Boundary Control Problems in Convective Heat Transfer with Lifting Function Approach and Multigrid Vanka-Type Solvers

Eugenio Aulisa¹, Giorgio Bornia¹,* and Sandro Manservisi²

¹ Department of Mathematics and Statistics, Texas Tech University, Lubbock, TX 79409-1042, USA.
² Dipartimento di Ingegneria Industriale, Università di Bologna, Bologna, Italy.

Received 13 September 2014; Accepted (in revised version) 23 January 2015

Abstract. This paper deals with boundary optimal control problems for the heat and Navier-Stokes equations and addresses the issue of defining controls in function spaces which are naturally associated to the volume variables by trace restriction. For this reason we reformulate the boundary optimal control problem into a distributed problem through a lifting function approach. The stronger regularity requirements which are imposed by standard boundary control approaches can then be avoided. Furthermore, we propose a new numerical strategy that allows to solve the coupled optimality system in a robust way for a large number of unknowns. The optimality system resulting from a finite element discretization is solved by a local multigrid algorithm with domain decomposition Vanka-type smoothers. The purpose of these smoothers is to solve the optimality system implicitly over subdomains with a small number of degrees of freedom, in order to achieve robustness with respect to the regularization parameters in the cost functional. We present the results of some test cases where temperature is the observed quantity and the control quantity corresponds to the boundary values of the fluid temperature in a portion of the boundary. The control region for the observed quantity is a part of the domain where it is interesting to match a desired temperature value.

AMS subject classifications: 49J20, 65N55
Key words: Boundary optimal control, multigrid and domain decomposition algorithms, heat and Navier-Stokes equations.

1 Introduction

The optimal control of the heat and Navier-Stokes equations is an interesting subject due to a wide range of applications. In particular, coupled thermo-fluid mechanics optimal

*Corresponding author. Email addresses: eugenio.aulisa@ttu.edu (E. Aulisa), giorgio.bornia@ttu.edu (G. Bornia), sandro.manservisi@unibo.it (S. Manservisi)
control problems are very challenging due to their difficulties both in the theoretical formulation and in the construction of the computational algorithm. The implementation of a computationally feasible and robust boundary control algorithm is a nontrivial task. The first nontrivial issue is the definition and numerical implementation of the boundary control in a large functional space which is naturally associated to the volume variables. A control boundary constraint or a cumbersome numerical implementation can seriously limit the set of possible solutions. For this reason in many circumstances it is desirable to transform a boundary optimal control problem into a distributed problem through the lifting function approach of nonhomogeneous boundary conditions. With the lifting function approach, boundary controls are defined in the appropriate function boundary spaces restriction of the natural state volume spaces. Another challenging task in the study of optimal control problems is the numerical solution of the optimality system. The coupled implicit solution of the state, adjoint and control equations may allow a stronger and more robust solution process, as the oscillations that would be induced in the case of uncoupled solution become negligible. The uncoupled solution of the state and the adjoint system brings to limits in the choice of the penalty parameter for the tracking term in the objective functional. In fact in the limit of this parameter tending to infinity the solution tends to lose smoothness and the uncoupled algorithm cannot converge. For a strong and robust optimization the optimality system must be solved in a coupled way. To this end, we propose the use of multigrid algorithms with domain decomposition Vanka-type smoothers. With these the problem is split into small blocks of finite element subdomains with a small number of degrees of freedom, so that an optimal solution is computed by solving the fully coupled state-adjoint system on each subdomain. In this way large penalty parameters for the tracking term can be used and a better matching with the desired state can be attained.

Various works have been presented in literature for this class of problems, see for example [13–15] and references therein. The analysis of a two-dimensional problem for the Boussinesq equations is treated in [15]. Neumann boundary optimal control problems for the stationary Boussinesq equations including solid media are considered in [13]. In [14], a linear feedback control problem for the Boussinesq equations is studied. In this paper, we consider a similar optimal control problem for the heat and Navier-Stokes equations in which velocity influences temperature but the effects of temperature on the velocity and pressure fields, such as buoyancy, are neglected. This type of one-way coupling occurs in the case where forced convection is the dominant physical mechanism for heat transfer. The control is performed through the boundary conditions of temperature on all or part of the boundary. These boundary controls are implemented in the form of lifting functions, since this method brings several advantages from theoretical and computational points of view [3, 17].

The implementation of boundary controls is a nontrivial task, especially when one wants to search for controls in natural trace spaces. In [18, 19] the trace space $H^{1/2}(Γ)$ is used for the control, but this involves the introduction of a norm on this space that is realized by means of a stabilized hypersingular boundary integral operator. This con-
struction is only applicable for elliptic constraint equations and if a fundamental solution is known. For the discretization, finite element and boundary element approximations are considered. In [12], wavelet coordinates are used to introduce a representation of an original elliptic abstract control problem. In the wavelet framework, this allows to obtain an equivalent cost functional in which the norm of the wavelet representation of the control stems from measuring the original control in the natural space $H^{1/2}(\Gamma)$. In our present work we penalize the cost functional with lifting functions and we use standard norms on volume spaces. Our approach is intended to work for general operators and not to be restricted to elliptic operators.

We briefly introduce some basic notations and definitions for the functional analytic framework. Let $\Omega$ be a subset of $\mathbb{R}^N$ with Lipschitz continuous boundary $\Gamma$. When $E \subset \Omega$, $|E|$ denotes the Lebesgue measure of the set $E$. When $F \subset \Gamma$, $\sigma(F)$ is the surface measure of $F$. We denote as $H^m(\Omega)$ the usual Sobolev spaces of square integrable functions with respect to the Lebesgue measure in $\mathbb{R}^N$ and with square integrable weak derivatives up to order $m$. We use the boldface notation $H^m(\Omega)$ for vector-valued functions. We denote with $(\cdot, \cdot)_m$ the scalar product in $H^m(\Omega)$ and the induced norm is $\| \cdot \|_m = \sqrt{(\cdot, \cdot)_m}$ [1]. Let $H^0(\Omega)$ denote the closure of $C_0^\infty(\Omega)$ with respect to the norm $\| \cdot \|_m$ and $H^{-m}(\Omega)$ denote the dual space of $H^0(\Omega)$. Also, we define the spaces

$$L^2_0(\Omega) = \left\{ p \in L^2(\Omega) \mid \int_\Omega p \, dx = 0 \right\}$$

(1.1)

and, given any subset $\Gamma_s \subset \Gamma$, we denote the space $H^1_{\Gamma_s}(\Omega)$ as

$$H^1_{\Gamma_s}(\Omega) = \{ u \in H^1(\Omega) \mid \gamma_{\Gamma_s} u = 0 \},$$

(1.2)

where $\gamma_{\Gamma_s}$ is the trace restriction operator on $\Gamma_s$. Let us introduce the bilinear and trilinear forms. We define

$$a_T(T, v) = \int_\Omega \nabla T \cdot \nabla v \, d\Omega \quad \forall T, v \in H^1(\Omega),$$

(1.3)

$$c_T(u, T, v) = \int_\Omega (u \cdot \nabla) T \cdot v \, d\Omega \quad \forall T, v \in H^1(\Omega), u \in H^1(\Omega).$$

(1.4)

For the Navier-Stokes equations, we set

$$a(u, v) = \int_\Omega \nabla u : \nabla v \, dx, \quad d(v, q) = -\int_\Omega q \nabla \cdot v \, dx,$$

(1.5)

$$c(u, v, w) = \int_\Omega (u \cdot \nabla) v \cdot w \, dx.$$  

(1.6)

Also, let us introduce the nondimensional quantities

$$Re = \frac{\rho U L}{\mu}, \quad Pr = \frac{\mu c_p}{k}, \quad Pe = Re Pr,$$

(1.7)
where \( Re \) is the Reynolds number, \( Pr \) is the Prandtl number, \( Pe \) is the Péclet number, and \( \rho, U, L, \mu, c_p, k \) denote the reference values for density, velocity, length, dynamic viscosity, specific heat at constant pressure and thermal conductivity of the fluid, respectively.

Hence, in the Navier-Stokes equations we pick \( f \in H^{-1}(\Omega), g \in H^{1/2}(\Gamma) \) and we seek for \((u,p) \in H^1(\Omega) \times L^2(\Omega)\) such that

\[
\frac{1}{Re} a(u,v) + c(u,u,v) + d(v,p) = f(v) \quad \forall v \in H^1_0(\Omega),
\]

\[
d(u,q) = 0 \quad \forall q \in L^2(\Omega),
\]

\[
u = g \quad \text{on} \: \Gamma,
\]

where \( g \) satisfies the compatibility condition \( \int_{\Gamma} g \cdot n d\Gamma = 0 \). Let us consider the temperature equation. In order to formulate the optimal control problem, for the temperature Dirichlet boundary conditions we subdivide the boundary into two parts, \( \Gamma_d \) and \( \Gamma_c = \Gamma \setminus \Gamma_d \).

