An Essential Extension of the Finite-Energy Condition for Extended Runge-Kutta-Nyström Integrators when Applied to Nonlinear Wave Equations

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Abstract. This paper is devoted to an extension of the finite-energy condition for extended Runge-Kutta-Nyström (ERKN) integrators and applications to nonlinear wave equations. We begin with an error analysis for the integrators for multi-frequency highly oscillatory systems $y'' + My = f(y)$, where $M$ is positive semi-definite, $\|M\| \gg \|\partial f / \partial y\|$, and $\|M\| \gg 1$. The highly oscillatory system is due to the semi-discretisation of conservative, or dissipative, nonlinear wave equations. The structure of such a matrix $M$ and initial conditions are based on particular spatial discretisations. Similarly to the error analysis for Gauschi-type methods of order two, where a finite-energy condition bounding amplitudes of high oscillations is satisfied by the solution, a finite-energy condition for the semi-discretisation of nonlinear wave equations is introduced and analysed. These ensure that the error bound of ERKN methods is independent of $\|M\|$. Since stepsizes are not restricted by frequencies of $M$, large stepsizes can be employed by our ERKN integrators of arbitrary high order. Numerical experiments provided in this paper have demonstrated that our results are truly promising, and consistent with our analysis and prediction.

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Key words: Finite-energy condition, multi-frequency highly oscillatory system, error analysis, ERKN method, nonlinear wave equation.

1 Motivation

It is known that the study of numerical methods for solving highly oscillatory problems has become increasingly important in recent decades. A major source of these problems...
is from the spatial discretisation of nonlinear wave equations, such as the Klein-Gordon equation which has received a great deal of attention in both its numerical and analytical aspects. In this paper, we pay attention to an essential extension of the finite-energy condition for ERKN integrators and applications to nonlinear wave equations.

We commence with a system of multi-frequency highly oscillatory second-order differential equations

\[
\begin{align*}
    y'' + My &= f(y), & t \in [t_0, T], \\
    y(t_0) &= y_0, & y'(t_0) = y'_0,
\end{align*}
\]

(1.1)

where \( M \in \mathbb{R}^{d \times d} \) is a positive semi-definite matrix (not necessarily diagonal nor symmetric, in general), \( ||M|| \gg ||\frac{\partial f}{\partial y}|| \), and \( ||M|| \gg 1 \). This type of problem occurs in many aspects of science and engineering, among which the spatial discretisation of nonlinear wave equations by finite difference methods or spectral methods provides a large number of practical applications. In dealing with these oscillatory problems, the adapted Runge-Kutta-Nyström (ARKN) methods and ERKN integrators were respectively proposed by Franco [4] and Yang et al [42] as developments of classical Runge-Kutta-Nyström (RKN) methods. As shown in the literature (see e.g. [4, 6, 22, 40]), based on the internal stages of traditional RKN methods, the ARKN methods adopt a new form of updates given by

\[
\begin{align*}
    y_{n+1} &= \phi_0(V)y_n + h\phi_1(V)y'_n + h^2 \sum_{i=1}^{s} \bar{B}_i(V)f(Y_i), \\
    y'_{n+1} &= -hM\phi_1(V)y_n + \phi_0(V)y'_n + h\sum_{i=1}^{s} B_i(V)f(Y_i),
\end{align*}
\]

where \( \phi_0, \phi_1, \bar{B}_i \) and \( B_i \) are matrix-valued functions, whereas, totally differently from the ARKN methods, in light of the variation-of-constants formula for (1.1), the ERKN methods not only adopt a new form of updates, but also adopt a new form of internal stages given by

\[
Y_i = \phi_0(C_i^2 V)y_n + C_i h\phi_1(C_i^2 V)y'_n + h^2 \sum_{j=1}^{s} A_{ij}(V)f(Y_j),
\]

to achieve a high level of harmony with the oscillatory structure of the problem (1.1). The well-known examples of explicit ERKN integrators are Gaußchi-type methods of order two [7–11, 13]. As we will show by (2.10) in Section 2, the Gaußchi-type method can be displayed by a Butcher tableau, which is just in the form of ERKN methods. From this observation, ERKN integrators also can be thought of as generalized Gaußchi-type methods.

Another type of numerical method for solving the oscillatory problem is the exponentially (or functionally) fitted methods, such as the exponentially fitted Runge-Kutta (EFRK) method [27], the exponentially fitted Runge-Kutta-Nyström method (EFRKN) (see e.g. [5]) and the functionally-fitted energy-preserving method [19]. As stated in the
literature, the application of these methods highly depends on the choice of a fitted frequency $\omega$. For example, when solving the problem (1.1), both EFRK methods and EFRKN methods require the symmetry of $M$, which will provide the preferred fitted frequencies by the diagonalization of the matrix $M$. Fortunately, however, the symmetry of $M$ is not necessary for ERKN methods as we have mentioned above. This means the broader use of ERKN methods for solving oscillatory problems once $M$ is not symmetric. An essential relation between ERKN methods and exponentially fitted methods is that the ERKN methods are consistent with the exponentially fitted Runge-Kutta-Nyström methods (EFRKN) provided that $M$ is symmetric. This point has been definitely claimed by Wu et al. [37]. It is noted that the energy-preserving methods have also been developed as numerical approaches to solving oscillatory Hamiltonian PDEs (see, e.g. [2, 18, 19, 41]) in recent years.

