+1}.
\]

Now, let us give the expression of basis functions \( \phi^1_k \) and \( \phi^2_k \) (similar to \( \phi^1_k \) and \( \phi^2_k \) in any element \( K \)) on the edge \( \hat{e}_k \in \hat{K} \ (k=1,2,3) \) by using the above notations.
Proof. For the basis functions $\phi^1_k$ and $\phi^2_k$, we can obtain

$$\phi^1_k = \begin{bmatrix} \lambda_{k+1} \eta_{k-1} - \lambda_k \eta_k + \lambda_{k+2} \xi_{k-2} + \lambda_k \xi_{k+2} \\ -\lambda_{k+1} \xi_{k-2} - \lambda_k \xi_{k+2} \end{bmatrix}, \quad \phi^2_k = \begin{bmatrix} \lambda^2_k \\ -\lambda_k^2 \end{bmatrix}.$$  

Remark 2.1. We can also describe $\phi^1_k$ as follows:

$$\phi^1_k = \begin{bmatrix} -\hat{y} \\ -\hat{x} \end{bmatrix}, \quad \phi^2_k = \begin{bmatrix} 1 - \hat{y} \\ 1 - \hat{x} \end{bmatrix}, \quad \phi^3_k = \begin{bmatrix} -\hat{y} \\ -\hat{x} \end{bmatrix}.$$

Let $\hat{O}(\hat{x}_0, \hat{y}_0)$ be the middle point of edge $\hat{e}_1$. Let us denote

$$\hat{y}_{k+1} = \frac{\hat{y}_{k+1} + \hat{y}_k + 2 \hat{y}_{k+1} - 4 \hat{y}_0}{4!}, \quad \hat{y}_{k-1} = \frac{2 \hat{y}_{k-1} + \hat{y}_k + \hat{y}_{k+1} - 4 \hat{y}_0}{4!},$$

$$\hat{x}_{k+1} = \frac{\hat{x}_{k+1} + \hat{x}_k + 2 \hat{x}_{k+1} - 4 \hat{x}_0}{4!}, \quad \hat{x}_{k-1} = \frac{2 \hat{x}_{k-1} + \hat{x}_k + \hat{x}_{k+1} - 4 \hat{x}_0}{4!}.$$  

Now, we present some integrals in $\hat{K}$.

Lemma 2.2. Let $\hat{f}$ be a function defined in $\hat{K}$. Then

$$\int_{\hat{K}} \hat{f} \ast \hat{\lambda}_{k-1} \ d\hat{K} = \begin{cases} \hat{X}_{k-1}, & \hat{f} = \hat{x} - \hat{x}_0, \\ \hat{y}_{k-1}, & \hat{f} = \hat{y} - \hat{y}_0, \end{cases}$$

$$\int_{\hat{K}} \hat{f} \ast \hat{\lambda}_{k+1} \ d\hat{K} = \begin{cases} \hat{X}_{k+1}, & \hat{f} = \hat{x} - \hat{x}_0, \\ \hat{y}_{k+1}, & \hat{f} = \hat{y} - \hat{y}_0. \end{cases}$$
Proof. Let us prove \( \hat{f} = \hat{x} - \hat{x}_0 \) and \( k = 2 \) as an example. Then
\[
\int_K (\hat{x} - \hat{x}_0) \hat{\lambda}_1 \, d\hat{K} = \int_K (\hat{x} - 0) \hat{x} \, d\hat{K} = \frac{1}{12} = \hat{X}_{2-1}.
\]
The others can be obtained similarly.

\[\square\]

2.2 Preliminary lemmas in \( K \)

In this subsection, we state some simple analysis in \( K \). As shown in Fig. 1 (right), \( K \) has vertices \( A_k(x_k, y_k) \) \((k = 1, 2, 3)\), oriented counterclockwise, and corresponding local node index \( k \). Let \( \{e_k\}_{k=1}^3 \) denote the edges of element \( K \), \( \{l_k\}_{k=1}^3 \) the edge lengths and \( \{t_k\}_{k=1}^3 \) the unit tangent vectors. Let \( O(x_0, y_0) \) be the middle point of edge \( e_1 \). Let \( |K| \) denote the area of triangle \( K \). Let us introduce some geometry parameters
\[
\xi_k = x_{k+1} - x_{k-1}, \quad \eta_k = y_{k+1} - y_{k-1}. \tag{2.3}
\]
Denote
\[
M_k^x = \left( \frac{x_{k+1} - x_k}{2!}, \frac{x_{k+1} - x_0}{2!} \right), \quad N_k^x = \frac{\xi_k}{6} + \frac{1}{2}(x_{k-1} - x_0)(x_{k+1} - x_0),
\]
\[
M_k^y = \left( \frac{y_{k+1} - y_k}{2!}, \frac{y_{k+1} - y_0}{2!} \right), \quad N_k^y = \frac{\eta_k}{6} + \frac{1}{2}(y_{k-1} - y_0)(y_{k+1} - y_0),
\]
\[
H_k = -\frac{\xi_k \eta_k}{3} + \frac{1}{2}(x_{k+1} - x_0)(y_{k+1} + y_{k-1}) - \frac{y_0}{2}(x_{k-1} + x_{k+1} - 2x_0).
\]
Moreover, the following quadrature formulas [5]
\[
\int_{e_k} \lambda_k^\alpha \lambda_{k-1}^\beta \, ds = \frac{\alpha! \beta! l_k}{(\alpha + \beta + 1)!}, \quad \forall e_k \in K, \tag{2.5}
\]
\[
\int_{K} \lambda_k^\alpha \lambda_j^\beta \lambda_l^\gamma \, dK = \frac{\alpha! \beta! \gamma!}{(\alpha + \beta + \gamma + 1)!} |K|, \tag{2.6}
\]
hold in any \( K \). In Lemma 2.3, we collect some simple identities.

Lemma 2.3. Let \( f \) and \( g \) be functions defined in \( K \), then
\[
\int_{e_k} f (\lambda_k^\alpha - \lambda_{k-1}^\alpha) \, ds = \begin{cases} \frac{\lambda_k^\alpha}{3}, & f = x - x_0, \\ \frac{\lambda_k^\alpha}{3}, & f = y - y_0, \end{cases} \tag{2.7}
\]
\[
\int_{e_k} f \, ds = \begin{cases} M_k^x \ast l_k, & f = x - x_0, \\ M_k^y \ast l_k, & f = y - y_0, \end{cases} \tag{2.8}
\]
\[
\int_{e_k} g \, ds = \begin{cases} N_k^x \ast l_k, & g = \frac{1}{2}(x - x_0)^2, \\ N_k^y \ast l_k, & g = \frac{1}{2}(y - y_0)^2, \\ H_k \ast l_k, & g = (x - x_0)(y - y_0). \end{cases} \tag{2.9}
\]
Proof. One of identities in (2.9) follows from (2.5) and the fact \( x = x_{k-1} \lambda_{k-1} + x_{k+1} \lambda_{k+1} \) on \( e_k \):

\[
\int_{e_k} \frac{1}{2} (x - x_0)^2 \, ds = \int_{e_k} \frac{1}{2} (x_{k-1} \lambda_{k-1} + x_{k+1} \lambda_{k+1} - x_0)^2 \, ds
\]

\[
= \frac{1}{2} \int_{e_k} \left[ x_{k-1}^2 \lambda_{k-1}^2 + 2x_{k-1}x_{k+1} \lambda_{k-1} \lambda_{k+1} + x_{k+1}^2 \lambda_{k+1}^2 + 2x_0(x_{k-1} \lambda_{k-1} + x_{k+1} \lambda_{k+1}) + x_0^2 \right] \, ds
\]

\[
= \left[ \frac{x_{k-1} - x_{k+1}}{2} + \frac{x_0}{2} (x_{k-1} + x_{k+1}) \right] * l_k.
\]

All the others follow from the same pattern. \( \square \)

Now for any \( u = (u_1, u_2) \in H(curl; K) \), we need to define the corresponding Nédélec interpolations \( \Pi^l_h u \in V^l_h \) \((l = 1, 2)\). The first type Nédélec interpolation function \( \Pi^1_h u \in V^1_h \) satisfies

\[
\int_{e_k} (u - \Pi^1_h u) \cdot t_k = 0, \quad k = 1, 2, 3.
\]

(2.10)

The second type Nédélec interpolation function \( \Pi^2_h u \in V^2_h \) satisfies

\[
\int_{e_k} (u - \Pi^2_h u) \cdot t_k q = 0, \quad \forall q \in P_1(e_k), \quad k = 1, 2, 3.
\]

(2.11)

We refer to [21, Definition 5.33 and Definition 8.9] for the details on the two types of interpolation. Note that they were originally designed in the setting of 3-D space and it can be shown readily that it holds true for 2-D case. In Lemma 2.4, we present some identities for barycentric coordinates that are used in the proof of Lemma 2.5.