The Dirichlet datum can be split into two functions \( h_d \) and \( h_c \) defined on \( \Gamma_d \) and \( \Gamma_c \) respectively. With the function \( h_d \) on \( \Gamma_d \) we enforce fixed Dirichlet conditions, while the function \( h_c \) is variable in order to perform the control on the system. Therefore, given \( f_T \in H^{-1}(\Omega) \) we seek \( T \in H^1(\Omega) \) such that

\[
\frac{1}{Pe} a_T(T,v) + c_T(u,T,v) = <f_T,v> \quad \forall v \in H^1_0(\Omega),
\]

\[
T = h_d \quad \text{on} \: \Gamma_d, \quad T = h_c \quad \text{on} \: \Gamma_c,
\]

where the quantity \( f_T \) is a heat source that may come from viscous dissipation or other physical effects. Let us enforce Dirichlet temperature conditions with a lifting function. So, we simply introduce \( T_0 \in H^1(\Omega) \) such that

\[
T_0 = h_d \quad \text{on} \: \Gamma_d, \quad T_0 = h_c \quad \text{on} \: \Gamma_c.
\]

Hence, we can express the temperature field as \( T = T^* + T_0 \) and, for a given \( T_0 \in H^1(\Omega) \) such that (1.13) is satisfied, we seek \( T^* \in H^1_0(\Omega) \) such that

\[
\frac{1}{Pe} a_T(T^*,v) + c_T(u,T^*,v) = <f_T,v> - \frac{1}{Pe} a_T(T_0,v) + c_T(u,T_0,v) \quad \forall v \in H^1_0(\Omega).
\]

The heat equation has the following strong form

\[
- \frac{1}{Pe} \Delta T^* + (u \cdot \nabla)T^* = f_T + \frac{1}{Pe} \Delta T_0 - (u \cdot \nabla)T_0, 
\]

\[
T^* = 0 \quad \text{on} \: \Gamma, 
\]

\[
T_0 = h_d \quad \text{on} \: \Gamma_d, \quad T_0 = h_c \quad \text{on} \: \Gamma_c
\]

and for the Navier-Stokes equations we can write

\[
(u \cdot \nabla)u = \frac{1}{Re} \Delta u - \nabla p + f \quad \text{in} \: \Omega,
\]

\[
\nabla \cdot u = 0 \quad \text{in} \: \Omega,
\]

\[
u = g \quad \text{on} \: \Gamma.
\]
We remark that in the system (1.15)-(1.20) the coupling between the Navier-Stokes and temperature equations is a one-way coupling, in the sense that velocity influences temperature through convection while we do not consider possible influences of temperature in the Navier-Stokes equations, such as buoyancy or variable density.

Now, let $U_{ad}$ be the set of admissible targets and $T_0$ be the control. We formulate the optimal control problem as follows.

**Problem 1.** Given a desired temperature $T_d \in U_{ad}$ and a pair $(u, p)$ satisfying the system (1.8)-(1.10), find a minimizer of the objective functional

$$
\mathcal{J}(T^*, T_0) = \frac{\alpha}{2} \int_{\Omega} |T^* + T_0 - T_d|^2 d\Omega + \frac{\beta}{2} \int_{\Omega} T_0^2 d\Omega + \frac{\gamma}{2} \int_{\Omega} |\nabla T_0|^2 d\Omega,
$$

(1.21)

where $\alpha$, $\beta$ and $\gamma$ are positive constants and the variables $T^*$ and $T_0$ are subject to the state equation (1.15)-(1.17).

Of course, the target $T_d$ need not be a solution of the state equation (1.15). For instance, we may choose $U_{ad} = L^2(\Omega)$. Clearly, the goal is to match the fluid temperature to the desired profile $T_d$. The $\beta$-terms are used to limit the size of the control $T_0$. The values of the constants $\alpha, \beta$ and $\gamma$ can be chosen to adjust the relative importance of the terms in the functional. Large values of $\alpha$ allow for better temperature matching whereas small values result in poorer matching. The presence of the $\gamma$-term is motivated by the need to have $T_0$ in $H^1$ so that the optimal control problem is well-posed.

The lifting function can be considered an extension of the temperature field from the boundary to the interior. The optimal solution is then searched by exploring all possible extended functions. This approach brings several advantages from both a theoretical and a computational point of view. In fact in this case the boundary control algorithm can be solved by robust distributed control techniques over the inner part of the domain. Furthermore, the optimal boundary controls can be determined in natural half-integer Sobolev spaces as traces of the optimal extended functions. Also, in a direct boundary approach one would have to deal with fractional Sobolev norms which are much less convenient from a numerical point of view than the $H^1$-norm of the lifting function. This approach has also a disadvantage: the number of controlled points increases since the interior has much more points than the boundary. This drawback can be limited by restricting the controlled area to a small region close to the boundary.

The plan of the paper is as follows. In Section 2 we address the existence results for the optimal control problem under study, and we illustrate the advantageous features of the lifting function approach for the treatment of boundary controls. Section 3 is devoted to the description of the local multigrid algorithm for the solution of the optimality system, which is considered in combination with domain decomposition Vanka-type smoothers. Section 4 illustrates the results of several computations aimed at describing the numerical properties of the proposed algorithm.


2 Existence results

2.1 Existence of a constraint solution

In this paragraph we claim the basic results that are needed for the existence of a minimizer and for the subsequent first-order necessary condition. The state system consists in the temperature equation (1.11), in which the velocity satisfies the Navier-Stokes system (1.8)-(1.10). The existence of a weak solution to (1.8)-(1.10) with nonhomogeneous Dirichlet boundary conditions can be proved by using the Hopf lifting construction [9, 20]. The operators of the temperature equation (1.11) do not satisfy the coercivity property due to the presence of the convective terms and therefore the usual Lax-Milgram setting cannot be applied. This problem falls within the class of non-coercive elliptic partial differential equations. Coercivity can be obtained only by adding further regularity hypotheses on the velocity field.

Nevertheless, it is possible to claim the existence of the state solutions for the non-coercive elliptic case if the velocity field \( u \) is in \( L^2(\Omega) \) [8]. This existence result is obtained not in the Lax-Milgram setting but by using a Leray-Schauder Topological Degree argument [7]. We recall the result, which holds for weak solutions of a convection-diffusion equation of the following type

\[
-\text{div}(A^T \nabla T) + (u \cdot \nabla)T + bT = f \quad \text{in } \Omega, \\
T = T_1 \quad \text{on } \Gamma_1, \\
A^T \nabla T \cdot n + \lambda T = T_2 \quad \text{on } \Gamma_2.
\]

Here, \( \Gamma_1 \) and \( \Gamma_2 \) are measurable subsets of \( \Gamma \) such that \( \sigma(\Gamma_1 \cap \Gamma_2) = 0 \), and \( \Gamma = \Gamma_1 \cup \Gamma_2 \). Condition (2.2) is a Dirichlet boundary condition, while (2.3) is referred to as Fourier boundary condition.

**Theorem 2.1.** Let \( N_* = N \) when \( N \geq 3 \), \( N_* \in \{2, \infty\} \) when \( N = 2 \). Consider the following hypotheses:

1. \( \Gamma_1 \) and \( \Gamma_2 \) are measurable subsets of \( \Gamma \) such that \( \sigma(\Gamma_1 \cap \Gamma_2) = 0 \) and \( \Gamma = \Gamma_1 \cup \Gamma_2 \);
2. \( A : \Omega \to \mathbb{R}^{N \times N} \) is a measurable matrix-valued function which satisfies these two properties:
   - \( \exists \alpha_A > 0 \) such that \( A(x)\xi \cdot \xi \geq \alpha_A |\xi|^2 \) for a.e. \( x \in \Omega \) and for all \( \xi \in \mathbb{R}^N \);
   - \( \exists \Lambda_A > 0 \) such that \( \|A(x)\| \leq \Lambda_A \) for a.e. \( x \in \Omega \);
3. \( b \in L^{N_*/2}(\Omega), b \geq 0 \) a.e. on \( \Omega \);
4. \( \lambda \in L^{N_*-1}(\partial \Omega), \lambda \geq 0 \) \( \sigma \)-a.e. on \( \Gamma \);
5. \( u \in L^{N_*}(\Omega) \);
6. \( f \in \left(H^1_{\Gamma_1}(\Omega)\right)^* \) (the superscript \( ^* \) denotes the dual).
7. $\exists b_0 > 0, \exists E \subset \Omega$ such that $b \geq b_0$ in $E$, $\exists \lambda_0 > 0, \exists S \subset \Gamma_2$ such that $\lambda \geq \lambda_0$ in $S$, and either $\sigma(\Gamma_1) > 0$ or $|E| > 0$ or $\sigma(S) > 0$.

Then, there exists a unique solution $T \in H^1(\Omega)$ of (2.1).

Proof. The proof of this result is based on a Leray-Schauder Topological Degree argument and can be found in [8].

We remark that the hypothesis (7) in the previous theorem is usually referred to as the coercivity of the non-convective parts in (2.1). Based on the previous result, we can state the existence of solutions to the state equations (1.14).

**Theorem 2.2.** The Navier-Stokes system in (1.8) with Dirichlet boundary conditions has at least one solution $(u, p) \in H^1(\Omega) \times L^2_0(\Omega)$, which is unique when $Re < Re_c$, for some $Re_c > 0$ depending on the data. Given a velocity field $u$ satisfying (1.8) there exists a unique solution $T^* \in H^1_0(\Omega)$ to (1.14).