As a type of structure-preserving algorithm, ERKN integrators display some important properties in dealing with the multi-frequency and highly oscillatory system (1.1). The most notable one is that the multi-frequency and highly oscillatory homogeneous equation $y'' + My = 0$ can be exactly integrated by both the updates and the internal stages of ERKN integrators. Another one to merit our attention is their superiorities over the classical RKN methods in numerical behaviour, such as the local truncation error, the global error, the dispersion and the dissipation. However, these superiorities in errors for ERKN integrators are usually supported by different applications, noting that only a few studies of theoretical analysis (see, e.g. [8–10, 13, 17, 31]) have been conducted, in which some further restrictions are needed on the variables $y, y''$, and the right-hand side function $f$ of the system (1.1).

Meanwhile, it should be noticed that Gautschi-type methods of order two have been successfully applied to oscillatory second-order differential equations [7,13] among which the Ferm-Pasta-Ulam problem [12] is the most notable one. An observation from the Ferm-Pasta-Ulam problem leads to the finite-energy condition, which plays a very important role in effectively solving oscillatory second-order differential equations. For example, it is shown that long-time-step can be used for these explicit methods when applied to oscillatory differential equations (see e.g. [7]). The long-time energy conservation for these methods also can be achieved with this condition (see e.g. [11]). The most important result is that under the finite-energy condition, the error bound of Gautschi-type methods of order two is independent of $\|M\|$ (see, e.g. [8–10]). This point is crucial to the underlying multi-frequency and highly oscillatory problem.

With this observation and the fact that Gautschi-type methods are special ERKN methods of order two, we further investigate an extension of the finite-energy condition in this paper. Using an analogical methodology, we devote ourselves to nonlinear wave equations, for which the appropriate spatial discretisation plays a similar role to the finite-energy condition for ERKN methods. Furthermore, the finite-energy condition for ERKN integrators could be naturally derived for nonlinear wave equations by suitable spatial discretisation, whose differentiation matrix $M$ is symmetric and positive semi-definite. We then prove that the error bound of ERKN methods is entirely indepen-
dent of $\|M\|$ when applied to the semi-discrete wave equations. This result is completely consistent with those stated in [8–10]. Moreover, another promising result is that large stepsizes are allowed for explicit ERKN methods. This point is soundly supported by the numerical experiments in this paper, where the corresponding classical RKN methods with the same stepsizes could hardly be employed.

This paper is organized as follows. In Section 2, we briefly summarize the ERKN integrators in dealing with the multi-frequency highly oscillatory problems, and some achieved results on error analysis are included as well. In Section 3, we make an analysis on the estimation of error bounds independent of $\|M\|$ for ERKN integrators when applied to conservative or dissipative nonlinear wave equations. We conduct numerical experiments in Section 4 and the numerical results greatly support our theoretical analysis stated in this paper. Conclusions are given in the last section.

Throughout this paper, the dimension of the system (1.1) is $d$, i.e., $y,y' \in \mathbb{R}^d$.

## 2 Preliminaries

In this section, in need of context we summarize the results on ERKN integrators for the second-order oscillatory autonomous system (1.1). First, we introduce the following unconditionally convergent matrix-valued functions

$$\phi_j(V) := \sum_{k=0}^{\infty} \frac{(-1)^k V^{2k+j}}{(2k+j)!}, \quad j = 0, 1, \cdots.$$  \hfill (2.1)

Then the definition of ERKN integrators is given below.

**Definition 2.1** (see, e.g. [38]). An $s$-stage ERKN integrator for the initial value problem (1.1) is defined by

\[
\begin{align*}
Y_i &= \phi_0(C_i^2 V)y_n + C_i h \phi_1(C_i^2 V)y'_n + h^2 \sum_{j=1}^{s} A_{ij}(V) f(Y_j), \quad i = 1, \cdots, s, \\
y_{n+1} &= \phi_0(V)y_n + h \phi_1(V)y'_n + h^2 \sum_{i=1}^{s} B_i(V) f(Y_i), \\
y'_{n+1} &= -h M \phi_1(V)y_n + \phi_0(V)y'_n + h \sum_{i=1}^{s} \bar{B}_i(V) f(Y_i),
\end{align*}
\]  \hfill (2.2)

where $C_1, \cdots, C_s$ are real constants, $B_i(V), \bar{B}_i(V)$ for $i = 1, \cdots, s$, and $A_{ij}(V)$ for $i, j = 1, \cdots, s$ are matrix-valued functions of $V \equiv h^2 M$.

In this paper, we conveniently denote the coefficients of an ERKN integrator in uppercase $(C,B,\bar{B},A)$. Some useful properties related to the unconditionally convergent matrix-valued functions $\phi_j(V)$ for $j = 0, 1, \cdots$ are established in [38], and summarized below.
Proposition 2.1. The matrix-valued functions $\phi_j(M)$ defined by (2.1) satisfy:

1. $\lim_{M \to 0} \phi_j(M) = \frac{1}{j!} I$ for $j = 0, 1, \cdots$, where $I$ is the identity matrix;

2. All $\phi_j(M)$ for $j = 0, 1, \cdots$, are bounded when $M$ is positive semi-definite, i.e. $\|\phi_j(M)\| \leq \bar{c}$, where $\bar{c}$ is a constant depending on $\|M\|$ in general. However, an important and special case is that $\bar{c}$ is independent of $\|M\|$ provided $M$ is symmetric and positive semi-definite;

3.\[
\begin{aligned}
&\int_0^1 \frac{(1-\zeta)x^2(1-\zeta)^2M}{j!} d\zeta = \phi_{j+2}(x^2M), \quad x \in \mathbb{R}, \\
&\int_0^1 \frac{\phi_0(x^2(1-\zeta)^2M)}{j!} d\zeta = \phi_{j+1}(x^2M), \quad x \in \mathbb{R}.
\end{aligned}
\]

We then quote the following theorem, which is related to the order conditions for ERKN integrators.