**Lemma 2.4.** Let \( \lambda_n \) be as in (1.4) and \( \tau_k = \overrightarrow{A_{k+1}A_{k-1}} \). Then

\[
\nabla \lambda_n \cdot \tau_k = \begin{cases} 
-1, & n = k+1, \\
1, & n = k-1, \\
0, & n = k.
\end{cases}
\]

**Proof.** Let us just prove the case \( k = 2 \) and \( n = k - 1 \) as an example.

\[
\nabla \lambda_1 \cdot \tau_2 = \frac{1}{2|K|} \left[ \begin{array}{c} \eta_1 \\
-\xi_2 \\
\end{array} \right] \cdot \left[ \begin{array}{c} -\xi_2 \\
-\eta_2 \\
\end{array} \right] = 1.
\]

The other cases can be proved similarly. \( \square \)
Let $\Pi_1^h u$ and $\Pi_2^h u$ be as defined in (2.10) and (2.11) respectively. Now, we present the relation between them.

**Lemma 2.5.** Let $\phi_1^k$ and $\phi_2^k$ be as in (1.4) and (1.5), respectively. For any function $u \in H(curl;K)$. The Nédélec interpolation can be written as

\[
\text{(i)} \quad \Pi_1^h u = \sum_{k=1}^{3} u_k^1 \phi_1^k, \quad \Pi_2^h u = \sum_{k=1}^{3} u_k^2 \phi_2^k,
\]

where the degrees of freedom (DoFs) are

\[
u_k^1 = \frac{1}{l_k} \int_{e_k} u \tau_k ds, \quad \nu_k^2 = \frac{3}{l_k} \int_{e_k} u \tau_k (\lambda_{k+1} - \lambda_{k-1}) ds, \quad \tau_k = \overrightarrow{A_{k+1} A_{k-1}}.
\]

**Proof.** (i) By definition, $\Pi_1^h u$ can be written in the following form:

\[
\Pi_1^h u = \sum_{k=1}^{3} c_k^1 \phi_1^k,
\]

where $c_k^1$ are the unknown DoFs. It follows from Lemma 2.4 that

\[
\Pi_1^h u \cdot \tau_1 = c_1^1 (\lambda_2 + \lambda_3) - c_2^1 \lambda_1 - c_3^1 \lambda_1.
\]

It is clear that $\lambda_2 + \lambda_3 = 1$ and $\lambda_1 = 0$ on $e_1$. This combined with (2.10) yields that

\[
\frac{1}{l_1} \int_{e_1} u \cdot \tau_1 ds = \frac{1}{l_1} \int_{e_1} \Pi_1^h u \cdot \tau_1 ds = \frac{1}{l_1} \int_{e_1} c_1^1 ds = c_1^1.
\]

Thus, $c_1^1 = u_1^1$. Similarly, we obtain that $c_k^1 = u_k^1 (k = 2,3)$.

(ii) By definition, $\Pi_2^h u$ can be written in the following form:

\[
\Pi_2^h u = \sum_{k=1}^{3} d_k^1 \phi_1^k + \sum_{k=1}^{3} d_k^2 \phi_2^k,
\]

where $d_k^1, d_k^2$ are DoFs needed to be obtained. Now Lemma 2.4 and (2.11) gives

\[
\frac{1}{l_1} \int_{e_1} \Pi_2^h u \cdot \tau_1 q ds = \frac{d_1^1}{l_1} \int_{e_1} \phi_1^1 \cdot \tau_1 q ds + \frac{d_2^1}{l_1} \int_{e_1} \phi_2^1 \cdot \tau_1 q ds + \frac{d_3^1}{l_1} \int_{e_1} \phi_3^1 \cdot \tau_1 q ds.
\]

Note that the function $q \in P_1 (e_1)$ (linear function on edge $e_1$) can be written as $q = a \lambda_2 + b \lambda_3$ ($a, b$ are unknown constants). Substituting $q$ into (2.13), we deduce that

\[
\frac{1}{l_1} \int_{e_1} \phi_1^1 \cdot \tau_1 q ds = \frac{a + b}{2}, \quad \frac{1}{l_1} \int_{e_1} \phi_2^1 \cdot \tau_1 q ds = \frac{a - b}{6}.
\]

First, choosing $\frac{a + b}{2} = 1$ and $\frac{a - b}{6} = 0$, then $a = b = 1$, thus $d_1^1 = u_1^1$. Next, choosing $\frac{a + b}{2} = 0$ and $\frac{a - b}{6} = 1$, we obtain $a = -b = 3$, then $d_2^1 = u_2^1$. Similarly, $d_k^1 = u_k^1$ and $d_k^2 = u_k^2 (k = 2,3)$. □
Now we present the error estimate for the second type Nédélec interpolation $\Pi_2^h u$ shown by Monk in [21, Theorem 8.15 ($k = 1$)]:

**Lemma 2.6.** Assume $u \in (H^2(\Omega))^2$. Then we have

$$||u - \Pi_2^h u||_0 + ||\nabla \times (u - \Pi_2^h u)||_0 \leq Ch^2||u||_{(H^2(\Omega))^2}.$$  

Let $n = 1, 2$. Let $M_i^x, M_i^y, N_i^x, N_i^y, H_k$ be as in (2.4). Denote

$$A_n = u_n(O) + M_i^x \frac{\partial u_n}{\partial x}(O) + M_i^y \frac{\partial u_n}{\partial y}(O),$$

$$C_n = N_i^x \frac{\partial^2 u_n}{\partial x^2}(O) + H_k \frac{\partial^2 u_n}{\partial x \partial y}(O) + N_i^y \frac{\partial^2 u_n}{\partial y^2}(O),$$

$$(2.15)$$

$$D_n = \frac{\partial^2 u_n}{\partial x(O)} \frac{\zeta_k}{2} + \frac{\partial^2 u_n}{\partial y(O)} \frac{\eta_k}{2}.$$  

Now let us present another expression of $u_1^k$ and $u_2^k$ in (2.12).

**Lemma 2.7.** Let $u_1^k$ be as in (2.12). Assume $u \in (C^3(\Omega))^2$. Then it holds that

(i) $u_1^k = -\zeta_k A_1^k - \eta_k A_2^k + O(h^3)||\nabla^2 u||_\infty,$

(ii) $u_1^k = -\zeta_k A_1^k - \eta_k A_2^k - \zeta_k C_1^k - \eta_k C_2^k + O(h^4)||\nabla^3 u||_\infty.$

**Proof.** (i) From (2.12), we know

$$u_1^k = \frac{1}{l_k} \int_{\xi_k^1} u \cdot \tau_k \, ds = -\zeta_k \int_{\xi_k^1} u_1 \, ds + \frac{\eta_k}{l_k} \int_{\xi_k^2} u_2 \, ds.$$  

By the 1-th order Taylor expansion of $u$ at $O$ and Lemma 2.3, we conclude (i).

(ii) The 2-th order Taylor expansion of $u$ at $O$ together with Lemma 2.3 derive it. $\square$

**Lemma 2.8.** Let $u_2^k$ be as in (2.12). Assume $u \in (C^2(\Omega))^2$, Then

$$u_2^k = -\zeta_k D_1^k - \eta_k D_2^k + O(h^3)||\nabla^2 u||_\infty.$$  

**Proof.** From (2.12), we obtain

$$u_2^k = \frac{3}{l_k} \int_{\xi_k} u \cdot \tau_k * (\lambda_{k+1} - \lambda_{k-1}) \, ds$$

$$= -\frac{3\zeta_k}{l_k} \int_{\xi_k} u_1 * (\lambda_{k+1} - \lambda_{k-1}) \, ds - \frac{3\eta_k}{l_k} \int_{\xi_k} u_2 * (\lambda_{k+1} - \lambda_{k-1}) \, ds.$$  

Lemma 2.3 and the 1-th order Taylor expansion of $u$ at $O$ conclude the lemma. $\square$
To derive the basis functions on general triangle $K$, we introduce the affine mapping $F: \hat{K} \rightarrow K$, which is given by $x = B_K \hat{x} + b_K$, for $\hat{x} \in \hat{K}$. Note that the Nédeléc basis function $\phi$ can be transformed by Piola transformation [21]: $\phi = B_K^{-T} \cdot \eta F^{-1}$, which is not different from the basis function in standard $H^1(\Omega)$-conforming FEM. It is easy to calculate that

$$B_K^{-T} = \frac{1}{2 |K|} \begin{bmatrix} \eta_1 & \eta_2 \\ -\xi_1 & -\xi_2 \end{bmatrix}. $$

For clarity, we use it clearly in Lemma 2.9.