Proof. The existence of a weak solution $(u, p) \in H^1(\Omega) \times L^2_0(\Omega)$ to (1.8) with nonhomogeneous Dirichlet boundary conditions is well-known [9, 20]. Now, it is straightforward to check that we can fit the temperature state equations (1.14) within the framework of (2.1). First, thanks to the Sobolev compact embeddings $H^1(\Omega) \hookrightarrow L^q(\Omega)$ which hold for $1 \leq q < \infty$ when $N=2$ and for $1 \leq q \leq 6$ when $N=3$, we have that $u \in H^1(\Omega)$ verifies the hypothesis (5) both with $N=2$ and with $N=3$. Then, for the matrix $A^T$ we have $A^T = \frac{1}{Pe} I$, where $I$ is the identity matrix, so that (2) is trivially verified. Also, we notice that it suffices to have a non-zero measure Dirichlet boundary, $\sigma(\Gamma_1) > 0$, to avoid the conditions (3) and (4), which would require $b > 0$ on a set with strictly positive Lebesgue measure and $\lambda > 0$ on a set with strictly positive surface measure. In our case we have $b = 0$, (1.14) is a particular case of (2.1) and Theorem 2.2 completes the proof.

**Remark 2.1.** We remark that, if one needs to use the Lax-Milgram setting, existence can be proven by assuming some condition on the velocity field $u$. For instance, in the case of fully Dirichlet boundary conditions ($\Gamma_2 = \emptyset$), one can have coercivity with

$$-\frac{1}{2} \text{div} u + b \geq 0 \quad \text{in} \quad \mathcal{D}'(\Omega).$$

This condition of additional regularity on the velocity field is not needed in the Leray-Lions setting.

### 2.2 Existence of an optimal solution

We define the set of admissible controls $C_{ad}$ as

$$C_{ad} = \{ T_0 \in H^1(\Omega) : T_0 = h_d \quad \text{on} \quad \Gamma_d \},$$

where $h_d \in H^{1/2}(\Gamma_d)$ is a given function. The set of admissible states $A_{ad}$ is defined as

$$A_{ad} = \{ T^* \in H^1_0(\Omega) : T^* \text{ solves (2.13)} \text{ with } T_0 \in C_{ad} \}.$$
Remark 2.2. The functions included in the set $C_{ad}$ can be defined over the subdomain $\Omega_c \subset \Omega$ arbitrary small that includes the boundary $\Gamma_d$. This can be used to limit the controlled area.

Given these definitions, the optimal control problem consists of seeking $T^* \in A_{ad}$ and $T_0 \in C_{ad}$ such that the cost functional (1.21) is minimized for an admissible desired temperature profile $T_d \in U_{ad}$. We now state the existence of a global minimizer in the set of admissible solutions $A_{ad} \times C_{ad}$, i.e. the set of admissible states and controls.

Theorem 2.3. Given $T_d \in U_{ad}$, there exists a solution $(T^*,T_0) \in A_{ad} \times C_{ad}$ of the optimal control problem (1).

Proof. The proof follows standard techniques [6, 11]. By Theorem (2.2) the state system has a solution, therefore the set of admissible solutions $A_{ad} \times C_{ad}$ is nonempty. We define

$$M := \inf_{(T^*,T_0) \in A_{ad} \times C_{ad}} J(T^*,T_0).$$

Clearly $M$ exists by the greatest lower bound property of real numbers, since the set $A_{ad} \times C_{ad}$ is nonempty, and $M \geq 0$ since the objective functional $J(T^*,T_0)$ is non-negative. Thus let $\{(T^*_n,T_{0,n})\}$ be a minimizing sequence in $A_{ad} \times C_{ad}$ for $J(T^*,T_0)$, i.e.,

$$\lim_{n \to \infty} J(T^*_n,T_{0,n}) = M.$$

Now we show that the minimizing sequence $\{(T^*_n,T_{0,n})\}$ is uniformly bounded. The convergence of the sequence $J(T^*_n,T_{0,n})$ implies that it is bounded in $\mathbb{R}$, hence from the definition of the objective functional we have that $\|T_{0,n}\|_1$ is uniformly bounded, i.e., the sequence $\{T_{0,n}\}$ is uniformly bounded in $C_{ad}$. Then by the continuous dependence of the state solution on the data we also have that $\|T^*_n\|_1$ is uniformly bounded, i.e. the sequence $\{T^*_n\}$ is uniformly bounded in $A_{ad}$. Since $A_{ad} \times C_{ad}$ is a reflexive Banach space, we can extract a subsequence $\{(T^*_m,T_{0,m})\}$ that converges weakly to some $(\tilde{T}^*,\tilde{T}_0)$ in $H^1(\Omega) \times H^1_0(\Omega)$.

Now we pass the weak limit inside the constraint operator in order to show that $(\tilde{T}^*,\tilde{T}_0) \in A_{ad} \times C_{ad}$, i.e. the weak limit of the subsequences satisfies the constraints. This is straightforward due to the linearity of the state equations with respect to the arguments $T^*_m$ and $T_{0,m}$. Therefore, the set $A_{ad} \times C_{ad}$ is closed in the weak topology. In order to see that $(\tilde{T}^*,\tilde{T}_0)$ is also a solution for the optimal control problem we finally have to show that it minimizes the functional. In fact, by the weak lower semi-continuity of the functional, we have

$$M \leq J(\tilde{T}^*,\tilde{T}_0) \leq \liminf_{m \to \infty} J(T^*_m,T_{0,m}) = \lim_{m \to \infty} J(T^*_m,T_{0,m}) = M.$$

Therefore $J(\tilde{T}^*,\tilde{T}_0) = M$, i.e., the infimum $M$ of the functional is attained at the point $(\tilde{T}^*,\tilde{T}_0)$, which is indeed a global minimizer. \qed
**Remark 2.3.** We remark that, for the previous result to hold, we need to have $\beta$ and $\gamma$ strictly positive. This yields the boundedness of the minimizing sequence $T_{0,n}$ in $C_{ad}$. The regularization terms can be avoided in the case of inequality constraints on the control function $T_0$.

### 2.3 Advantages of the lifting function approach

For a better understanding of the proposed formulation for boundary control problems, let us point out the differences occurring between a standard direct boundary control approach and the lifting function one. A direct boundary control formulation would be as follows:

**Problem 2.** Seek for a minimizer of the objective functional

$$J_b(T,h_c) = \frac{\alpha_b}{2} \int_{\Omega} |T - T_d|^2 \, d\Omega + \frac{\beta_b}{2} \int_{\Gamma_c} h_c^2 \, d\Omega + \frac{\gamma_b}{2} \int_{\Gamma_c} |\nabla_s h_c|^2 \, d\Omega,$$

where $\nabla_s$ denotes the surface Laplacian operator.

This direct implementation presents some disadvantages that can be circumvented with the lifting function approach. First of all, due to the presence of the $\beta_b$ and $\gamma_b$ penalty boundary integrals in the objective functional (2.7), we see that the solution of the boundary control equation would belong to $H^1(\Gamma_c)$. Hence, it would be smoother than the half-integer space $H^{1/2}(\Gamma_c)$, which is a natural trace space for $H^1(\Omega)$, the space for the temperature $T$ variable. Therefore, a direct boundary approach would yield a smoother boundary temperature than needed, thus excluding less regular solutions. Instead, with the lifting function approach one computes an optimal control function $T_0$ in $H^1(\Omega)$ and then one can determine the optimal boundary control simply by trace restriction.

Also, the optimality condition for the control variable $h_c$ arising from the objective functional (2.7) is an equation defined on the boundary control subregion $\Gamma_c$. When the control region $\Gamma_c$ is not single-connected, i.e. consisting of a set of disjoint boundary portions, a different control equation should be implemented for each boundary part, leading to a rather cumbersome implementation. On the other hand, with the lifting function equation one can deal with controls in disjoint parts of the boundary with a single equation defined on the overall fluid domain, thus obtaining an easier representation.

Moreover, the introduction of the surface Laplacian term in (2.7) leads to a differential rather than algebraic optimality condition, so that boundary conditions on the control function must be enforced. At the conjunction between the $\Gamma_c$ and $\Gamma_d$ parts, in most cases the boundary conditions to be imposed on the control function $h_d$ are not physically meaningful and they must be enforced with some amount of arbitrariness.

Concerning the difference between the functional (2.7) and (1.21), we finally have to remark that the $\beta_b$ and $\gamma_b$ terms in (2.7) actually limit the size of the $H^1(\Gamma)$ norm of the boundary control function, while the $\beta$ and $\gamma$ terms in (1.21) limit the $H^1(\Omega)$ norm of the lifting function $T_0$. This means that in the second case not only the degrees of freedom for
the boundary control are involved, but also the degrees of freedom inside the domain, which are used for auxiliary purposes. If the addition of the inner degrees of freedom is computationally too expensive, the extension of the boundary conditions can be limited to any desired subvolume.

Also, we remark that Problem (1) and Problem (2) are not equivalent, in the sense that a solution \((\tilde{T}, \tilde{h}_c)\) of Problem (2) need not be equal to \((\tilde{T}^*, \gamma_{\Gamma_c}, \tilde{T}_0)\). In fact, \(\tilde{h}_c \in H^1(\Gamma_c)\) while \(\gamma_{\Gamma_c, \tilde{T}_0} \in H^{1/2}(\Gamma_c)\) (the existence of a solution to Problem (2) can be proven much in the same way as for Problem (1)).