Theorem 2.1 (see [43]). An $s$-stage ERKN integrator (2.2) is of order $p$ if and only if the following conditions

\[
\begin{aligned}
&\sum_{i=1}^s \bar{B}_i \Phi_i(\tau) = \frac{\rho(\tau)!}{\gamma(\tau) s(\tau)} \phi_{\rho(\tau)+1} + O(h^{p-\rho(\tau)}), \quad \forall \tau \in \text{SSENT}_m, \quad m \leq p-1, \\
&\sum_{i=1}^s B_i \Phi_i(\tau) = \frac{\rho(\tau)!}{\gamma(\tau) s(\tau)} \phi_{\rho(\tau)} + O(h^{p-\rho(\tau)+1}), \quad \forall \tau \in \text{SSENT}_m, \quad m \leq p,
\end{aligned}
\]

are satisfied.

Here, the definitions and properties associated with the order $\rho(\tau)$, the sign $s(\tau)$, the density $\gamma(\tau)$, and the weight $\Phi_i(\tau)$ can be found in [43]. Similarly to what the authors have done in [14, 31], we also admit the following two assumptions throughout the error analysis in this paper, but restrict them to a more relax setting.

Assumption 2.1. It is supposed that the solution $y(t)$ of (1.1) and its derivative $y'(t)$ are sufficiently smooth and uniformly bounded with respect to $t$.

Assumption 2.2. It is supposed that all the occurring derivatives $f^{(k)}(y)$ (with respect to $y$) of $f(y)$ are uniformly bounded.

Remark 2.1. We should note that the uniform bound is established with certain norm. For the vectors $y(t), y'(t)$ and the vector-valued function $f(y)$, we naturally use the Euclidean norm $\|\cdot\|_2$. Meanwhile, each derivative $f^{(k)}(y)$ can be regarded as a $k$-linear mapping [12]

\[
f^{(k)}(y) : \underbrace{\mathbb{R}^d \times \cdots \times \mathbb{R}^d}_{k-fold} \to \mathbb{R}^d,\]

(2.5)
where \( d \) is the dimension of the problem (1.1). In this sense, we can take the induced norm \( \| | \cdot | \| \)
\[
\| | f^{(k)}(y) | | = \sup \{ \| f^{(k)}(y)(v_1, \cdots, v_k) \|_2 : \| v_i \|_2 = 1, i = 1, \cdots, k \},
\] (2.6)
subordinated to the Euclidean norm \( \| \cdot \|_2 \) for \( f^{(k)}(y) \). As a special case where \( f''(y) \) is
a negative semi-definite matrix, we have known that the induced norm \( \| | \cdot | | \) is just the
spectral norm. In the remainder of this paper, we will denote all the occurring norms by
a uniform \( \| \cdot \| \) for convenience, if there is no any confusion.

An earlier error analysis has been made in part by Wang et al. [31] for explicit ERKN
methods. We briefly summarize the results below.

**Theorem 2.2** (see [31]). *Under some suitable assumptions, the explicit ERKN methods converge
for \( 0 \leq nh \leq T - t_0 \) when applied to the problem (1.1). In particular, the numerical solution and its
derivative satisfy the following error bounds
\[
\| e_n \| \leq \hat{C}_1 h^p,
\]
\[
\| e'_n \| \leq \hat{C}_2 h^p,
\]
where \( \hat{C}_1 \) and \( \hat{C}_2 \) depend on \( T \) and \( \| M \| \), and they are independent of \( h \) and \( n \). However, if \( M \) is
symmetric and positive semi-definite, then \( \hat{C}_1 \) is also independent of \( \| M \| \).

The result really sounds promising, though a stronger restriction on the function \( f \) is
used, that is, all occurring derivatives of \( f \) (especially the total derivatives of \( f(y(t)) \) with
respect to \( t \)) are assumed to be uniformly bounded. The stronger restricted assumption
exactly leads to that the norm of high-order derivatives (with respective to \( t \)) of the exact
solution \( y(t) \) is independent of \( \| M \| \). Hence the global error bounds are independent of
\( \| M \| \). Another issue of the pervious result is that the \( s \)-stage explicit ERKN integrators of
order \( p \) are restricted to those with \( s \leq 3 \) and \( p \leq 3 \), and then the accuracy of the numerical
methods is limited.

If the \( m \)-th derivatives of the exact solution \( y(t) \) to the problem (1.1) are bounded by
\[
y^{(m)}(t) = O(\| M \|^k), \quad m, k \in \mathbb{N}^+,
\] (2.7)
then both the local truncation error bound and the global error bound must be dependent
on \( \| M \| \). In the case of multi-frequency highly oscillatory ODEs, (2.7) always holds with
the given initial conditions \((y_0, y'_0)\). In this sense, the global error bound for ERKN inte-
grators depends on \( \| M \| \) since the derivatives of \( f(y) \) are involved with \( M \). Fortunately,
however, when \( M \) is symmetric and positive semi-definite, the earlier seminal work on
error analysis had discussed the so-called finite-energy condition (see, e.g. [7–11, 13]):
\[
\frac{1}{2} \| y'(t) \|^2 + \frac{1}{2} \| \Omega y(t) \|^2 \leq \frac{1}{2} K^2, \quad (\Omega^2 = M).
\] (2.8)
With condition (2.8), it has been proved that the global error bound for the Gaustchi-type exponential integrators of order two is independent of $\|M\|$. The next theorem confirms this point.