**Lemma 2.9.** Let $\{\phi_k^1\}_{k=1}^3$ be as in (1.4). They have the following explicit expression at the midpoint $O$:

$$\phi_1^1(O) = \frac{1}{2 |K|} \begin{bmatrix} -\frac{1}{2} \eta_1 - \eta_2 \\ \frac{1}{2} \xi_1 + \xi_2 \end{bmatrix}, \quad \phi_2^1(O) = \frac{1}{2 |K|} \begin{bmatrix} \frac{1}{2} \eta_1 \\ -\xi_1 \end{bmatrix},$$

$$\phi_3^1(O) = \frac{1}{2 |K|} \begin{bmatrix} -\frac{1}{2} \eta_1 \end{bmatrix}, \quad \nabla \cdot \phi_k^1 = \frac{\eta_k}{|K|}.$$

**Proof.** By Remark 2.1, we obtain $\phi_1^0(\hat{O}) = \begin{bmatrix} -\frac{1}{2} \end{bmatrix}. Then

$$\phi_1^1(O) = B_K^{-T} \phi_1^0(\hat{O}) = \frac{1}{2 K} \begin{bmatrix} -\frac{1}{2} \eta_1 - \eta_2 \\ \frac{1}{2} \xi_1 + \xi_2 \end{bmatrix}.$$

For $k = 2,3$, $\phi_k^1$ is obtained from the same technique. By Lemma 1.2 in [13], we obtain $\nabla \cdot \phi_k^1 (k=1,2,3)$. □

Let $\hat{\xi}_k, \hat{\eta}_k$ be as in (2.1), $\hat{X}_{k-1}, \hat{X}_{k+1}, \hat{Y}_{k-1}, \hat{Y}_{k+1}$ in (2.2), and $\xi_k, \eta_k$ in (2.3). Denote

$$S_k = \eta_1 - \hat{\eta}_k - \frac{\hat{\xi}_k}{3!}, \quad T_k = \frac{\xi_1 - \hat{\xi}_k}{3!} + \frac{\hat{\xi}_k}{3!},$$

$$I^\hat{Y}_{k,j} = \hat{\eta}_j - \hat{\eta}_{k+1} + \hat{\eta}_{k+1} - \hat{\eta}_{k-1}, \quad I^\hat{X}_{k,j} = \frac{\hat{\xi}_j}{3!} - \frac{\hat{\xi}_{k+1}}{3!} - \frac{\hat{\xi}_{k-1}}{3!} + \frac{\hat{\xi}_{k+2}}{3!} + \frac{\hat{\xi}_{k-2}}{3!} + \frac{\hat{\xi}_{k+3}}{3!} + \frac{\hat{\xi}_{k-3}}{3!}.$$

Now, let us present the expression of $\int_K \phi_k^2 \cdot \phi_1^1 dK$.

**Lemma 2.10.** Let $\hat{\phi}_1^1 = (\phi_{1,1}^1, \phi_{1,2}^1)$ be basis on the edge $\hat{e}_1 \in \hat{K}$. Let $\hat{\phi}_1^1 = B_K^{-T} \hat{\phi}_1^1$ be the basis function on $e_1 \subset K$. Let $\phi_k^2$ be as in (1.5). Then it holds that

$$\int_K \phi_k^2 \cdot \phi_1^1 dK = \frac{\phi_{1,1}^{1,1}(\hat{O})}{2 |K|} (\eta_1 S_k + \xi_1 T_k) + \frac{\phi_{1,2}^{1,2}(\hat{O})}{2 |K|} (\eta_2 S_k + \xi_2 T_k)$$

$$+ \frac{\partial_\eta \phi_{1,1}^{1,1}(\hat{O})}{2 |K|} \left[ \eta_1 \left( \eta_1 \hat{Y}_{k,j} - \eta_2 \hat{X}_{k,j} \right) + \eta_2 \left( \hat{\xi}_1 \hat{Y}_{k,j} - \hat{\xi}_2 \hat{X}_{k,j} \right) \right]$$

$$+ \frac{\partial_{\xi_1} \phi_{1,1}^{1,2}(\hat{O})}{2 |K|} \left[ \eta_1 \left( \eta_1 \hat{Y}_{k,j} - \eta_2 \hat{X}_{k,j} \right) + \eta_2 \left( \hat{\xi}_1 \hat{Y}_{k,j} - \hat{\xi}_2 \hat{X}_{k,j} \right) \right].$$
Proof. By Piola transformation, we have \( \phi_1^i = B_K^{-T} \phi_1^i \), this together with Lemma 2.1 and \(|B_K| = 2|K|\), we deduce

\[
\int_K \phi_k^j \cdot \phi_1^i \, dK = \int_K B_K^{-T} \phi_k^j \cdot (B_K^{-T} \phi_1^i) \cdot |B_K| \, dK = |B_K| \frac{1}{2|K|} \left[ \eta_1 - \xi_1 \right] \left[ \eta_2 - \xi_2 \right] \left[ \Lambda_k^0 \right] dK
\]

\[
= \frac{1}{2|K|} \int_K \phi_1^{1,1} \left[ \eta_1 (\eta_1 \Lambda_k^0 - \eta_2 \Lambda_k^0) + \xi_1 (\xi_1 \Lambda_k^0 - \xi_2 \Lambda_k^0) \right] dK + \frac{1}{2|K|} \int_K \phi_1^{1,2} \left[ \eta_2 (\eta_1 \Lambda_k^0 - \eta_2 \Lambda_k^0) + \xi_2 (\xi_1 \Lambda_k^0 - \xi_2 \Lambda_k^0) \right] dK.
\]

Due to Remark 2.1, the Taylor expansion of \( \phi_1^i \) are

\[
\phi_1^{1,1} = \phi_1^{1,1} (\hat{O}) + \partial_0 \phi_1^{1,1} (\hat{O})(\hat{y} - y_0), \\
\phi_1^{1,2} = \phi_1^{1,2} (\hat{O}) + \partial_0 \phi_1^{1,2} (\hat{O})(\hat{x} - x_0),
\]

which together with Lemma 2.2 can conclude this lemma. \( \square \)

### 2.3 Superconvergence analysis of Nédélec interpolation

In this subsection, we will present the superconvergence of Nédélec interpolation at the midpoint of the interior edge under the \( L^\infty \) norm. As shown in Fig. 1 (right), let \( K \) be as in Subsection 2.2. The triangle \( K' \) has vertices \( A_4(x_4,y_4), A_3(x_3,y_3), A_2(x_2,y_2) \) and corresponding local index \( 1', 2', 3' \), then the coordinates can be also presented as \( A_4(x'_4,y'_4), A_3(x'_3,y'_3), A_2(x'_2,y'_2) \). Let \( \omega = K \cup K' \) be a patch sharing the interior edge \( A_2A_3 \). Let \( \hat{O} \) be the midpoint of \( A_2A_3 \). Let \( \delta_x = x_1 + x_4 - x_2 - x_3, \delta_y = y_1 + y_4 - y_2 - y_3 \). It is shown in [4] that \( \delta_x = O(h^2), \delta_y = O(h^2) \). Let us introduce some geometric parameters \( \xi'_k, \eta'_k \) similar to \( \xi_k, \eta_k \) in (2.3) and also give the relation between them:

\[
\xi'_1 = x'_2 - x'_3 = -\xi_1, \quad \eta'_1 = y'_2 - y'_3 = -\eta_1, \\
\xi'_2 = x'_3 - x'_1 = -\xi_2 - \delta_x, \quad \eta'_2 = y'_3 - y'_1 = -\eta_2 - \delta_y, \\
\xi'_3 = x'_1 - x'_2 = -\xi_3 + \delta_x, \quad \eta'_3 = y'_1 - y'_2 = -\eta_3 + \delta_y.
\]

Denote

\[
M^{x'}_k = \left( \frac{x'_k - 1}{21} + \frac{x'_{k+1}}{21} - x_0 \right), \quad N^{x'}_k = \frac{\xi'_k}{6} + \frac{1}{2}(x'_k - x_0)(x'_{k+1} - x_0), \\
M^{y'}_k = \left( \frac{y'_k - 1}{21} + \frac{y'_{k+1}}{21} - y_0 \right), \quad N^{y'}_k = \frac{\eta'_k}{6} + \frac{1}{2}(y'_k - y_0)(y'_{k+1} - y_0), \\
H'_k = \frac{\xi'_k y'_k}{3} - \frac{\xi'_{k+1} y'_{k+1}}{6} + \frac{1}{2}(x'_k - x_0)(y'_{k+1} + y'_{k-1}) - \frac{y_0}{2}(x'_{k+1} + x'_{k-1} - 2x_0).
\]