### 2.4 First order necessary condition and the optimality system

In order to determine an optimality system whose solutions are candidate solutions for the optimal control problem, we use the well-known technique of Lagrange multipliers for problems constrained by partial differential equations \([10, 21]\). To this end, let us consider the constraint operator \(M: H^1_0(\Omega) \times H^1(\Omega) \to H^{-1}(\Omega)\) \([11]\). For given \(u\) solving the system (1.18), the constraint operator is defined as

\[
M(T, T_0) = f,
\]

for all functions \(v \in H^1_0(\Omega)\) and boundary conditions on the control \(T_0 = h_d\) on \(\Gamma_d\). In other words, the constraints (1.14) may be expressed as \(M(T, T_0) = f\). Since in this work the coupling is one-way, the function \(u\) must be regarded as an external datum, so that \((u, p)\) is a pair of external functions.

The Fréchet differential \(M'\) of the operator \(M\) is the continuous linear operator \(M': H^1_0(\Omega) \times H^1_f(\Omega) \to H^{-1}(\Omega)\) such that

\[
< q, v > = \frac{1}{P_e} a_T(T^*, v) + c_T(u, T^*, v) + \frac{1}{P_e} a_T(T_0, v) + c_T(u, T_0, v)
\]

for all test functions \(v \in H^1_0(\Omega)\) and with a given \(u\) solving the system (1.18). Due to the existence and uniqueness result for the state equations, the operator \(M'\) is an isomorphism, hence surjective, and its range is closed.

After the introduction of the constraint operator, one can deal with the optimal control problem by adopting well-known Lagrange multiplier techniques. To do this, one introduces the augmented objective functional

\[
\mathcal{L}(T^*, T_0, \lambda) = a_J(T^*, T_0) + < \lambda, M(T^*, T_0) >,
\]

where \(\lambda \in H^1_0(\Omega)\) and \(< \cdot, \cdot >\) denotes the duality pairing between \(H^1_0(\Omega)\) and its dual \(H^{-1}(\Omega)\). Then, one may claim the following first order necessary condition by using standard techniques (for instance, see \([11]\)).
Theorem 2.4. Let \((\overline{T}^*, \overline{T}_0)\) be a local minimizer for the objective functional \(J(T^*, T_0)\). There exists \((a, \lambda) \in \mathbb{R} \times H^1(\Omega)\) such that

\[
\mathcal{L}'(\overline{T}^*, \overline{T}_0, \lambda) = a J'(\overline{T}^*, \overline{T}_0) \cdot (\delta T^*, \delta T_0)
+ < \lambda, M' \cdot (\delta T^*, \delta T_0) >= 0 \quad \forall (\delta T^*, \delta T_0) \in (H^1_0(\Omega) \times H^1_{\Gamma_d}(\Omega)),
\]

where the superscript ’ denotes Fréchet differentiation. Furthermore, since \(M'(T^*, T_0) \cdot (\delta T, \delta T_0)\) is surjective, we have \(a \neq 0\) so that one may set \(a = 1\).

From (2.11) one derives the following adjoint and control equations. First, we notice that

\[
J'(T^*, T_0) \cdot (\delta T^*, \delta T_0) = \alpha (T^* + T_0 - T_d, \delta T^*) + \alpha (T^* + T_0 - T_d, \delta T_0)
+ \beta (T_0, \delta T_0) + \gamma (\nabla T_0, \nabla \delta T_0).
\]

(2.12)

By splitting the independent variations, for the adjoint equation we gather the variations \(\delta T^*\) and seek \(\lambda \in H^1_0(\Omega)\) such that

\[
\frac{1}{Pe} a_T(\lambda, \delta T^*) + c_T(u, \delta T^*, \lambda) + \alpha (T^* + T_0 - T_d, \delta T_0) = 0 \quad \forall \delta T^* \in H^1_0(\Omega).
\]

(2.13)

The control equation is given by gathering the variations \(\delta T_0\), to obtain

\[
\beta (T_0, \delta T_0) + \gamma a_T(T_0, \delta T_0) + \frac{1}{Pe} a_T(\lambda, \delta T_0) + c_T(u, \delta T_0, \lambda)
+ \alpha (T^* + T_0 - T_d, \delta T_0) = 0 \quad \forall \delta T_0 \in H^1_{\Gamma_d}(\Omega).
\]

(2.14)

3 Numerical algorithm

The global system can be split into two parts: the Navier-Stokes equations and the temperature optimality system. The Navier-Stokes system is decoupled from the optimality system and can be solved beforehand. Its linearization is performed using the fixed point approximation strategy. Once the velocity field is known, it can be explicitly used into the optimality system that is therefore linear. For the solution of both the linearized Navier-Stokes and the temperature optimality systems we intend to use the Local Multigrid algorithm which is described in Section 3.1. The basic idea is to combine the major features of geometric multigrid along with a-priori local mesh refinement.

Concerning the optimality system, the Local Multigrid algorithm is an example of collective smoothing multigrid (CSMG) approach [4]. The coupling in the optimality system between the state, adjoint and control variables is realized at the smoothing step level, in order to achieve typical multigrid efficiency and robustness with respect to changes in the regularization constants \(a, \beta, \gamma\) of the cost functional [4]. For the multigrid smoothing
of both systems, we investigate the use of Vanka-type smoothers, a discussion on which
is addressed in Section 3.2.

The discretization of the optimality system and of the Navier-Stokes equations is performed in a finite element framework. We approximate the optimality system variables $T^*, \lambda$ and $T_0$ by choosing a finite element space $Y_h \subset H^1(\Omega)$. For the velocity vector $u$ and the pressure variable $p$, we choose a pair of finite dimensional spaces $(X_h,M_h)$, such that $X_h \subset H^1(\Omega)$ and $M_h \subset L^2(\Omega)$, respectively. The construction of the spaces $Y_h,X_h,M_h$ is illustrated in the following section within the framework of the Local Multigrid algorithm.

We also recall that the pair of spaces $X_h$ and $M_h$ has to be chosen appropriately in such a way that the discrete inf-sup condition is satisfied. This guarantees the stability of the approximation of the Navier-Stokes equations [9].

We denote $X_{0,h} = X_h \cap H^1_0(\Omega)$, $M_{0,h} = M_h \cap L^2_0(\Omega)$, $Y_{0,h} = Y_h \cap H^1_0(\Omega)$; moreover, for any subset $\Gamma_s \subset \Gamma$, we set $Y_{\Gamma_s,h} = Y_h \cap H^1_{\Gamma_s}(\Omega)$. We seek a solution $(T^*_h, \lambda_h, T_{0,h}) \in Y_{0,h} \times Y_{0,h} \times Y_{0,h}$ that satisfies the discrete optimality system

\begin{align}
\frac{1}{Pe} a_T(T^*_h, w_h) + c_T(u_h, T^*_h, w_h) + \frac{1}{Pe} a_T(T_{0,h}, w_h) + c_T(u_h, T_{0,h}, w_h) = < f, w_h >, & \tag{3.1} \\
\frac{1}{Pe} a_T(\lambda_h, \delta T^*_h) + c_T(u_h, \delta T^*_h, \lambda_h) + a(T^*_h + T_{0,h} - T_{d,h}, \delta T^*_h) = 0, & \tag{3.2} \\
\beta(T_{0,h}, \delta T_{0,h}) + \gamma a_T(T_{0,h}, \delta T_{0,h}) + \frac{1}{Pe} a_T(\lambda_h, \delta T_{0,h}) + c_T(u_h, \delta T_{0,h}, \lambda_h) \\
& + a(T^*_h + T_{0,h} - T_{d,h}, \delta T_{0,h}) = 0, & \tag{3.3} \\
T^*_h = 0 & \text{ on } \Gamma, \tag{3.4} \\
\lambda_h = 0 & \text{ on } \Gamma, \tag{3.5} \\
T_{0,h} = h_d & \text{ on } \Gamma_d. \tag{3.6}
\end{align}

for all test functions $(w_h, \delta T^*_h, \delta T_{0,h}) \in Y_{0,h} \times Y_{0,h} \times Y_{\Gamma_d,h}$. The pair $(u_h, p_h) \in X_h \times M_{0,h}$ satisfies the discrete Navier-Stokes equations

\begin{align}
a(u_h, v_h) + c(u_h, u_h, v_h) + d(v_h, p_h) = < f, v_h >, & \tag{3.7} \\
d(u_h, q_h) = 0, & \tag{3.8} \\
u_h = g_h & \text{ on } \Gamma, \tag{3.9}
\end{align}

for all test functions $(v_h, q_h) \in X_{0,h} \times M_{0,h}$.

We remark that the boundary conditions for $T^*_h$ and $\lambda_h$ are Dirichlet homogeneous due to the temperature decomposition $T_h = T^*_h + T_{0,h}$. The non-homogeneous boundary conditions, either fixed or variable for the control, are all taken into account with the lifting function $T_{0,h}$. Hence, after solving the optimality system for the variables $(T^*_h, \lambda_h, T_{0,h})$ one can retrieve the total temperature field $T_h$ by the sum $T_h = T^*_h + T_{0,h}$.

### 3.1 Local Multigrid algorithm

In this section we describe the Local Multigrid algorithm (LMG) used to discretize and solve the temperature optimality system and the Navier-Stokes system. The domain $\Omega$
is first divided into several non-overlapping subregions, for each of which we introduce a finite element discretization. The different subregions may have different mesh sizes and the mesh may not be conforming at the interface between adjacent subregions. Nevertheless, continuity for each variable on the common boundaries can be enforced in a natural and straightforward way by an appropriate definition of the spaces in the geometric multigrid framework.