**Theorem 2.3** (see [8]). In (1.1), let $M = \Omega^2$ be an arbitrary symmetric positive semi-definite matrix. Suppose that $f, f_y$ and $f_{yy}$ are bounded in Euclidean norm or the norms induced by the Euclidean norm, respectively. Moreover, it is assumed that the solution $y(t)$ satisfies the finite-energy condition (2.8). Then, under suitable assumptions for the even analytic functions $\Psi, \Psi_0$ and $\Psi_1$, we have the following estimation of global errors

$$
\|y(t_n) - y_n\| \leq Ch^2, \quad t_n \in [t_0, T],
$$

(2.9)

for the Gaustchi-type exponential integrators, where the constant $C$ only depends on $T, K, \tilde{C}, \|f\|, \|f_y\|$ and $\|f_{yy}\|$, and $\tilde{C}$ is a small constant independent of $\|M\|$.

Since the Gaustchi-type integrators of order two described therein are naturally explicit ERKN integrators of order two with the Butcher tableau

$$
c | A(V) \\
| 1 | b(V)^T

0 | 0 | 0 \\
\frac{1}{2} \Psi(V) | 0 | 0 \\
\frac{1}{2} \Psi_0(V) | \frac{1}{2} \Psi_1(V)
$$

(2.10)

where $\Psi, \Psi_0$ and $\Psi_1$ are matrix-valued functions respectively fixed for different methods, we also make an attempt to find a suitable condition so that the error bound for ERKN integrators is independent of $\|M\|$ when applied to nonlinear wave equations. With this idea and observation, below we will concentrate our attention on nonlinear wave equations, which can be converted to the form of (1.1) by an appropriate spatial discretisation.

### 3 Error analysis for ERKN integrators when applied to nonlinear wave equations

We now consider the wave equation

$$
\begin{cases}
\frac{\partial^2 u(x,t)}{\partial t^2} - a^2 \Delta u(x,t) = f(u), & x \in D, \quad t_0 \leq t \leq T, \\
u(x,t_0) = \varphi(x), & u_t(x,t_0) = \psi(x), \quad x \in D, \\
B(x)u(x,t) = 0, & x \in \partial D,
\end{cases}
$$

(3.1)

where $a$ means horizontal propagation speed of the wave motion, $D$ is a spatial domain with boundary $\partial D$, $\Delta$ is the Laplacian operator and $B(x)$ is a linear boundary operator.
Here, we suppose that the problem (3.1) is well-posed and satisfies the conditions described in very recent papers \([33, 34]\). As shown in \([33, 34]\), the exact solution \(u(x,t)\) and its derivative \(u_t(x,t)\) satisfy

\[
\begin{align*}
  u(x,t) &= \phi_0((t-t_0)^2a^2\Delta)\varphi(x) + (t-t_0)\phi_1((t-t_0)^2a^2\Delta)\psi(x) \\
  &\quad + \int_{t_0}^{t}(t-\zeta)\phi_1((t-\zeta)^2a^2\Delta)f(\zeta)d\zeta, \\
  u_t(x,t) &= (t-t_0)a^2\Delta\phi_1((t-t_0)^2a^2\Delta)\varphi(x) + \phi_0((t-t_0)^2a^2\Delta)\psi(x) \\
  &\quad + \int_{t_0}^{t}\phi_0((t-\zeta)^2a^2\Delta)f(\zeta)d\zeta,
\end{align*}
\]  

(3.2)

under some regular conditions, where \(f(\zeta) = f(u(x,\zeta))\), and the operator determined by the Laplacian-valued functions \(\phi_j(\Delta)\) are defined by

\[
\phi_j(\Delta) := \sum_{k=0}^{\infty} \frac{\Delta^k}{(2k+j)!}, \quad j = 0, 1, \cdots.
\]

(3.3)

It can be observed that (3.3) is obtained from replacing \(z\) by \(\Delta\) in the functions

\[
\phi_j(z) = \sum_{k=0}^{\infty} \frac{z^k}{(2k+j)!}, \quad j = 0, 1, 2, \cdots,
\]

(3.4)

and all \(\phi_j(z)\) for \(j = 0, 1, 2, \cdots\) are bounded for any \(z \leq 0\).

It is known that \(\Delta\) is not defined for every \(v \in L^2(D)\). In order to model boundary conditions, it is required to restrict ourselves to the case where \(\Delta\) is defined on the domain \(\Omega(\Delta) \subset L^2(D)\), and the underlying boundary condition is satisfied. We then consider the linear differential operator \(\mathcal{A}\) defined by \(\mathcal{A}v(x) = a^2\Delta v(x)\). The operator has a complete system of orthogonal eigenfunctions in the Hilbert space \(L^2(D)\). The operator on \(L^2(D)\) induces a corresponding operator on \(\ell^2\) due to the isomorphism between \(L^2\) and \(\ell^2\). An elementary analysis which is similar to that for the exponential operator presented by Hochbruck and Ostermann \([15]\) can make sure that the Laplacian-valued functions \(\phi_j(\Delta)\) are bounded operators with respect to the norm \(\|\cdot\|_{L^2(D)\rightarrow L^2(D)}\). The details of the analysis can be found in Appendix of this paper.