(2.18)
Lemma 2.11. Assume that \( \phi \) and \( \psi \) c e. Similarly, it follows that

Remark 2.2. Let \( S_1 = \{1,2,3\} \), an integer set related to global node index in \( K \). Let \( S_2 = \{2,3,4\} \), an integer set related to global node index in \( K' \). Let \( c_{ij}^1, c_{ij}^2 \) (\( i,j \in S_1 \)) be the DoFs along with edge \( e_{ij} \) in \( K' \). The relation between \( c_{ij}^m \) and \( u_k^m \) (\( m = 1,2 \)), \( c_{ij}^m \) and \( u_k^m \) (\( m = 1,2 \)) respectively are as follows:

\[
\begin{align*}
    c_{12}^1 &= u_1^1, & c_{23}^1 &= u_1^4, & c_{13}^1 &= -u_2^1, & c_{23}^2 &= u_2^3, & c_{23}^3 &= u_2^1, & c_{13}^3 &= u_3^2, \\
    c_{34}^1 &= -u_3^1, & c_{24}^1 &= -u_4^1, & c_{24}^2 &= u_1^2, & c_{24}^3 &= u_2^2, & c_{24}^4 &= u_3^2,
\end{align*}
\]

where \( c_{12}^1 = c_{23}^1, c_{23}^2 = c_{23}^3 \). Let \( \psi_{ij}^1, \psi_{ij}^2 \) be Nédélec basis functions on edge \( e_{ij} \).

Remark 2.3. Let \( \psi_{ij}^1, \psi_{ij}^2 \) be Nédélec basis function on edge \( e_{ij} \). The relation between \( \psi_{ij}^m \) and \( \phi_{ij}^m \) (\( m = 1,2 \)), \( \psi_{ij}^m \) and \( \phi_{ij}^m \) (\( m = 1,2 \)) respectively are as follows:

\[
\begin{align*}
    \psi_{12}^1 &= \phi_{13}^1, & \psi_{23}^1 &= \phi_{13}^3, & \psi_{13}^1 &= -\phi_{12}^1, & \psi_{12}^2 &= \phi_{13}^2, & \psi_{23}^2 &= \phi_{13}^1, & \psi_{13}^2 &= \phi_{23}^1, & \psi_{13}^3 &= \phi_{13}^1, \\
    \psi_{34}^1 &= -\phi_{34}^1, & \psi_{23}^3 &= -\phi_{12}^1, & \psi_{12}^4 &= \phi_{13}^2, & \psi_{23}^4 &= \phi_{13}^1, & \psi_{13}^4 &= \phi_{23}^1, & \psi_{23}^4 &= \phi_{23}^1, & \psi_{23}^5 &= \phi_{23}^2.
\end{align*}
\]

Now we deduce the superconvergence of Nédélec interpolation at \( O \) in \( L^\infty \) norm.

Lemma 2.11. Assume that \( u \in (C^3(\overline{\omega}))^2 \). Then

\[
\begin{align*}
    &\Pi_h^1\mathbf{u}(O)|_K + \Pi_h^2\mathbf{u}(O)|_K = 2\mathbf{u}(O) + \mathcal{O}(h^2)(||\partial u||_\infty + ||\partial^2 u||_\infty), \\
    &\nabla \times \Pi_h^1\mathbf{u}(O)|_K + \nabla \times \Pi_h^2\mathbf{u}(O)|_K \\
    &= 2\nabla \times \mathbf{u}(O) + \mathcal{O}(h^2)(||\partial^2 u||_\infty + ||\partial^3 u||_\infty).
\end{align*}
\]

Proof. (i) By Lemma 2.5 and Lemma 2.9, we obtain

\[
\Pi_h^1\mathbf{u}(O)|_K = c_{12}^1 \psi_{12}^1(O) + c_{23}^1 \psi_{23}^1(O) + c_{13}^1 \psi_{13}^1(O)
\]

\[
= u_1^1 \phi_1^1(O) + u_2^1 \phi_2^1(O) + u_3^1 \phi_3^1(O)
\]

\[
= \frac{1}{2K} \left[ \frac{1}{2} \eta_1 (u_1^1 - u_2^1 - u_3^1) - \eta_2 u_1^1 \right] \left[ \Pi_h^1 \mathbf{u}(O)|_K \right] \\
= \frac{1}{2K} \left[ \Pi_h^1 \mathbf{u}(O)|_K \right].
\]

Similarly, it follows that

\[
\Pi_h^1\mathbf{u}(O)|_{K'} = \frac{1}{2K'} \left[ \Pi_h^1 \mathbf{u}(O)|_{K'} \right].
\]
By Lemma 2.7 (i), we have
\[
\Pi_h^{1,1} u(O)\big|_K = u_1(O) - \frac{\eta_1}{4|K|} \left[ (\zeta_2 M_2^x - \zeta_3 M_3^x) \frac{\partial u_1}{\partial x}(O) + (\zeta_2 M_2^y - \zeta_3 M_3^y) \frac{\partial u_1}{\partial y}(O) \right]
+ (\eta_2 M_2^x - \eta_3 M_3^x) \frac{\partial u_1}{\partial x}(O) + (\eta_2 M_2^y - \eta_3 M_3^y) \frac{\partial u_1}{\partial y}(O) \right) + O(h^2) \|\partial^2 u\|_\infty,
\]
(2.19)
where \( M_2^x, M_3^y \) are denoted in (2.4). Similarly, we obtain
\[
\Pi_h^{1,1} u(O)\big|_{K'} = u_1(O) - \frac{\eta_1'}{4|K'|} \left[ (\zeta_2 M_2^x - \zeta_3 M_3^x) \frac{\partial u_1}{\partial x}(O) + (\zeta_2 M_2^y - \zeta_3 M_3^y) \frac{\partial u_1}{\partial y}(O) \right]
+ (\eta_2 M_2^x - \eta_3 M_3^x) \frac{\partial u_1}{\partial x}(O) + (\eta_2 M_2^y - \eta_3 M_3^y) \frac{\partial u_1}{\partial y}(O) \right) + O(h^2) \|\partial^2 u\|_\infty,
\]
(2.20)
where \( M_2^x, M_3^y \) are defined in (2.18). Adding (2.19) and (2.20) together, we deduce
\[
\Pi_h^{1,1} u(O)\big|_K + \Pi_h^{1,1} u(O)\big|_{K'} = 2u_1(O) + U_1^x \frac{\partial u_1}{\partial x}(O) + U_1^y \frac{\partial u_1}{\partial y}(O)
+ U_2^x \frac{\partial u_2}{\partial x}(O) + U_2^y \frac{\partial u_2}{\partial y}(O) + O(h^2) \|\partial^2 u\|_\infty,
\]
(2.21)
where \( U_1^x, U_1^y, U_2^x \) and \( U_2^y \) are denoted as the coefficients in front of \( \frac{\partial u_1}{\partial x}(O), \frac{\partial u_1}{\partial y}(O), \frac{\partial u_2}{\partial x}(O) \)
and \( \frac{\partial u_2}{\partial y}(O) \), respectively. From (2.3) and (2.17), we know that
\[
U_1^x = -\frac{\eta_1}{4|K|} (\zeta_2 M_2^x - \zeta_3 M_3^x) - \frac{\eta_1'}{4|K'|} (\zeta_2 M_2^x - \zeta_3 M_3^x) = -\frac{\eta_1 \zeta_2 \zeta_3}{4|K|} - \frac{\eta_1' \zeta_2' \zeta_3'}{4|K'|}
= -\frac{\eta_1 \zeta_2 \zeta_3 |K'| + \zeta_2' \zeta_3' |K'| - \zeta_2 \zeta_3' |K'| - \zeta_2' \zeta_3 |K|}{|K||K'|}
= O(h^2).
\]
Similarly, it holds that \( U_1^y = O(h^2), U_2^x = O(h^2), U_2^y = O(h^2) \). Substituting these estimates into (2.21), we obtain
\[
\Pi_h^{1,1} u(O)\big|_K + \Pi_h^{1,1} u(O)\big|_{K'} = 2u_1(O) + O(h^2) (\|\partial u\|_\infty + \|\partial^2 u\|_\infty).
\]
Similarly, we can deduce
\[
\Pi_h^{1,2} u(O)\big|_K + \Pi_h^{1,2} u(O)\big|_{K'} = 2u_2(O) + O(h^2) (\|\partial u\|_\infty + \|\partial^2 u\|_\infty).
\]
(ii) By Lemma 2.9 and Lemma 2.7 (ii), we get
\[
\nabla \times \Pi_k u(O) |_{K} = \nabla \times u(O) + \frac{1}{|K|} \left[ \sum_{k=1}^{3} (-\xi_k) C_k^1 \right.
\]
\[
+ \sum_{k=1}^{3} (-\eta_k) C_k^2 \left. \right] + \mathcal{O}(h^2) \| \partial^3 u \|_{\infty}. \tag{2.22}
\]