In Fig. 1 an example of geometrically nonconforming mesh partitioning is generated for a square domain $\Omega$. In the region close to the right and bottom boundary sides a progressive mesh refinement is constructed as follows. By starting at the coarse level $l = 0$, we discretize the entire domain $\Omega \equiv \Omega^0$ into a collection of finite elements (quadrilaterals or triangles in two dimensions; hexahedra, wedges or tetrahedra in three dimensions). A geometrically conforming coarse triangulation $T_{h}^{0}$ is then generated throughout the en-
tire domain $\Omega^0$. Based on a simple element midpoint refinement, successive level meshes $\mathcal{T}^l_h$ are built recursively on the corresponding subregions $\Omega^l$ up to the top level $l=n$, until the physics of the considered problem is well resolved. Finer subregions are included in coarser ones, namely $\Omega^0 \supseteq \Omega^1 \supseteq \cdots \supseteq \Omega^n$. This choice of local mesh refinement is very efficient, since only a small number of degrees of freedom is added during the refinement process, which drastically reduces the computational cost with respect to the case of uniform mesh refinement.

Each level mesh $\mathcal{T}^l_h$ is not defined on the whole domain $\Omega$, but only on the subregion $\Omega^l$, where the refinement up to level $l$ is performed. Every triangulation $\mathcal{T}^l_h$ is geometrically conforming within its own domain $\Omega^l$. In the example of Fig. 1, the refined subregions $\Omega^l$ are thinner regions closer to the right and bottom sides. If we consider the nonconforming mesh given by the union of all the triangulations $\mathcal{T}^l_h$, the nodes of the coarser mesh are always included in the nodes of the finer mesh on adjacent boundaries. This is the key point for this kind of discretization and it is always true for different level meshes generated by using midpoint refinement.

For each level $l = 0, 1, \cdots, n$, let $V^l_{loc} \subset H^1_0(\Omega^l)$ be a continuous finite-element space built on the triangulation $\mathcal{T}^l_h$ (for instance, Lagrange piecewise-linear or piecewise-quadratic). Clearly, functions in $V^l_{loc}$ have zero trace on $\partial \Omega^l$. Let $V^l$ be the extension by zero of $V^l_{loc}$ over the whole domain $\Omega$. Notice that each element $u^i \in V^l$ is continuous due to the zero trace on the boundary $\partial \Omega^l$. Also, for all levels $l$ we have $V^l \subset H^1_0(\Omega)$, i.e., these spaces are conforming in $H^1_0(\Omega)$ by their definition. Moreover, $V^{l-1} \subsetneq V^l$ and $V^i \cap V^j \neq \{0\}, i,j = 0,1,\cdots,n$, since finer functions can generate coarser ones. We define the space $V_l$ as the algebraic sum

$$V_l = \sum_{k=0}^l V^k = \left\{ v \mid v = \sum_{k=0}^l v^k, v^i \in V^i, i = 0,1,\cdots,l \right\}.$$  

(3.10)

Clearly, this space is conforming in $H^1_0(\Omega)$ as the algebraic sum of conforming subspaces and it is defined on the geometrically nonconforming triangulation

$$\mathcal{T}_{h,l} = \bigcup_{k=0}^l \mathcal{T}^k_h.$$  

(3.11)

Therefore we have $V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n$, $\mathcal{T}_{h,0} \subseteq \mathcal{T}_{h,1} \cdots \mathcal{T}_{h,n}$ and for each level $l$ two families of spaces:

1. $V^l$, built over a geometrically conforming triangulation on $\Omega^l$, with zero trace on $\partial \Omega^l$ and extension by zero in $\Omega \setminus \Omega^l$;

2. $V_l$, built over a geometrically nonconforming triangulation up to level $l$.

In the case of uniform mesh refinement, we have $\Omega^0 \equiv \Omega^1 \equiv \cdots \equiv \Omega^n$. If we denote the uniformly refined spaces with the superscript $\sim$, we recover $\tilde{V}_l \equiv \tilde{V}^l$ for all levels, because the refinement is performed over the whole domain $\Omega$. Also we have $\tilde{V}^0 \subseteq \tilde{V}^1 \subseteq \cdots \tilde{V}^n$ [5].
Comparing the local with the uniform case, it is clear that $V_l \subseteq \tilde{V}_l$ for all levels $l$, which reflects the reduced number of degrees of freedom obtained with local refinement. We define the index $n_{MG}$ to be the highest level index $l$ for which $V_l = V^l$. Note that $n_{MG} \geq 0$ since on the coarsest level ($l = 0$) we built a conforming triangulation all over the domain $\Omega$. Moreover if $n = n_{MG}$ then at each level we have a conforming triangulation, and the standard geometric multigrid spaces are recovered.

We can now define the inter-space transfer operators as follows. The coarse-to-fine operator

$$I_{l-1}^l : V_{l-1} \to V_l$$

is taken to be the natural injection. In other words

$$I_{l-1}^l u = u, \quad \forall u \in V_{l-1}.$$ (3.13)

The fine-to-coarse inter-space transfer operator

$$I_{l}^{l-1} : V_l \to V_{l-1}$$

is defined to be the transpose of $I_{l-1}^l$, with respect to the inner product $(\cdot, \cdot)$. In other words

$$(I_{l}^{l-1} w_l, u_{l-1}) = (w_l, I_{l-1}^l u_{l-1}) \quad \forall u_{l-1} \in V_{l-1}, \, w_l \in V_l.$$ (3.15)

Note that although the inter-space operators are defined similarly to the intergrid operators used in the standard geometric multigrid method [5], they differ due to the different spaces involved ($V_l$ instead of $\tilde{V}_l$).

We also define the prolongation operator

$$P_l : V^l \to V_l$$

to be the natural injection, in other words $P_l v = v, v \in V^l$, and the restriction operator

$$R^l : V_l \to V^l$$

to be the transpose of $P_l$, with respect to the inner product $(\cdot, \cdot)$ such that

$$(R^l w_l, u^l) = (w_l, P_l u^l) \quad \forall u^l \in V^l, \, w_l \in V_l.$$ (3.18)

The Local Multigrid algorithm works similarly to a geometric multigrid algorithm with the exception that the spaces for the unknown variables, the residuals, the error correction and the corresponding test functions are redefined according to the local mesh refinement process [5]. For the same reason the inter-level (fine-to-coarse and coarse-to-fine), the restriction and the prolongation operators are non-standard. Below we present a schematic description for the solution of the model problem

$$a(u_h, v_h) = (f, v_h) \quad \forall v_h \in V_h,$$ (3.19)
where $V_h := V_n$ as defined in (3.10), $a(\cdot,\cdot)$ is a differential operator, $f$ is an external force/source term and they both depend on the physics of particular problem at hand. First we describe the $l^{th}$ level iteration, then the full algorithm.

**The $l^{th}$ Level Iteration.** $L MG(l,z_0,g)$ is the approximate solution of

$$a(z,v_l) = (g,v_l) \quad \forall v_l \in V_l$$

obtained at the $l^{th}$ level with initial guess $z_0 \in V_l$.

For $l = 0$, $L MG(l,z_0,g)$ is the solution of the problem

$$a(L MG(l,z_0,g),v_0) = (g,v_0) \quad \forall v_0 \in V_0,$$

obtained with a direct method.

For $l > 0$, $L MG(l,z_0,g)$ is obtained recursively in 3 steps.

1. **Pre-smoothing Step.** For $1 \leq j \leq m_1$, let

$$z_j = z_{j-1} + P_l \left( S^l \right)^{-1} R^l r_j,$$

where $z_j \in V_l$ and $r_j \in V_l$ satisfies

$$(r_j,v_l) = (g,v_l) - a(z_{j-1},v_l) \quad \forall v_l \in V_l.$$  

Here, $S^l : V^l \rightarrow V^l$ is some invertible smoothing operator that will be defined later and it depends on the specific equation system.

2. **Error Correction Step.** Let $q_0 = 0$, $\bar{g} := I_{l-1} r_{m_1}$ and $q_1 = L MG(l-1,q_0,\bar{g})$. Then

$$z_{m_1+1} = z_{m_1} + I_{l-1} q_1.$$  

3. **Post-smoothing Step.** For $m_1 + 2 \leq j \leq m_1 + m_2 + 1$, let

$$z_j = z_{j-1} + P_l \left( S^l \right)^{-1} R^l r_j.$$

Then the output of the $l^{th}$ level iteration is

$$L MG(l,z_0,g) := z_{m_1+m_2+1},$$

where $m_1$ and $m_2$ are positive integers.

**The Full Local Multigrid algorithm.** The solution $u_h \in V_h$ of the problem

$$a(u_h,v_h) = (f,v_h) \quad \forall v_h \in V_h$$

is obtained with $L MG(0,u_h,\bar{g})$ and then proceeds as described above.
is obtained with the following nested iterations. For $l=0$, $u_0 \in V_0$ is the solution of
\[
a(u_0,v_0) = (f,v_0) \quad \forall v_0 \in V_0,
\]
(3.28)

obtained with a direct method.