On the basis of the above discussion, in what follows, we first define \(v(t)\) as the function that maps \(x\) to \(u(x,t)\):

\[
v(t) := [x \mapsto u(x,t)].
\]

Then the system (3.1) can be formulated by an abstract second-order ordinary differential equation on the infinity-dimensional function space \(L^2(D)\),

\[
\begin{align*}
  u''(t) - \mathcal{A}u(t) &= f(u(t)), \quad t_0 < t \leq T, \\
  u(t_0) &= \varphi(x), \quad u'(t_0) = \psi(x).
\end{align*}
\]  

(3.5)
Approximating the operator $-\mathcal{A}$ in (3.5) by the differentiation matrix $M$ (with some manipulation if necessary), we obtain an initial value problem of ODEs:

\[
\begin{cases}
U''(t) + MU = f(U), & t \in [t_0, T], \\
U(t_0) = U_0, & U'(t_0) = U'_0.
\end{cases}
\]  

(3.6)

For the applications of ERKN integrators to (3.6), it is required that the differentiation matrix $M$ is positive semi-definite. It is easy to fulfill this requirement, since the suitable difference discretisation (see, e.g. [21]) and the well-known Fourier or Chebyshev pseudospectral spatial discretisation could yield the positive semi-definite differentiation matrix $M$. In what follows, we assume that the spatial approximation used here is consistent with the original PDEs and $U(t)$ is convergent to $u(t) \equiv u(x,t)$. Note that the matrix $M$ and the initial values $(U_0, U'_0)$ of (3.6) are concordantly and simultaneously generated by the underlying spatial discretisation.

Another point should be especially emphasized that if we replace the operator $-\mathcal{A}$ by the differentiation matrix $M$ in (3.2), then the solution $U(t)$ of (3.6) and its derivative $U'(t)$ can be exactly yielded as follows:

\[
\begin{cases}
U(t) = \phi_0((t-t_0)^2M)U_0 + (t-t_0)\phi_1((t-t_0)^2M)U'_0 + \int_{t_0}^{t} (t-\zeta)\phi_1((t-\zeta)^2M)f(\zeta)d\zeta, \\
U'(t) = -(t-t_0)M\phi_1((t-t_0)^2M)U_0 + \phi_0((t-t_0)^2M)U'_0 + \int_{t_0}^{t} \phi_0((t-\zeta)^2M)f(\zeta)d\zeta,
\end{cases}
\]

(3.7)

where $f(\zeta) = f(U(x,\zeta))$.

We now turn to an important property of the conservative nonlinear wave equations (3.5). By denoting

\[ H(u) = -\int_0^u f(s)ds, \]

we obtain that

\[ E = E(t) = \frac{1}{2} \int_D ((u_t)^2 + a^2 (u_x)^2 + 2H(u))dx, \]

(3.8)

is conserved during the evolution of the wave equation, when suitable boundary conditions, such as periodic or homogeneous Dirichlet or Neumann boundary conditions are prescribed. For another case where $f$ is explicitly involved with $t$, i.e. $f = f(t,u)$, it is noticed that $E(t)$ is no longer conserved and the wave equation is then dissipative. Thus a suitable spatial discretisation for (3.1) can give an approximate relation between the continuous energy and the discrete energy for (3.6)

\[ \frac{1}{2} \|U'(t)\|^2 + \frac{1}{2} \|\Omega U(t)\|^2 \approx \frac{1}{2} \int_D ((u_t)^2 + a^2 (u_x)^2)dx, \quad (\Omega^2 = M). \]

(3.9)

Then the finite-energy condition (2.8) is easily satisfied, since the constant $K$ in (2.8) can now be roughly selected as

\[ K = \sqrt{2(|E(t_0)| + |\int_D H(u(t_0))dx|)}. \]
for both the conservative and dissipative nonlinear wave equations. Here it should be noticed that though the constant $K$ depends also on the boundary condition according to its formula, this could be neglected by virtue of Assumption 2.1. This point is the inner essence of our fundamental error analysis on ERKN methods for efficiently solving conservative or dissipative nonlinear wave equations.

**Theorem 3.1.** **Under Assumption 2.1 and Assumption 2.2, if the matrix $M$ in (3.6) is symmetric positive semi-definite and the spatial discretisation is consistent with the original equation and convergent to the exact solution, then the spatial discretisation error satisfies**

$$
\|U(t) - u(t)\| \leq C_2 \tau^k, \quad (3.10)
$$

*where $\tau$ is the maximal spatial stepsize, $C_2$ is a constant depending on $T$ but independent of $\|M\|$, and $k$ is a positive integer number depending on the spatial discretisation.*

**Proof.** Let $r_\tau$ be the natural restriction operator on the space grids and $\varrho(\tau)$ stands for the maximal distance in the grids. We denote $u_\tau(t) = r_\tau u(x, t)$. By denoting

$$
u''_\tau(t) = \frac{d^2u_\tau(t)}{dt^2} = r_\tau u_{tt}(x, t),
$$

we introduce the space truncation error

$$
\delta(t) = -Mu_\tau(t) + f(u_\tau(t)) - u''_\tau(t) = -a^2r_\tau \Delta u(x, t) - Mu_\tau(t). \quad (3.11)
$$

It is trivial to obtain that $\|\delta(t)\| \to 0$ as $\varrho(\tau) \to 0$ uniformly in $t$ due to the consistency of the spatial discretisation with the wave equation. It follows from (3.11) that $\delta(t)$ is only involved with the spatial discretisation. Hence it can be estimated by

$$
\|\delta(t)\| \leq C \tau^k, \quad (3.12)
$$

*where $k$ is some positive integer number depending on the spatial discretisation, and $C$ is a constant independent of $\|M\|$.*

If we denote

$$
\eta(t) = U(t) - u_\tau(t),
$$

then the spatial discretization error $\eta(t)$ is the solution of the system

$$
\eta''(t) = -MU(t) + f(U) + Mu_\tau(t) - f(u_\tau(t)) + \delta(t),
$$

that is

$$
\eta''(t) + M\eta(t) = G(t)\eta(t) + \delta(t), \quad (3.13)
$$

where

$$
G(t) = \int_0^1 F'(u_\tau(t) + \theta\eta(t))d\theta,
$$
and $F'(\cdot)$ is the Jacobian of $f(\cdot)$. The application of the variation-of-constants formula to (3.13) yields

$$
\eta(t) = \phi_0((t-t_0)^2M)\eta_0 + (t-t_0)\phi_1((t-t_0)^2M)\eta'_0 \\
+ \int_{t_0}^t (t-\zeta)\phi_1((t-\zeta)^2M)\tilde{G}(\zeta)d\zeta,
$$

(3.14)

where $\eta_0 = \eta(t_0)$, $\eta'_0 = \eta'(t_0)$ and $\tilde{G}(\zeta) = G(\zeta)\eta(\zeta) + \delta(\zeta)$.