Similarly, it follows that
\[
\nabla \times \Pi_{k'} u(O) |_{K'} = \nabla \times u(O) + \frac{1}{|K'|} \left[ \sum_{k=1}^{3} (-\xi'_{k'}) C_{k'}^{1l} \right.
\]
\[
+ \sum_{k=1}^{3} (-\eta'_{k'}) C_{k'}^{2l} \left. \right] + \mathcal{O}(h^2) \| \partial^3 u \|_{\infty}. \tag{2.23}
\]

By adding (2.22) and (2.23) together, we get
\[
\nabla \times \Pi_k u(O) |_{K} + \nabla \times \Pi_{k'} u(O) |_{K'}
\]
\[
= 2 \nabla \times u(O) + es\text{st}_1 + es\text{st}_2 + \mathcal{O}(h^2) \| \partial^3 u \|_{\infty}, \tag{2.24}
\]

where
\[
es\text{st}_1 = \frac{1}{|K|} \left[ \sum_{k=1}^{3} (-\xi_k) C_k^1 \right] + \frac{1}{|K'|} \left[ \sum_{k=1}^{3} (-\xi'_{k'}) C_{k'}^{1l} \right],
\]
\[
es\text{st}_2 = \frac{1}{|K|} \left[ \sum_{k=1}^{3} (-\eta_k) C_k^1 \right] + \frac{1}{|K'|} \left[ \sum_{k=1}^{3} (-\eta'_{k'}) C_{k'}^{1l} \right].
\]

From (2.15), we have
\[
es\text{st}_1 = \left[ \frac{1}{|K|} \sum_{k=1}^{3} (-\xi_k) N_k^x \right] \frac{\partial^2 u_1}{\partial x^2}(O) + \left[ \frac{1}{|K'|} \sum_{k=1}^{3} (-\xi'_{k'}) N_{k'}^x \right] \frac{\partial^2 u_1}{\partial x'^2}(O)
\]
\[
+ \left[ \frac{1}{|K|} \sum_{k=1}^{3} (-\xi_k) N_k^y \right] \frac{\partial^2 u_1}{\partial x \partial y}(O) + \left[ \frac{1}{|K'|} \sum_{k=1}^{3} (-\xi'_{k'}) N_{k'}^y \right] \frac{\partial^2 u_1}{\partial x' \partial y'}(O)
\]
\[
:= U_1^{xx} \frac{\partial^2 u_1}{\partial x^2}(O) + U_1^{xy} \frac{\partial^2 u_1}{\partial x \partial y}(O) + U_1^{yx} \frac{\partial^2 u_1}{\partial y^2}(O), \tag{2.25}
\]

where \(H'_k\) is defined in (2.18). From (2.17), we obtain
\[
U_1^{xx} = \mathcal{O}(h^2), \quad U_1^{xy} = \mathcal{O}(h^2), \quad U_1^{yx} = \mathcal{O}(h^2). \tag{2.26}
\]

Substituting (2.26) into (2.25), we have
\[
es\text{st}_1 = \mathcal{O}(h^2) \| \partial^3 u \|_{\infty}. \tag{2.27}
\]
Similarly, it holds that
\[ est^2 = O(h^2)||\partial^2 u||_\infty. \] (2.28)
Substituting (2.27) and (2.28) into (2.24), we conclude (ii).

**Remark 2.4.** If the triangulation is uniform, it holds that \( U_1^x = U_1^y = U_2^x = U_2^y = 0 \) in (2.21) and \( U_1^xx = U_1^xy = U_1^yy = 0 \) in (2.25), which can refer to [10].

## 3 Superconvergence result

In this section, we shall derive a superconvergence result for finite element approximation \( u_h \in V_h \) for problem (1.1)-(1.2). We begin with the following error estimate.

**Lemma 3.1.** Let \( \phi^1 = (\phi_1^{1,1}, \phi_1^{1,2}) \) be basis on \( \mathring{\hat{e}}_1 \in \hat{K} \). Let \( \phi^1 = B_K^{-T} \phi^1 \) be basis on \( e_1 \in K \). Then
\[
(\text{i}) \quad \phi^1_1(\hat{O}) \leq C ||\phi^1||_{L^2(K)}, \\
(\text{ii}) \quad \partial_y \phi^1_1(\hat{O}) \leq C ||\phi^1||_{L^2(K)}, \quad \partial_y \phi^1_2(\hat{O}) \leq C ||\phi^1||_{L^2(K)}.
\]

**Proof.** By the equivalence of norms on the reference element and Piola transformation, we have
\[
(\text{i}) \quad \phi^1_1(\hat{O}) \leq ||\phi^1||_{L^\infty} \leq ||\phi^1||_{L^2(K)} = \left( \int_K \phi^1 \cdot \phi^1 d\hat{K} \right)^{\frac{1}{2}} = \left( \int_K B_K^{-1} \phi^1 \cdot B_K \phi^1 \right)^{\frac{1}{2}}, \\
(\text{ii}) \quad \partial_y \phi^1_1(\hat{O}) \leq C ||\phi^1||_{L^2(K)}, \quad \partial_y \phi^1_2(\hat{O}) \leq C ||\phi^1||_{L^2(K)}.
\]
This completes the proof. \( \square \)

The superconvergence that we will obtain depend on magnitudes of \( u_k^2 \) and \( u_s^2 \).

**Lemma 3.2.** Assume that \( u \in (C^2(\Omega))^2 \). The Dofs \( u_k^2 \) in \( K' \) have the same form as \( u_k^2 \) in (2.12). And they have the following properties:
\[
(\text{i}) \quad u_k^2 = O(h^2)||\partial u||_\infty, \quad u_k^2 = O(h^2)||\partial u||_\infty, \quad \text{for all } k = 1, 2, 3, \\
(\text{ii}) \quad u_1^2 = u_1^2, \quad u_2^2 - u_3^2 = O(h^3)(||\partial u||_\infty + ||\partial^2 u||_\infty), \quad (s = 2, 3).
\]

**Proof.** (i) From Lemma 2.8, (i) holds.
(ii) On \( e_{23} \), we obviously get the first result \( c_{23}^2 = u_1^2 = c_{23}^2 = u_1^2 \).

The second result can be obtained by Lemma 2.8, for example,
\[
u_2^2 = -\frac{\zeta_2^2}{2} \partial_s u_1(0) - \frac{\zeta_2^2}{2} \partial_s u_1(O) - \frac{\eta_2^2}{2} \partial_s u_2(0) - \frac{\eta_2^2}{2} \partial_s u_2(O) + O(h^3)||\partial^2 u||_\infty, \quad (3.1)
\]
\[
u_3^2 = -\frac{\zeta_3^2}{2} \partial_s u_1(0) - \frac{\zeta_3^2}{2} \partial_s u_1(O) - \frac{\eta_3^2}{2} \partial_s u_2(0) - \frac{\eta_3^2}{2} \partial_s u_2(O) + O(h^3)||\partial^2 u||_\infty. \quad (3.2)
\]
Subtracting (3.2) from (3.1), together with (2.17), we obtain \( u_2^2 - u_2^2 = O(h^3)(||\partial u||_\infty + ||\partial^2 u||_\infty) \).

The superclose result playing an essential role in the superconvergence relies on the following error estimate.