For $1 \leq l \leq n$, the approximate solutions $u_l \in V_l$ are obtained recursively from
\[
u_{l,0} = I_{l-1}^l u_{l-1},
\]
(3.29)
\[
u_{l,j} = LMG(l,u_{l-1},f), \quad 1 \leq j \leq r,
\]
(3.30)
\[
u_l = u_{l,r}.
\]
(3.31)

Finally, the solution of the problem (3.19) is $u_h = u_n$. The spaces $Y_h$ for the variables $\bar{T}_h$, $\lambda_h$ and $T_0,h$ in the discretized temperature optimality system, as well as the spaces $(X_h,M_h)$ for $u_h$ and $p_h$ are built up to the finest level $n$ like the space $V_n$ defined by (3.10).

### 3.2 Domain decomposition Vanka-type smoothing

With the exception of the coarse level where a direct solver is used, the pre-smoothing and the post-smoothing steps in the LMG algorithm are done with Vanka-type iterative smoothers, whose purpose is to reduce the residual norm. The Vanka-type class of smoothers can be considered as a block Gauss-Seidel method, where each block consists of a small number of degrees of freedom (for details see [2, 16, 22, 23]). In each step a large number of small linear systems is solved. Each system corresponds to all the degrees of freedom associated with a block of elements in a compact subdomain. The blocks are overlapping and their union covers the whole domain. The Vanka-smoother can be regarded as an overlapping domain decomposition method, where each subsystem can be solved using the most appropriate direct or iterative solver.

Note that in the pre- and post-smoothing steps we defined the smoothing operator as $S_l^l : V_l \rightarrow V_l$. Since $V_l$ is built as a continuous extension by zero of $V_{loc}^l$ to the whole domain $\Omega$, then without loss of generality we can consider the equivalent operator $S_l^l_{loc} : V_{loc}^l \rightarrow V_{loc}^l$. The space $V_{loc}^l$ is associated to the conforming triangulation $T_h^l$ on $\Omega$.

For conforming finite elements the Navier-Stokes block may consist of all the elements containing a certain set of pressure vertices. Thus, in this case a smoothing step with a Vanka smoother consists of a loop over all the blocks, solving only the equations involving the degrees of freedom (dofs) related to the elements that are around the pressure vertices. The velocity and pressure dofs are solved many times in one smoothing step. Examples of computations with Vanka-smoothers can be found in [2, 22, 23].

Fig. 2 shows an example of a minimal Vanka-block for a 2D structured Quad9/Quad4 Taylor-Hood finite element triangulation. We use the same strategy for non-structured 2D and 3D triangulations. The solution steps are summarized, with reference to Fig. 2, as follows:
1. The internal elements are identified: one darker element located in the middle.

2. A search for all the pressure dofs associated with the internal elements is performed: the 4 solid circles.

3. A search for all the velocity dofs associated to the internal elements and the neighboring ones (the shaded elements) is done: the 49 solid crosses.

4. The equation system associated with all the found dofs \((4 + 49 \times 2 = 102)\) is extracted from the global system, and solved using a direct solver.

The equation system associated with the dofs to be solved involves a number of pressure and velocity dofs that are not solved. In this case the 32 dashed circles for the pressure, and the 80 dashed crosses for the velocity are considered known from the previous solution. Their contribution to the equation system is taken into account in the right hand side of the equation as an explicit term calculated with the last updated values.

The number of internal elements used in each Vanka-block can be fixed depending on the solution strategy. A larger number of internal elements implies a bigger size of the domain for each Vanka-block. This in turn yields a smaller number of Vanka-blocks needed to span the whole domain, which improves the convergence properties. However, the number of dofs increases linearly with the Vanka-block size, thus the computational time increases with a cubic factor, since a direct solver is used. For our two-dimensional simulations, we found that a good compromise between these contrasting factors is obtained with a number of 256 quadratic internal elements. Also notice that in our algorithm the limiting case of a unique Vanka block containing the entire level subdomain corresponds to a direct solver smoother for the LMG algorithm.

For the temperature optimality system we solve only for the quadratic dofs associated to the internal and neighboring elements. We perform the solution of the optimality system in a fully coupled way block by block. In the case described in Fig. 2, only the 49 solid crosses must be considered for each unknown, for a total of \(49 \times 3 = 147\) dofs per block for the 3 scalar variables \((T^*, \lambda, T_0)\). The coupled implicit solution of the state, costate and
control equations allows a better solution process. In fact, numerical oscillations would be induced in the case of uncoupled solution, even with the use of gradient-type algorithms [3]. These oscillations are removed in case of coupled solution for the optimality system.

**Remark 3.1.** In solving the Vanka-block equation systems we have used direct solvers. Iterative solvers such as GMRES with Schur complement preconditioner (for the Navier-Stokes block), and GMRES with ILU preconditioner (for the temperature optimality block) are also possible. The results are somehow comparable in terms of computational time, since the dimensions of the systems to be solved are relatively small. Direct solvers seem to be more robust with respect to iterative ones but at this moment there is no clear advantage in using ones rather than the others. In this work we only use direct solvers and we plan to investigate the convergence and robustness properties of the iterative solvers for the Vanka-blocks in future papers.

### 4 Numerical results

#### 4.1 Multigrid smoother comparison

In this part we compare the performances of two different smoothers in the multigrid algorithm: the Vanka-type smoother previously discussed in Section 3.2 and a GMRES smoother with ILU preconditioner. Since we focus on the role played by the smoothers, we consider for this series of tests a standard geometric multigrid algorithm, which corresponds to our Local Multigrid algorithm in the special case of uniform mesh refinement [5]. We study the solution of the optimality system for different target regions \( \Omega_c \) and a given boundary control region \( \Gamma_c \). In this way we can also evaluate how the location of the target region with respect to the boundary control region \( \Gamma_c \) affects the solution of the system.

Let us consider a two-dimensional domain \( \Omega = \{(x,y): x \in [0,1], y \in [0,2]\} \) as shown in Fig. 3. From now on and unless otherwise stated, we assume unit reference values and refer to non-dimensional quantities. We perform the calculations for three different cases, denoted as Case A, B and C, corresponding to three target regions shown in Fig. 3 and given by \( \Omega_{c,A} = \{(x,y): x \in [0.25,0.75], y \in [0.4,0.8]\} \), \( \Omega_{c,B} = \{(x,y): x \in [0.25,0.75], y \in [0.8,1.2]\} \) and \( \Omega_{c,C} = \{(x,y): x \in [0.25,0.75], y \in [1.2,1.6]\} \), respectively. Over each of these target regions we want to steer the temperature profile towards the constant value \( T_d = 0.9 \).

The boundary region where Dirichlet boundary conditions of the temperature field are controlled is given by \( \Gamma_c = \{(x,y): x \in [0.25,0.75], y = 0\} \). Concerning the boundary with fixed Dirichlet temperature conditions we set \( T = T_0 = 0 \) on \( y = 1 \) and \( T = T_0 = 1 \) elsewhere. For the Navier-Stokes equations, we set an inflow velocity profile \( g = (0,0.375) \) on the boundary \( \Gamma_c \) and a pressure outflow condition on the line \( y = 1 \). Hence, \( \Gamma_c \) acts both as a boundary control region and as a mass injection region. No slip boundary conditions are enforced on the remaining part of the boundary. In this test we want to study the
Figure 3: Three cases with mass injection from below and different target regions $\Omega_c$: $y \in [0.4, 0.8]$ (Case A), $y \in [0.8, 1.2]$ (Case B), $y \in [1.2, 1.6]$ (Case C).

The numerical solution of the optimality system for different choices of the position of the control boundary $\Gamma_c$ and the target regions $\Omega_{c,A}, \Omega_{c,B}$ and $\Omega_{c,C}$ respectively. The relation between the target region and the boundary control region is different for each optimal control problem under study. We set $\beta = \gamma = 1$ so that the objective functional becomes

$$ J(T^*, T_0) = \alpha \int_{\Omega_c} |T^* + T_0 - T_d|^2 d\Omega + \frac{1}{2} \int_{\Omega} \nabla T_0^2 d\Omega + \frac{1}{2} \int_{\Omega} |\nabla T_0|^2 d\Omega. \quad (4.1) $$

We remark that, due to the presence of the indicator function $\chi_{\Omega_c}$ of the domain $\Omega_c$, the integral in the $\alpha$ term is restricted to the target region. On the other hand, the penalty terms must be extended to the whole domain $\Omega$, in order to regularize the control and limit its size. This penalty must involve the entire fluid domain, or at least a part of the fluid domain whose boundary contains all the boundary control regions.

The numerical solution of the optimality system is very challenging since many parameters are involved. The computation of a candidate optimal solution for this problem is by no means trivial. A variety of features may be studied of both theoretical and practical nature. For instance, the choice of the values for $\alpha$, $\beta$ and $\gamma$ is very important for various reasons. A higher value of $\alpha$ implies a lower error with respect to the target profile. The values of these constants lead to different values for the adjoint and control variables, so that a different local minimum may be computed. We remark that the $\gamma$ constant must be set to a non-zero value not only to have solution in $H^1$, but also in order to have a sufficient smoothness on the control. Starting at the coarse level $l = 0$, we discretize the computational domain with a mesh that consists of $n_0 = 4 \times 20$ quadratic elements. The mesh is uniformly refined up to the level $l = 5$ by using the midpoint refinement strategy. Thus, at each level the number of elements is given by $n_l = n_0 \times 4^l$. 
Table 1: Values of the temperature error $E_T = \int_{\Omega} |T - T_d|^2 d\Omega$ for Cases A, B and C, for various values of $\alpha$ and for different types of smoothers.