It follows from Assumption 2.2 that $\|F'| \leq L$, where $L$ is a constant independent of $\|M\|$. Then on noting that $\eta(t_0) = U(t_0) - u_r(t_0) = 0$ and $\eta'(t_0) = U'(t_0) - u'_r(t_0) = 0$, we obtain

$$
\|\eta(t)\| = \| \int_{t_0}^t (t-\zeta)\phi_1((t-\zeta)^2M)\tilde{G}(\zeta)d\zeta \| \\
\leq L\|\eta(t)\|\| (t-t_0)^2 \| \int_0^1 (1-\zeta)\phi_1((1-\zeta)^2(t-t_0)^2M)d\zeta \| \\
+ \|\delta(t)\|\| (t-t_0)^2 \| \int_0^1 (1-\zeta)\phi_1((1-\zeta)^2(t-t_0)^2M)d\zeta \| \\
\leq L|T-t_0|^2\|\eta(t)\|\|\phi_2((t-t_0)^2M)\| \\
+ C|T-t_0|^2\|\phi_2((t-t_0)^2M)\| \tau^k.
$$

(3.15)

The last inequality follows from (2.3) of Proposition 2.1 and (3.12). Since $M$ is symmetric and positive semi-definite, $\|\phi_2((t-t_0)^2M)\|$ is uniformly bounded and independent of $\|M\|$. On noting Proposition 2.1. Consequently, (3.15) gives $\|\eta(t)\| \leq C_2 \tau^k$, where $C_2$ is dependent on $T$ but independent of $\|M\|$. This completes the proof.

As the mesh partition in the space discretisation increases for (3.5), $\|M\|$ will increase in (3.6), and the larger $\|M\|$ is, the higher accuracy will be gained in space approximations. This implies that

$$
U(t) \to u_r(t) \quad \text{as} \quad \|M\| \to \infty.
$$

(3.16)

Since the derivatives of $u_r(t)$ are entirely independent of $\|M\|$, the exact solution $U(t)$ to (3.6) and its high-order derivatives (with respect to $t$) are also independent of $\|M\|$ due to the convergence stated by (3.16). This leads to the uniform boundness and the independence of $\|M\|$ for all occurring derivatives of $\tilde{f}(t) = f(U)$, which has been stated by Assumption 1 in [31]. With this insight into the underlying problem and following the analysis in [31], we present the following important result on the global error bounds for ERKN integrators when applied to nonlinear wave equations (3.1).

**Theorem 3.2.** Let $M$ be real symmetric and positive semi-definite. Suppose that initial-boundary value problems (3.1) are well-posed and conservative (or dissipative), and a proper spatial discretisation for (3.1) results in (3.6). Then, applying a $p$-th order ERKN integrator to the semi-discrete problem (3.6), we have the global error bound

$$
\|U_n - u(t_n)\| \leq \|U_n - U(t_n)\| + \|U(t_n) - u(t_n)\| \leq C_1 h^p + C_2 \tau^k,
$$

(3.17)
where $C_1$ and $C_2$ are dependent on $T$ but independent of $\|M\|$, $h$ and $\tau$ are time and spatial stepsizes respectively, and $k$ is a positive integer determined by the spatial discretisation.

**Proof.** We begin with the fact that the high-order derivatives of $U(t)$ are independent of $\|M\|$ as (3.16) claims. We denote $\tilde{f}(t) = f(U(t))$. Since the high-order total derivatives $\tilde{f}^{(j)}(t) = \frac{d^j}{dt^j} f(U(t))$ are the combinations of $f^{(i)}(U), f^{(i)}(U)$, and $U^{(i)}(t)$, it can be derived that $\tilde{f}^{(j)}(t)$ are bounded and independent of $\|M\|$ based on Assumption 2.2. From the variation-of-constants formula (3.7) for (3.6) and the definition of the ERKN integrator (2.2), under the local assumptions $U_n = U(t_n)$ and $U_n' = U'(t_n)$, the local truncation error $T_{ERKN}^1$ satisfies

$$T_{ERKN}^1 = U(t_n + h) - U_{n+1} = h^2 \int_0^1 (1 - \xi) \phi_1((1 - \xi)V) \tilde{f}(\xi) d\xi - h^2 \sum_{i=1}^s \bar{B}_i(V) f(U_i')$$

$$= \sum_{j=0}^{\infty} h^{j+2} \left( \phi_{j+2}(V) - \sum_{k=1}^s \bar{B}_k(V) \frac{c_j}{j!} \right) \tilde{f}^{(j)}(t_n)$$

$$= \hat{C} h^{p+1} + \sum_{j=p}^{\infty} h^{j+2} \left( \phi_{j+2}(V) - \sum_{k=1}^s \bar{B}_k(V) \frac{c_j}{j!} \right) \tilde{f}^{(j)}(t_n),$$