**Lemma 3.3.** Assume \( u \in (C^2(\Omega))^2 \). Then

(i) \[
\int_K \phi_{12}^2 \cdot \psi_{23}^1 dK + c_{34}^2 \int_{K'} \phi_{34}^2 \cdot \psi_{23}^1 dK
\leq Ch^3( ||\partial u||_\infty + ||\partial^2 u||_\infty)(||\phi_1^1||_{L^2(K)} + ||\phi_1^1||_{L^2(K')}),
\]

(ii) \[
\int_K \phi_{13}^2 \cdot \psi_{23}^1 dK + c_{23}^2 \int_{K'} \phi_{24}^2 \cdot \psi_{23}^1 dK
\leq Ch^3( ||\partial u||_\infty + ||\partial^2 u||_\infty)(||\phi_1^1||_{L^2(K)} + ||\phi_1^1||_{L^2(K')}),
\]

(iii) \[
\int_K \phi_{23}^2 \cdot \psi_{23}^1 dK + c_{23}^2 \int_{K'} \phi_{23}^2 \cdot \psi_{23}^1 dK
\leq Ch^3( ||\partial u||_\infty + ||\partial^2 u||_\infty)(||\phi_1^1||_{L^2(K)} + ||\phi_1^1||_{L^2(K')}).
\]

**Proof.** (i) By Remark 2.3, it follows that

\[
c_{12}^2 \int_K \psi_{12}^2 \cdot \psi_{23}^1 dK + c_{34}^2 \int_{K'} \phi_{34}^2 \cdot \psi_{23}^1 dK
= u_2^2 \int_K \phi_{3}^2 \cdot \phi_{1}^1 dK - u_2^2 \int_{K'} \phi_{3}^2 \cdot \phi_{1}^1 dK.
\]

From Lemma 2.10, we obtain

\[
\int_K \phi_{3}^2 \cdot \phi_{1}^1 dK = \phi_{1}^1(\hat{\mathcal{O}}) \frac{\eta_1 \eta_2 + \eta_1 \eta_2 + \zeta_1 \zeta_2}{2|K|} + \frac{\partial_2 \phi_{1}^1(\hat{\mathcal{O}}) - \eta_1 \eta_2 - \zeta_1 \zeta_2}{2|K|}
\]

\[
+ \phi_{1}^2(\hat{\mathcal{O}}) \frac{\eta_1 \eta_2 + \eta_1 \eta_2 + \zeta_1 \zeta_2}{2|K|} + \frac{\partial_2 \phi_{1}^2(\hat{\mathcal{O}}) - \eta_1 \eta_2 + 2\eta_1^2 + \zeta_1 \zeta_2 + 2\zeta_2^2}{2|K|}.
\]

Similarly, we get

\[
\int_{K'} \phi_{3}^2 \cdot \phi_{1}^1 dK = \phi_{1}^1(\hat{\mathcal{O}}) \frac{\eta_1 \eta_2 + \eta_1 \eta_2 + \zeta_1 \zeta_2}{2|K'|} + \frac{\partial_2 \phi_{1}^1(\hat{\mathcal{O}}) - \eta_1 \eta_2 - \zeta_1 \zeta_2}{2|K'|}
\]

\[
+ \phi_{1}^2(\hat{\mathcal{O}}) \frac{\eta_1 \eta_2 + \eta_1 \eta_2 + \zeta_1 \zeta_2}{2|K'|} + \frac{\partial_2 \phi_{1}^2(\hat{\mathcal{O}}) - \eta_1 \eta_2 + 2\eta_1^2 + \zeta_1 \zeta_2 + 2\zeta_2^2}{2|K'|}.
\]

Substituting (3.4) and (3.5) into (3.3), we get

\[
c_{12}^2 \int_K \psi_{12}^2 \cdot \psi_{23}^1 dK + c_{34}^2 \int_{K'} \phi_{34}^2 \cdot \psi_{23}^1 dK
\leq Ch^3( ||\partial u||_\infty + ||\partial^2 u||_\infty)(||\phi_1^1||_{L^2(K)} + ||\phi_1^1||_{L^2(K')}),
\]

where we use (2.3), Lemma 3.1 and Lemma 3.2. The others follow the same pattern.
In the following lemma, we present an important estimate.

**Lemma 3.4.** As shown in Fig. 1 (right), let $\omega = K \cup K'$ be a patch sharing the common edge $e_1$. Define
\[
\psi_{23} := \begin{cases} 
\psi_{23}^1 = \phi_1^1, & \text{in } K, \\
\psi_{23}^{1'} = -\phi_1^{1'}, & \text{in } K'. 
\end{cases}
\]
Assume $u \in (C^2(\Omega))^2$. Then
\[
\int_\omega (\Pi_h^2 u - \Pi_h^1 u) \cdot \psi_{23} \, dK 
\leq Ch^3 \left( ||\partial u||_\infty + ||\partial^2 u||_\infty \right) \left[ ||\phi_1^1||_{L^2(K)} + ||\phi_1^{1'}||_{L^2(K')} \right].
\]

**Proof.** Denote
\[
ERR = \int_\omega (\Pi_h^2 u - \Pi_h^1 u) \cdot \psi_{23} \, dK.
\]
By Lemma 2.5, it follows that
\[
ERR = c_{12} \int_K \psi_{12}^2 \cdot \psi_{23}^1 \, dK + c_{34} \int_{K'} \phi_{24}^2 \cdot \psi_{23}^1 \, dK + c_{13} \int_K \psi_{23}^1 \cdot \psi_{23}^1 \, dK
\]
\[
+ c_{24} \int_{K'} \phi_{24}^2 \cdot \psi_{23}^1 \, dK + c_{25} \int_K \psi_{23}^1 \cdot \psi_{23}^1 \, dK + c_{23} \int_{K'} \phi_{23}^2 \cdot \psi_{23}^1 \, dK.
\]
Using Remark 2.3 and Lemma 3.3, we obtain
\[
ERR \leq Ch^3 \left( ||\partial u||_\infty + ||\partial^2 u||_\infty \right) \left[ ||\phi_1^1||_{L^2(K)} + ||\phi_1^{1'}||_{L^2(K')} \right].
\]
This completes the proof. \qed

Using the Stokes formula and the definition of $\Pi_h^1 u$, it is easy to see that

**Lemma 3.5.** [12] For all $u \in H(\text{curl}; K)$ and a const $q$, we get
\[
\int_K \nabla \times (u - \Pi_h^1 u) \ast q \, dK = 0.
\]

**Lemma 3.6.** For any $\sum_{i=1}^3 \alpha_i \phi_i^1 = \phi_h \in V_h^1$, we have
\[
\sum_{i=1}^3 ||\alpha_i \phi_i^1||_{L^2(K)} \leq C ||\phi_h||_{L^2(K)}.
\]
Proof. By the Piola transformation and norm equivalence in the reference element, we have

\[
\sum_{i=1}^{3} ||\alpha_i \phi_1^i||_{L^2(K)} = \sum_{i=1}^{3} \left( \int_K |\alpha_i \phi_1^i|^2 dK \right)^{\frac{1}{2}} = \sum_{i=1}^{3} \left( \int_K |B_K^{-T} \phi_1^i|^2 |B_K| dK \right)^{\frac{1}{2}}
\]

\[
\leq \sum_{i=1}^{3} ||B_K^{-T}||_K |B_K|^{\frac{1}{2}} \left( \int_K |\alpha_i \phi_1^i|^2 dK \right)^{\frac{1}{2}} \leq ||B_K^{-T}||_K ||B_K||^{\frac{1}{2}} \cdot C \left( \sum_{i=1}^{3} ||\alpha_i \phi_1^i||_{L^2(K)} \right)^{\frac{1}{2}}
\]

\[
= C ||B_K^{-T}||_K ||B_K||^{\frac{1}{2}} \left( \int_K \left( \sum_{i=1}^{3} \alpha_1^i B_K^T \phi_1^i \right)^2 \frac{1}{|B_K|} dK \right)^{\frac{1}{2}}
\]

\[
\leq C ||B_K^{-T}||_K ||B_K||^{\frac{1}{2}} ||B_K|| \frac{1}{|B_K|^{\frac{1}{2}}} \left( \int_K \left( \sum_{i=1}^{3} \alpha_1^i \phi_1^i \right)^2 dK \right)^{\frac{1}{2}},
\]

which concludes the proof. \(\square\)

With the above analysis, we give an important estimate that will be used later.

Lemma 3.7. Let \(u\) and \(u_h\) be the solution of (1.3) and (1.7), respectively. Denote \(\Phi_h = \Pi_1^h u - u_h\). Assume \(u \in (C^2(\Omega))^2\). Then

\[
\int_\Omega (u - \Pi_1^h u) \cdot \Phi_h \, d\Omega \leq Ch^2 (||u||_{(H^2(\Omega))^2} + ||\partial u||_{\infty} + ||\partial^2 u||_{\infty}) ||\Phi_h||_{L^2(\Omega)}.
\]

Proof. We begin with the identity

\[
\int_\Omega (u - \Pi_1^h u) \cdot \Phi_h \, d\Omega = \int_\Omega (u - \Pi_1^h u + \Pi_2^h u - \Pi_1^h u) \cdot \Phi_h \, d\Omega
\]

\[
= ERR_1 + ERR_2. \tag{3.6}
\]

The first term \(ERR_1\) can be estimated by Lemma 2.6 and

\[
ERR_1 = \int_\Omega (u - \Pi_1^h u) \cdot \Phi_h \, d\Omega \leq Ch^2 ||u||_{(H^2(\Omega))^2} ||\Phi_h||_0. \tag{3.7}
\]