<table>
<thead>
<tr>
<th></th>
<th>l = 4</th>
<th>l = 5</th>
<th>l = 4</th>
<th>l = 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$</td>
<td>$E_T$, Vanka</td>
<td>$E_T$, GMRES</td>
<td>$E_T$, Vanka</td>
<td>$E_T$, GMRES</td>
</tr>
<tr>
<td>$10^5$</td>
<td>2.86080e-04</td>
<td>2.86080e-04</td>
<td>2.85127e-04</td>
<td>2.85127e-04</td>
</tr>
<tr>
<td>$10^7$</td>
<td>1.40525e-04</td>
<td>1.40525e-04</td>
<td>1.40484e-04</td>
<td>1.40484e-04</td>
</tr>
<tr>
<td>$10^8$</td>
<td>1.33327e-04</td>
<td>1.33327e-04</td>
<td>1.33216e-04</td>
<td>1.33216e-04</td>
</tr>
<tr>
<td>$10^9$</td>
<td>1.12623e-04</td>
<td>1.12623e-04</td>
<td>none</td>
<td>none</td>
</tr>
<tr>
<td>$10^{10}$</td>
<td>1.06577e-04</td>
<td>1.06577e-04</td>
<td>none</td>
<td>none</td>
</tr>
</tbody>
</table>

Case B

<table>
<thead>
<tr>
<th></th>
<th>l = 4</th>
<th>l = 5</th>
<th>l = 4</th>
<th>l = 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$</td>
<td>$E_T$, Vanka</td>
<td>$E_T$, GMRES</td>
<td>$E_T$, Vanka</td>
<td>$E_T$, GMRES</td>
</tr>
<tr>
<td>$10^4$</td>
<td>1.10891e-04</td>
<td>1.10891e-04</td>
<td>1.10418e-04</td>
<td>1.10418e-04</td>
</tr>
<tr>
<td>$10^5$</td>
<td>2.75524e-05</td>
<td>2.75524e-05</td>
<td>2.75341e-05</td>
<td>2.75341e-05</td>
</tr>
<tr>
<td>$10^6$</td>
<td>2.54883e-05</td>
<td>2.54883e-05</td>
<td>2.54872e-05</td>
<td>2.54872e-05</td>
</tr>
<tr>
<td>$10^7$</td>
<td>2.54511e-05</td>
<td>2.54511e-05</td>
<td>2.54500e-05</td>
<td>2.54500e-05</td>
</tr>
<tr>
<td>$10^8$</td>
<td>2.53119e-05</td>
<td>none</td>
<td>2.53090e-05</td>
<td>none</td>
</tr>
<tr>
<td>$10^9$</td>
<td>2.40505e-05</td>
<td>none</td>
<td>2.40328e-05</td>
<td>none</td>
</tr>
<tr>
<td>$10^{10}$</td>
<td>none</td>
<td>none</td>
<td>none</td>
<td>none</td>
</tr>
</tbody>
</table>

Case C

<table>
<thead>
<tr>
<th></th>
<th>l = 4</th>
<th>l = 5</th>
<th>l = 4</th>
<th>l = 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$</td>
<td>$E_T$, Vanka</td>
<td>$E_T$, GMRES</td>
<td>$E_T$, Vanka</td>
<td>$E_T$, GMRES</td>
</tr>
<tr>
<td>$10^4$</td>
<td>1.49337e-04</td>
<td>1.49337e-04</td>
<td>1.49335e-04</td>
<td>1.49335e-04</td>
</tr>
<tr>
<td>$10^5$</td>
<td>1.49068e-04</td>
<td>1.49068e-04</td>
<td>1.49068e-04</td>
<td>1.49068e-04</td>
</tr>
<tr>
<td>$10^7$</td>
<td>1.49065e-04</td>
<td>none</td>
<td>1.49065e-04</td>
<td>none</td>
</tr>
<tr>
<td>$10^8$</td>
<td>1.49063e-04</td>
<td>none</td>
<td>1.49063e-04</td>
<td>none</td>
</tr>
<tr>
<td>$10^9$</td>
<td>1.49044e-04</td>
<td>none</td>
<td>1.49043e-04</td>
<td>none</td>
</tr>
<tr>
<td>$10^{10}$</td>
<td>1.48862e-04</td>
<td>none</td>
<td>1.48859e-04</td>
<td>none</td>
</tr>
</tbody>
</table>

In Table 1 we show the values of the temperature error, which constitutes the tracking term in the cost functional (4.1), namely $E_T = \int_{\Omega} |T - T_d|^2 d\Omega$, for the three target regions as a function of the $\alpha$ parameter. We compare the solution errors $E_T$ obtained at two different discretization levels, $l = 4$ and $l = 5$, and using the two different smoothers, Vanka and GMRES. As a stopping criterion for the solver we have chosen the $l_\infty$ norm of the difference between two solutions of two consecutive V-cycles to not exceed $10^{-4}$. If this target was not achieved in less than 15 V-cycles, we assumed the solver failed to
converge. It is clear that the \( \alpha_{\text{max}} \) parameter depends on the particular problem under consideration, being in general dependent on the target function, on the definition of both the target region \( \Omega_c \) and the boundary control region \( \Gamma_c \), along with all the other given data of the problem. A higher value of the \( \alpha \) parameter implies a smaller error \( E_T \) in tracking the desired field. On the other hand a higher value of \( \alpha \) brings to a less smooth solution, hence to a stiffer problem whose numerical solution eventually fails to converge. Nevertheless, the maximum value of \( \alpha \) cannot be known a priori nor estimated, and must be determined for each case by numerical investigation.

Clearly, each specific type of smoother used in the multigrid algorithm has a different maximum value of \( \alpha \) before experiencing lack of convergence. We observed that overall the Vanka smoother performs better than the GMRES one. In particular Table 1 shows that for the case A the two smoothers perform equally well, while in cases B and C the Vanka smoother is more robust, being capable of finding solutions with values of \( \alpha \) respectively 2 and 4 orders of magnitude larger than the GMRES ones. In order to study the effects of a change in the position of the target region \( \Omega_c \) with respect to the boundary control region \( \Gamma_c \) corresponding to Cases A, B and C, in Fig. 4 we plot the profiles of \( T^* \), \( T_0 \), \( T \) and \( \lambda \) along the longitudinal line \( x = 0.5 \) for Cases A, B and C and without control (Case 0), assuming \( \alpha = 10^4 \).
Figure 5: Boundary controls as restriction of the lifting function $T_0$ on the boundary line $y = 0$ with $\alpha = 10^4$ for Cases A, B, C and without control (Case 0).

Figure 6: Case B. Plots over the fluid domain $\Omega$ of the lifting function $T_0$ for $\alpha = 10^3, 5 \cdot 10^3, 10^4, 10^5$ (top to bottom, left to right).

In Fig. 5 we show the boundary controls as restriction of the lifting function on the boundary line $y = 0$ for Cases A, B and C with $\alpha = 10^4$ and temperature boundary value without control. In order to show the dependence on the $\alpha$ parameter, in Fig. 6 we plot
Figure 7: Case B. On the left: profiles of $T$ along the longitudinal line $x=0.5$ for $\alpha = 10^3, 5\cdot 10^3, 10^4, 10^5$ (circle, triangle, square and pentagon mark, respectively). On the right: a zoom on the target region.

Figure 8: Case B. On the left: boundary controls as restriction of the lifting function on the boundary line $y=0$. On the right: adjoint temperature $\lambda$ along the longitudinal line $x=0.5$. In both figures the plots are reported for $\alpha = 10^3, 5\cdot 10^3, 10^4, 10^5$ (circle, triangle, square and pentagon mark, respectively).

for Case B the lifting function $T_0$ over the domain $\Omega$ for $\alpha = 10^3, 5\cdot 10^3, 10^4, 10^5$. In Fig. 7 we show on the left the temperature profiles along the longitudinal line $x=0.5$ for Case B and $\alpha = 10^3, 5\cdot 10^3, 10^4, 10^5$. On the right we plot a zoom of these profiles on the target region $\Omega_c$. For the same values of $\alpha$, Fig. 8 also shows the boundary controls on the left and the corresponding adjoint temperature $\lambda$ profiles on the right.

4.2 Solution on different meshes

In the second part of this section we compare the solutions obtained using different non-conforming meshes. We consider only Case A as defined in the previous subsection, where the target region $\Omega_c$ is in between $0.4 \leq y \leq 0.8$. In Figs. 9 and 10 all the meshes considered in this test are shown. We report the results about the solution of the optimality
system in the finite element spaces constructed according to (3.10) on the nonconforming triangulations considered. We use the notation $l_i$ for conforming mesh obtained after $i$ midpoint refinements and $l_{ij}$ for 2-level nonconforming mesh obtained using the $l_i$ conforming mesh for $0 \leq y \leq 0.8$ and the $l_i$ conforming mesh for $0.8 \leq y \leq 2.0$. The label $l_{ij,k}$ is used for 3-level nonconforming mesh obtained using the $l_{ij}$ nonconforming mesh for $0 \leq y \leq 1.2$ and the $l_i$ conforming mesh for $1.2 \leq y \leq 2.0$. Finally $l_{ij,k,l}$ denotes a 4-level nonconforming mesh obtained using the $l_{ij,k}$ nonconforming mesh for $0 \leq y \leq 1.6$, and the $l_i$ conforming mesh for $1.6 \leq y \leq 2.0$.