(3.18)

where $\hat{C}$ is independent of $\|M\|$. The second identity of (3.18) follows from [31] and third identity holds because the ERKN integrator is of order $p$. Thus (3.18) gives

$$\|
T_{ERKN}^1\| \leq K_1 h^{p+1},$$

where $K_1$ is independent of $\|M\|$. Likewise, we can obtain

$$\|
T_{ERKN}^2\| = \|U'(t_n + h) - U'_{n+1}\| \leq K_2 h^{p+1},$$

where $K_2$ is independent of $\|M\|$. Consequently, it follows from Lemma 10.21 in [16] that

$$\|U_n - U(t_n)\| \leq \|U_n - U(t_n)\| + \|U_n' - U'(t_n)\| \leq C_1 h^p,$$

where $C_1 = \frac{K}{L} \left( \exp(L(T-t_0)) - 1 \right)$, $K = \max\{2K_1, 2K_2\}$, and $L$ is a Lipschitz constant independent of $\|M\|$. This shows that the constant $C_1$ is independent of $\|M\|$. Furthermore, the constant $C_2$ in (3.17) is only involved with the spatial discretisation, and hence independent of $\|M\|$ as stated in Theorem 3.1. This completes the proof.

The discussion about the convergence of the discretisation in space for (3.16) and that of the full discretisation for (3.17) can be found in [28], in which the authors discussed the convergence of Method of Lines (MOL) when applied to PDEs. Here we admit the potential convergence and pay our main attention to the analysis of global error bounds. This theorem reveals that the bound of the global error is independent of $\|M\|$ when
the semi-discrete wave equation (3.6) is numerically solved by ERKN integrators. It is noted that (3.6) is a multi-frequency oscillatory system. The result is a natural extension for that of the Gaustchi-type exponential integrators of order two [8–10], though the analysis for Gaustchi-type exponential integrators is carried out in a different approach. Furthermore, it gives a physical description for the finite-energy condition that this condition essentially originates from the proper spatial discretisation for the conservative or dissipative wave equations. Another point should be noticed that since one cannot have a similar result for classical RKN methods when applied to (3.6), it is not recommend to use them in practice because of the larger global error bound of RKN methods compared with ERKN integrators. Moreover, totally different from Gaustchi-type exponential integrators and the generalized Gaustchi-type exponential integrators, i.e. ERKN integrators, classical RKN methods are not designed specially for solving multi-frequency highly oscillatory problems, and the original idea of classical RKN methods takes no account of the oscillatory structure introduced by the term \( M\mu \) in (3.6). Finally, as stated in [38], ERKN methods always have larger stability region than the corresponding RKN methods in dealing with the oscillatory problem (3.6). Hence, large stepsizes are also allowed for ERKN methods when solving nonlinear wave equations, whereas the corresponding RKN methods may behave badly with the same stepsizes as those for ERKN methods. This point will be demonstrated by the numerical experiments in the next section.

4 Numerical experiments

In our numerical simulations, we prefer to use explicit ERKN methods rather than implicit ones due to the easy manipulation of the former, even though both of them have the same properties as stated in Theorem 3.2. Here, the key point is that implicit ERKN integrators require iteration solutions, whereas explicit ERKN integrators avoid the complexity brought by the iterative procedure. The explicit ERKN integrators are selected as follows:

- ERKN3s4: the three-stage symplectic ERKN method of order four given in [39];
- ERKN7s6: the seven-stage symplectic ERKN method of order six given in [23].

It is known that an ERKN method reduces to an RKN method when \( M \to 0 \). Thus the reduced ERKN methods are assigned as the corresponding RKN methods, which are denoted by RKN3s4 and RKN7s6, respectively. It will be observed from the numerical experiments that the numerical behaviour of ERKN methods are much better than that of the RKN methods. Although the Gaustchi-type exponential methods of order two mentioned in previous section are well known, we do not consider in the numerical comparison due to their limited accuracy.
Problem 1. Consider the Duffing equation (see e.g. [20])
\[
\begin{align*}
\ddot{q} + \omega^2 q &= k^2(2q^3 - q), \\
q(0) &= 0, \\
\dot{q}(0) &= \omega,
\end{align*}
\]

with \(0 \leq k < \omega\). It is a Hamiltonian system with the Hamiltonian
\[
H(p, q) = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2 + \frac{k^2}{2}(q^2 - q^4).
\]
The analytic solution is given by
\[
q(t) = sn(\omega t, k/\omega),
\]
where \(sn\) is the Jacobian elliptic function.

The problem is solved on the interval \([0,1000]\) with \(k = 0.03\). In this problem, we investigate the influence of \(\|M\|\) on the global errors for ERKN integrators in the case of traditional ODEs. Fig. 1 gives the numerical results by ERKN3s4 and ERKN7s6 with the different stepsizes when applied to Problem 1. As shown in Fig. 1, the larger \(\|M\|\) \((= \omega^2)\) becomes, the larger global errors are for both the two ERKN integrators. It is highly accordance with the conclusion that the global error bounds of ERKN methods are usually dependent on \(\|M\|\) when applied to the general oscillatory problem (1.1), where \(M\) is independent of the initial conditions \(y_0, y'_0\). A similar result is given in Fig. 2, where the same stepsizes are respectively used for the corresponding RKN methods. Note that the case of \(\omega = 20\) is not plotted in Fig. 2(b) for RKN7s6, since the scheme

\[\text{(a) The result for ERKN3s4} \quad \text{(b) The result for ERKN7s6}\]