In order to estimate \(ERR_2\), we introduce the following notations. Let \(N\) denote the number of triangles in triangulation \(\Gamma_h\). Let \(l\) be the set of all interior edges. Let \(w_l = K_l \cup K_l'\) be a patch sharing the common edge \(e_l \in l\) (\(l\) is its global index). Let

\[
\psi_l := \begin{cases} 
\psi_l^1 = \phi_k^l, & \text{in } K_l, \\
\psi_l'^1 = -\phi_k'^l, & \text{in } K_l' \end{cases}
\]

where $k$ is the local index. Then,

$$ERR_2 = \sum_{K \in T_I} \int_K (\Pi_h^2 u - \Pi_h^1 u) \cdot \Phi_h \, dK$$

$$= \sum_{\omega_i = K_i \cup K_i'} \omega_i \int_{\omega_i} (\Pi_h^2 u - \Pi_h^1 u) \cdot c_i \psi_i \, d\omega \quad \text{(by Lemma 3.4)}$$

$$\leq C h^3 (||\partial u||_\infty + ||\partial^2 u||_\infty) \sum_{\omega_i = K_i \cup K_i'} \sum_{i \in I} \sum_{k = 1}^3 ||c_k \Phi_h||_{L^2(K)}$$

$$\leq C h^3 (||\partial u||_\infty + ||\partial^2 u||_\infty) \sum_{K \in T_I} \sum_{k = 1}^3 ||c_k \Phi_h||_{L^2(K)} \quad \text{(by Lemma 3.6)}$$

$$\leq C h^3 (||\partial u||_\infty + ||\partial^2 u||_\infty) \sum_{K \in T_I} ||\Phi_h||_{L^2(K)} \quad \text{(by Cauchy-Schwarz inequality)}$$

$$\leq C h^2 (||\partial u||_\infty + ||\partial^2 u||_\infty) ||\Phi_h||_{0, \Omega}$$

(3.8)

where we use fact that $N = O(h^{-2})$ in the last step. Then, the lemma is concluded by substituting (3.7) and (3.8) into (3.6).

From the above discussion, we obtain the following superclose result.

**Theorem 3.1.** Assume that the domain $\Omega$ is covered by strongly regular triangulation. Let $\delta$ be as in Lemma 1.1. Under the same condition as Lemma 3.7, then we have

$$||\Pi_h^2 u - u_h||_{L^2(\Omega)} + ||\nabla \times (\Pi_h^2 u - u_h)||_{L^2(\Omega)}$$

$$\leq C \left( h^2 (||u||_{(H^{s+1}(\Omega))^2} + ||\partial u||_\infty + ||\partial^2 u||_\infty) + h^{s+\frac{1}{2}} ||e_h||_{H(curl,\Omega)} \right).$$

**Proof.** It follows from (1.3) that

$$||\Phi_h||_{H(curl,\Omega)}^2 = a(\Phi_h, \Phi_h) + (1 + \kappa_h^2)(\Phi_h, \Phi_h)$$

$$= a(\Pi_h^2 u - u_h, \Phi_h) + (1 + \kappa_h^2)(\Phi_h, \Phi_h)$$

$$= a(\Pi_h^2 u - u_h, \Phi_h) + (1 + \kappa_h^2)(\Phi_h, \Phi_h)$$

$$= a(\Pi_h^2 u - u, \Phi_h) + a(u - u_h, \Phi_h) + (1 + \kappa_h^2)(\Phi_h, \Phi_h)$$

$$= (\nabla \times (\Pi_h^2 u - u), \nabla \times \Phi_h) + (\Pi_h^2 u - u, \Phi_h) + (1 + \kappa_h^2)(u - u_h, \Phi_h)$$

$$:= \sum_{i = 1}^3 EST_i.$$  

(3.9)
By Lemma 3.5, we obtain the first term $EST_1 = 0$. The second term $EST_2$ can be estimated by Lemma 3.7 in the following way:

$$EST_2 \leq Ch^2 (\|u\|_{(H^2(\Omega))^2} + |\partial u|_\infty + |\partial^2 u|_\infty) \|\Phi_h\|_{L^2(\Omega)}.$$ 

The third term $EST_3$ can be estimated by Lemma 1.1 and we deduce that

$$EST_3 = (1 + \kappa_0^2)(u - u_h, \Phi_h) \leq \sup_{\nu_h \in \mathcal{V}_h} \frac{||\nu_h||_{H(curl, \Omega)}}{||\nu_h||_{H(curl, \Omega)} ||\Pi_h^1 u - u||_{H(curl, \Omega)}} \leq Ch^{2+\delta} ||u||_{H(curl, \Omega)} ||\Pi_h^1 u - u||_{H(curl, \Omega)}.$$ 

The proof is completed by Theorem 1.1 and substituting $EST_i (i = 1, 2, 3)$ into (3.9).

In order to obtain superconvergence result, we introduce some properties of gradient and curl operator in 2D.

**Lemma 3.8.** [1, Theorem 4] Let

$$v(x, y) = \begin{bmatrix} v_1(x, y) \\ v_2(x, y) \end{bmatrix} \in \mathcal{V}_h^1(K), \quad \hat{\Phi}(\hat{x}, \hat{y}) = \begin{bmatrix} \hat{\partial}_1(\hat{x}, \hat{y}) \\ \hat{\partial}_2(\hat{x}, \hat{y}) \end{bmatrix} \in \mathcal{V}_h^1(\hat{K})$$

and the relation between them is $v = B_K^{-T} \hat{\Phi}$. Then

(i) the gradient

$$Dv = \begin{bmatrix} \frac{\partial v_1}{\partial x} & \frac{\partial v_1}{\partial y} \\ \frac{\partial v_2}{\partial x} & \frac{\partial v_2}{\partial y} \end{bmatrix}$$

transforms according to $Dv = B_K^{-T} \hat{D}\Phi B_K^{-1}$, where

$$\hat{D}\Phi = \begin{bmatrix} \frac{\partial \hat{\partial}_1}{\partial x} & \frac{\partial \hat{\partial}_1}{\partial y} \\ \frac{\partial \hat{\partial}_2}{\partial x} & \frac{\partial \hat{\partial}_2}{\partial y} \end{bmatrix}$$

(ii) the curl operator transforms according to $\nabla \times v = |B_K^{-1}| \nabla \times \hat{\Phi}$.

Let $x_e$ be the middle point of the interior edge $e$. Let $N_e$ be the number of all interior edges. Let us denote by $R(u(x_e)) = \frac{1}{2}(u(x_e)|_K + u(x_e)|_{K'})$ the averaging operator of $u$ at $x_e$, where $K$ and $K'$ are two triangles sharing the interior edge $e$. We define the discrete $l^2$ norm

$$||u||_2 = \left( \frac{1}{N_e} \sum_{K \in \mathcal{T}_h} \sum_{x_e \in \partial K} u(x_e)^2 \right)^{\frac{1}{2}}.$$ 

**Theorem 3.2.** Under the same assumption as Theorem 3.1, let $\delta = \frac{1}{2}$ for smooth solution $u$ of (1.3). Then

(i) $||u - R(u_h)||_2 \leq Ch^2$,

(ii) $||\nabla \times u - R(\nabla \times u_h)||_2 \leq Ch^2$. 
Proof. (i) For any \( \mathbf{v}_h \in \mathbf{V}_h^1 \), using Piola transformation and the norm equivalence on the reference element, we have
\[
\sum_{K \in \Gamma_h} \sum_{x_e \in \partial K} R^2(\mathbf{v}_h(x_e)) \leq \sum_{K \in \Gamma_h} \sum_{x_e \in \partial K} \mathbf{v}_h^2(x_e) = \sum_{K \in \Gamma_h} \left| \sum_{x_e \in \partial K} (B_K^{-T} \mathbf{v}_h(\hat{x}_e))^2 \right| \\
\leq C \sum_{K \in \Gamma_h} ||B_K^{-T}||^2 \int_{\hat{K}} \mathbf{v}_h^2 d\hat{K} = C \sum_{K \in \Gamma_h} ||B_K^{-T}||^2 \int_{K} (B_K^{-1} \mathbf{v}_h)^2 \frac{1}{|B_K|} dK \\
\leq C \sum_{K \in \Gamma_h} ||B_K^{-T}||^2 ||\mathbf{v}_h||^2 \frac{1}{|B_K|} ||\mathbf{v}_h||_{L^2(K)}.
\]

Let \( \mathbf{v}_h = \Pi_h^1 \mathbf{u} - \mathbf{u}_h \). By Theorem 3.1 and the fact \(|B_K| = \mathcal{O}(h^2)\), \( N_e = \mathcal{O}(h^{-2}) \), we obtain
\[
||R(\Pi_h \mathbf{u} - \mathbf{u}_h)||_2 \leq C h^2. \tag{3.10}
\]

By Theorem 2.11, we find that
\[
||\mathbf{u} - R(\Pi_h \mathbf{u})||_2 \leq C h^2,
\]
which, together with (3.10) and triangular inequality, concludes (i).