The solution is computed using the Full Local Multigrid algorithm described in Section 3.1 together with the Vanka-type smoother illustrated in Section 3.2. In Table 2 we compare the results obtained with the mesh $l_4$ and with the 3 different nonconforming meshes $l_{4,3}$, $l_{4,3,2}$ and $l_{4,3,2,1}$. Note that the part of the domain containing the control region $\Gamma_c$ and the target region $\Omega_c$ is discretized for all the meshes at the finest refinement.
Table 2: Values of the temperature error $E_T = \int_\Omega |T - T_d|^2 d\Omega$ and computational times for several values of $\alpha,$ and for different types of nonconforming meshes.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$l_4$ $\text{nel} = 5120$</th>
<th>$l_{4,3}$ $\text{nel} = 2816$</th>
<th>$l_{4,3,2}$ $\text{nel} = 2432$</th>
<th>$l_{4,3,2,1}$ $\text{nel} = 2384$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10^3$</td>
<td>2.86080e-04</td>
<td>2.86080e-04</td>
<td>2.86080e-04</td>
<td>2.86080e-04</td>
</tr>
<tr>
<td>$10^4$</td>
<td>1.44570e-04</td>
<td>1.44570e-04</td>
<td>1.44570e-04</td>
<td>1.44570e-04</td>
</tr>
<tr>
<td>$10^5$</td>
<td>1.40525e-04</td>
<td>1.40525e-04</td>
<td>1.40525e-04</td>
<td>1.40525e-04</td>
</tr>
<tr>
<td>$10^6$</td>
<td>1.33327e-04</td>
<td>1.33327e-04</td>
<td>1.33327e-04</td>
<td>1.33327e-04</td>
</tr>
<tr>
<td>$10^7$</td>
<td>1.12623e-04</td>
<td>1.12623e-04</td>
<td>1.12623e-04</td>
<td>1.12623e-04</td>
</tr>
<tr>
<td>$10^8$</td>
<td>1.06577e-04</td>
<td>1.06577e-04</td>
<td>1.06577e-04</td>
<td>1.06577e-04</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$l_4$ $\text{nel} = 5120$</th>
<th>$l_{4,3}$ $\text{nel} = 2816$</th>
<th>$l_{4,3,2}$ $\text{nel} = 2432$</th>
<th>$l_{4,3,2,1}$ $\text{nel} = 2384$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10^2$</td>
<td>10.46</td>
<td>7.02</td>
<td>5.61</td>
<td>5.27</td>
</tr>
<tr>
<td>$10^3$</td>
<td>10.46</td>
<td>6.99</td>
<td>5.62</td>
<td>5.22</td>
</tr>
<tr>
<td>$10^4$</td>
<td>13.91</td>
<td>7.71</td>
<td>5.56</td>
<td>5.23</td>
</tr>
<tr>
<td>$10^5$</td>
<td>13.91</td>
<td>9.3</td>
<td>5.52</td>
<td>5.25</td>
</tr>
<tr>
<td>$10^6$</td>
<td>27.63</td>
<td>10.81</td>
<td>6.85</td>
<td>6.53</td>
</tr>
<tr>
<td>$10^7$</td>
<td>62.28</td>
<td>13.81</td>
<td>10</td>
<td>9.5</td>
</tr>
<tr>
<td>$10^8$</td>
<td>44.16</td>
<td>16.17</td>
<td>10.25</td>
<td>10</td>
</tr>
</tbody>
</table>

level. Similarly, in Table 3 we compare the results obtained with the mesh $l_5$ and with the 3 different nonconforming meshes $l_{5,4}, l_{5,4,3}$ and $l_{5,4,3,2}.$ Once again the part of the domain containing $\Gamma_c$ and $\Omega_c$ is always discretized at the top refinement level.

We compare the error $E_T$ and the computational times for several values of the parameter $\alpha.$ As pointed out before, the error $E_T$ decreases as the value of the parameter $\alpha$ increases. However, with $\alpha$ increasing the problem becomes numerically stiffer and the solver eventually fails to converge. In Tables 2 and 3 this behavior is obtained for both the conforming and the nonconforming cases, but it is more evident for the conforming ones. In particular as the value of $\alpha$ increases from $10^2$ to $10^8$ the computational time increases accordingly, but while for the conforming meshes $l_4$ and $l_5$ (left columns in the two tables), the computational time quadruples, before eventually breaking, for the nonconforming meshes $l_{4,3,2,1}$ and $l_{5,4,3,2}$ (right columns in the tables) it only doubles. Moreover in none of the considered cases the convergence fails to be achieved. An intermediate behavior is observed for the 2 central columns and in both tables.

For given $\alpha$ (left to right in both tables) there is no significant difference in the values of the error $E_T$ found. However the computational time significantly reduces, since the total number of elements, $\text{nel}$, decreases (moving left to right), ranging from 5120 to 2384 in Table 2, and from 20480 to 9536 in Table 3. The reduction in the computational time is
Table 3: Values of the temperature error $E_T = \int_\Omega (|T - T_d|^2 d\Omega)$ for various values of $\alpha$ and for different types of smoothers.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$l_5$ $\text{nel} = 20480$</th>
<th>$l_{5,4}$ $\text{nel} = 11264$</th>
<th>$l_{5,4,3}$ $\text{nel} = 9728$</th>
<th>$l_{5,4,3,2}$ $\text{nel} = 9536$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10^3$</td>
<td>2.85127e-04</td>
<td>2.85127e-04</td>
<td>2.85127e-04</td>
<td>2.85127e-04</td>
</tr>
<tr>
<td>$10^5$</td>
<td>1.40484e-04</td>
<td>1.40484e-04</td>
<td>1.40484e-04</td>
<td>1.40484e-04</td>
</tr>
<tr>
<td>$10^6$</td>
<td>1.33216e-04</td>
<td>1.33216e-04</td>
<td>1.33216e-04</td>
<td>1.33216e-04</td>
</tr>
<tr>
<td>$10^7$</td>
<td>none</td>
<td>none</td>
<td>1.12513e-04</td>
<td>1.12513e-04</td>
</tr>
<tr>
<td>$10^8$</td>
<td>none</td>
<td>none</td>
<td>1.06543e-04</td>
<td>1.06543e-04</td>
</tr>
</tbody>
</table>

Computational time

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$l_5$ $\text{nel} = 20480$</th>
<th>$l_{5,4}$ $\text{nel} = 11264$</th>
<th>$l_{5,4,3}$ $\text{nel} = 9728$</th>
<th>$l_{5,4,3,2}$ $\text{nel} = 9536$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10^2$</td>
<td>47.12</td>
<td>30.04</td>
<td>26.22</td>
<td>24.75</td>
</tr>
<tr>
<td>$10^3$</td>
<td>46.87</td>
<td>30.24</td>
<td>25.9</td>
<td>24.86</td>
</tr>
<tr>
<td>$10^4$</td>
<td>50.67</td>
<td>33.4</td>
<td>32.72</td>
<td>24.73</td>
</tr>
<tr>
<td>$10^5$</td>
<td>62.88</td>
<td>40.44</td>
<td>35.05</td>
<td>30.75</td>
</tr>
<tr>
<td>$10^6$</td>
<td>139.98</td>
<td>87.97</td>
<td>37.26</td>
<td>32.59</td>
</tr>
<tr>
<td>$10^7$</td>
<td>none</td>
<td>none</td>
<td>53.06</td>
<td>49.04</td>
</tr>
<tr>
<td>$10^8$</td>
<td>none</td>
<td>none</td>
<td>124.16</td>
<td>67.77</td>
</tr>
</tbody>
</table>

more evident for increasing values of the parameter $\alpha$. In particular in Table 3 for the last two values of $\alpha$ ($10^7$ and $10^8$), the algorithm does not converge for the meshes $l_5$ and $l_{5,4}$, while it converges for the two nonconforming meshes $l_{5,4,3}$ and $l_{5,4,3,2}$.

5 Conclusions

In this paper we have proposed the lifting function approach for the treatment of boundary optimal control problems involving the temperature and Navier-Stokes equations. With the lifting function approach boundary controls are determined in the natural trace function spaces of the volume variables, without stronger regularity requirements. Existence results for the state system can be obtained in the framework of non-coercive elliptic equations. Also, the numerical implementation is very similar to distributed control problems and it permits to avoid some difficulties related to standard direct boundary control approaches. We have proposed the use of a local multigrid algorithm for the numerical solution of the optimality system, where the smoothing steps are performed with domain decomposition Vanka-type smoothers. The numerical results for the optimality system show that candidate optimal solutions can be computed in an effective manner.
and with robustness with respect to the regularization parameters of the cost functional. Also, local mesh refinement can be included in order to reduce the computational effort. A convergence analysis of the proposed algorithm will be dealt with in future works.

We are currently working on a cut-off formulation that reduces the size of the domain for the optimality condition to a subset of $\Omega$ containing the boundary control region $\Gamma_c$. This formulation has two advantages. On one hand, it is clear that the degrees of freedom of the lifting function are reduced with a benefit on the computational time. On the other hand, for fixed values of $\alpha$, $\beta$ and $\gamma$ the penalty integrals are evaluated on a smaller domain. Then, it is our expectation that a better tracking can be achieved.

**Acknowledgments**

This work was supported by National Science Foundation grant DMS-1412796.

**References**