(ii) By Lemma 3.8 and the norm equivalence on the reference element, we have
\[
\sum_{K \in \Gamma_h} \sum_{x_e \in \partial K} R^2(\nabla \times \mathbf{v}_h(x_e)) \leq \sum_{K \in \Gamma_h} \sum_{x_e \in \partial K} (\nabla \times \mathbf{v}_h(x_e))^2 \\
= \sum_{K \in \Gamma_h} \left| \sum_{x_e \in \partial K} (B_K^{-1}\nabla \times \mathbf{v}_h(\hat{x}_e))^2 \right| \leq C \sum_{K \in \Gamma_h} ||B_K^{-1}||^2 \int_{\hat{K}} (\nabla \times \mathbf{v}_h)^2 d\hat{K} \\
= C \sum_{K \in \Gamma_h} ||B_K^{-1}||^2 \int_{K} \left( \frac{1}{|B_K|} \nabla \times \mathbf{v}_h \right)^2 \frac{1}{|B_K|} dK \leq C \sum_{K \in \Gamma_h} \frac{1}{|B_K|} ||\nabla \times \mathbf{v}_h||_{L^2(K)}^2.
\]

Let \( \mathbf{v}_h = \Pi_h^1 \mathbf{u} - \mathbf{u}_h \). By Theorem 3.1, the definition of discrete \( l_2 \) norm and the fact \(|B_K| = \mathcal{O}(h^2)\), \( N_e = \mathcal{O}(h^{-2}) \), we obtain
\[
||R(\nabla \times (\Pi_h \mathbf{u} - \mathbf{u}_h))||_2 \leq C h^2. \tag{3.11}
\]

By Theorem 2.11, we find that
\[
||\nabla \mathbf{u} - R(\nabla \times \Pi_h \mathbf{u})||_2 \leq C h^2,
\]
which, together with (3.11) and triangular inequality, concludes (ii). \( \square \)
4 Numerical example

In this section, we present numerical examples to support our theoretical analysis. For simplicity, we consider the domain $\Omega = [-1,1]^2$. To rigorously check the convergence rate, we construct the following analytical solution of (1.1)-(1.2) with $\kappa_0 = 1$ and

$$u = \begin{bmatrix} \cos(\pi x)\sin(\pi y) \\ -\sin(\pi x)\cos(\pi y) \end{bmatrix}. \quad (4.1)$$

It is easy to see that $u$ satisfy the condition $\nabla \cdot u = 0$ in $\Omega$ and the PEC boundary condition (1.2).

First, we solve the problem (1.7) on the uniform grids. The obtained errors in standard $L_2$ norm and the discrete $l_2$ norm are presented in Table 1 and Table 2, respectively. Table 1 records the classical convergence rate stated in Theorem 1.1. Results in Table 2 clearly show the superconvergence rates $O(h^2)$ both for $||u-R(u_h)||_{l_2}$ and $||\nabla \times u - \nabla \times R(u_h)||_{l_2}$, which are consistent with Theorem 2.2 in [10].

Table 1: Optimal results for uniform mesh on square domain.

| mesh size | $||u - u_h||_0$ order | $||\nabla \times (u - u_h)||_0$ order |
|-----------|----------------------|-------------------------------|
| $h = \frac{1}{4}$ | 0.32296819264 | 1.61378601590 |
| $h = \frac{1}{8}$ | 0.16064970187 | 1.0075 | 0.81852654046 | 0.9793 |
| $h = \frac{1}{16}$ | 0.08020203270 | 1.0022 | 0.41073947388 | 0.9948 |
| $h = \frac{1}{32}$ | 0.04008505673 | 1.0006 | 0.20555495661 | 0.9987 |
| $h = \frac{1}{64}$ | 0.02004051420 | 1.0001 | 0.10280065251 | 0.9997 |

Table 2: Superconvergence results for uniform mesh on square domain.

| mesh size | $||u - R(u_h)||_{l_2}$ order | $||\nabla \times (u - R(u_h)||_{l_2}$ order |
|-----------|-------------------|---------------------------------------|
| $h = \frac{1}{4}$ | 0.05962620464 | 1.24024800537 |
| $h = \frac{1}{8}$ | 0.01588192493 | 1.9086 | 0.06330424774 | 1.9242 |
| $h = \frac{1}{16}$ | 0.00405354930 | 1.9701 | 0.01609919856 | 1.9753 |
| $h = \frac{1}{32}$ | 0.00102113234 | 1.9909 | 0.00405015053 | 1.9909 |
| $h = \frac{1}{64}$ | 0.00025608212 | 1.9955 | 0.00101514631 | 1.9963 |

Next, we introduce that any convex quadrilateral can be partitioned by strongly regular mesh in the following way [5]: (i) a pair of opposite sides is equally divided into $M$ equal parts and another pair is equally divided into $N$ equal parts. (ii) let us connect these points of division by lines sequentially. After that we obtain $M \times N$ small quadrilaterals. (iii) A strongly regular triangulation for any convex quadrilateral is obtained.
after connecting those vertices, please see Fig. 2 ($M = 2$, $N = 2$) as an example. For simplicity, the strongly regular triangulation for trapezoidal region formed by $Q_1 = (-1, -1)$, $Q_2 = (1.5, -1)$, $Q_3 = (0.5, 1)$ and $Q_4 = (-0.5, 1)$ is also shown in Fig. 3. In order to measure the quality of the triangulation, we introduce the notation

$$Q_l = \left( \frac{1}{N_{e_1}} \sum_{e_1 \in I} (l_2 - l'_2)^2 + (l_3 - l'_3)^2 \right)^{\frac{1}{2}},$$

where $e_1$ is the interior edge sharing by the patch $\omega = K \cup K'$ like Fig. 1 (right). $l_2$ and $l_3$ are the lengths of edges $e_2, e_3 \in K$. $l'_2$ and $l'_3$ are the lengths of edges $e'_2, e'_3 \in K'$.

Due to the following two reasons: (i) the true solution $u$ we choose must satisfy $\nabla \cdot u = 0$ and (ii) the convergence theory was built on the PEC boundary condition (1.2), we just choose the true solution (4.1) and the square domain $\Omega = [-1, 1]^2$ to testify our superconvergence results on strongly regular mesh.

Finally, we resolve problem (1.7) on the strongly regular mesh shown in Fig. 4. They are obtained by perturbing $O(h^2)$ of the corresponding interior vertexes of the uniform mesh. The optimal error estimates are recorded in Table 3, which is consistent with Theorem 1.1. The quality of the triangulation and superconvergence are shown in Table 4. And we do observe superconvergence phenomenon for $||u - R(u_h)||_{l_2}$ and $||\nabla \times u - R(\nabla \times u_h)||_{l_2}$ in Table 4, which justify superconvergence in Theorem 3.2.
Figure 4: Strongly regular mesh on square domain: left: $h = \frac{1}{8}$, right: $h = \frac{1}{16}$.

Table 3: Optimal result for the strongly regular mesh on square region.

| mesh size | $||u-u_h||_0$ | order | $||\nabla \times (u-u_h)||_0$ | order |
|-----------|--------------|-------|-----------------------------|-------|
| $h = \frac{1}{4}$ | 0.49117834381 | 1.78410306772 |
| $h = \frac{1}{8}$ | 0.18780470267 | 1.3870 | 0.84323159785 | 1.0812 |
| $h = \frac{1}{16}$ | 0.08386200446 | 1.1631 | 0.41392311372 | 1.0266 |
| $h = \frac{1}{32}$ | 0.04055166785 | 1.0483 | 0.2059567179 | 1.0070 |

Table 4: Superconvergence for the strongly regular mesh on square region.

| mesh size | $||u-R(u_h)||_2$ | order | $||\nabla \times (u-R(u_h))||_2$ | order | Ql | order |
|-----------|-----------------|-------|-----------------------------|-------|----|-------|
| $h = \frac{1}{4}$ | 0.16745915613 | 0.4081430533 | 0.17699745190 |
| $h = \frac{1}{8}$ | 0.04359068499 | 1.9417 | 0.04874074561 | 1.8605 |
| $h = \frac{1}{16}$ | 0.01089695739 | 2.0001 | 0.02627202964 | 1.9888 | 0.0125633692 | 1.9559 |
| $h = \frac{1}{32}$ | 0.00271695969 | 2.0039 | 0.00659970399 | 1.9931 | 0.00317140041 | 1.9860 |
| $h = \frac{1}{64}$ | 0.00067808437 | 2.0025 | 0.00165462312 | 1.9959 | 0.00079551293 | 1.9952 |

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