Figure 1: The log-log plot of maximum global error \(GE\) against \(t\) with different \(\omega\) for Problem 1: (a) the result for ERKN3s4 with the stepsize \(h = 1/40\); (b) the result for ERKN7s6 with the stepsize \(h = 1/10\).
will be unstable with the stepsize \( h = 1/10 \). It is shown that the RKN methods give disappointed numerical solutions in high-frequency case (\( \omega = 20 \)), whereas the ERKN methods are robust and give satisfactory numerical solutions of high accuracy with the same stepsizes. Moreover, the results of ERKN methods are also much more accurate than those of classical RKN methods in the low-frequency case of \( \omega = 5 \). Finally, we plot the efficiency curves in Fig. 3(a) for all the four methods, where the ERKN methods show higher efficiency than their reduced RKN methods. As shown in Fig. 3(b), the
numerical convergence order of the selected numerical methods are 4.00, 5.99, 4.21 and 5.93 respectively for RKN3s4, RKN7s6, ERKN3s4 and ERKN7s6, which demonstrate the high consistence of their numerical convergence order with the theoretical convergence order.

Problem 2. [Breather soliton] We consider the well-known sine-Gordon equation (see e.g. [24])

\[
\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} - \sin u, \tag{4.1}
\]

on the region \(-10 \leq x \leq 10\) and \(-20 \leq t \leq 20\), with the initial conditions

\[
u(x, -20) = -4\arctan (c^{-1} \text{sech}(\kappa x) \sin(20\kappa)),
\]

\[
u_t(x, -20) = \frac{4\kappa \cos(20\kappa) \text{sech}(\kappa x)}{1 + c^{-2} \text{sech}^2(\kappa x) \sin^2(20\kappa)},
\]

and the boundary conditions

\[
u(-10, t) = \nu(10, t) = -4\arctan (c^{-1} \text{sech}(10\kappa) \sin(c\kappa t)),
\]

where \(\kappa = 1/\sqrt{1 + c^2}\). The exact solution is given by

\[
u(x, t) = -4\arctan (c^{-1} \text{sech}(\kappa x) \sin(c\kappa t)),
\]

which is known as the breather solution of the sine-Gordon equation.

For the spatial discretisation of Problem 2, we use the Chebyshev pseudospectral discretisation [26]. The parameter \(c\) is selected as \(c = 0.5\), and the total number of spatial mesh grids is denoted by \(N\). During the experiment, we choose various values of \(N\) for the two ERKN integrators, and hence the value of \(\| M \|\) varies with \(N\). As shown in Fig. 4, the larger \(N\) is, the larger \(\| M \|\) becomes. Fortunately, however, the large \(\| M \|\) is, the better accuracy gains for the global error. This result strongly supports our analysis in Section 3, which definitely shows that the global error bounds of ERKN integrators are independent of \(\| M \|\) when applied to the underlying nonlinear wave equations. For the stepsize \(h = 1/64\), the global errors of the two RKN methods are too large to plot here. This means that the two RKN methods are unstable with the stepsize \(h = 1/64\). To exclude the possible influence of the spatial discretisation on the accuracy of numerical solutions, we select \(N = 200\) and plot the efficiency curves in Fig. 5(a) by varying the stepsize \(h\), where the higher order ERKN method gives higher efficiency. Note that only the numerical results of the two ERKN methods are plotted in Fig. 5(a), since the reduced RKN methods give unstable numerical solutions for the large \(N\). The numerical convergence orders of the two ERKN methods are respectively 4.01 and 5.89 for ERKN3s4 and ERKN7s6, which are shown in Fig. 5(b).

Problem 3. [Single soliton] We consider the nonlinear Klein-Gordon equation (see e.g. [1])

\[
\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = -au + bu^3, \tag{4.2}
\]
Figure 4: The log-log plot of maximum global error $GE$ against $t$ with the time stepsize $h=1/64$ for Problem 2: (a) the results for ERKN3s4 for different pairs $(N, \|M\|)$: $(40,1.4677 \times 10^3)$, $(60,7.4169 \times 10^3)$, $(80,2.3426 \times 10^4)$; (b) the results for ERKN7s6 for different pairs $(N, \|M\|)$: $(50,3.5791 \times 10^3)$, $(70,1.3736 \times 10^4)$, $(90,3.7519 \times 10^4)$.

Figure 5: Numerical results for Problem 2 with $N = 200$: (a) the log-log plot of maximum global error $GE$ against the consumed CPU time; (b) the log-log plot of maximum global error $GE$ against the stepsize $h$.

on the region $-10 \leq x \leq 10$ and $0 \leq t \leq 10$, with the initial conditions

$$u(x,0) = \sqrt{\frac{2a}{b}} \text{sech}(\lambda x),$$

$$u_t(x,0) = c\lambda \sqrt{\frac{2a}{b}} \text{sech}(\lambda x) \tanh(\lambda x),$$
and the boundary conditions

\[ u(-10,t) = \sqrt{\frac{2a}{b}} \text{sech}(\lambda(-10-ct)), \]
\[ u(10,t) = \sqrt{\frac{2a}{b}} \text{sech}(\lambda(10-ct)), \]

where \( \lambda = \sqrt{\frac{a}{b^2-c^2}} \). The exact solution is given by

\[ u(x,t) = \sqrt{\frac{2a}{b}} \text{sech}(\lambda(x-ct)). \]

With regard to the spatial discretisation of this problem, we use the Chebyshev pseudospectral discretisation. In this experiment, we take the parameters \( a = 0.3, b = 1 \) and \( c = 0.25 \) similarly to those in [1]. Various values of \( N \) are selected for the two ERKN methods. As shown in Fig. 6, again the larger \( \| M \| \) always indicates the better accuracy for the global error, which is entirely similar to the result obtained in Problem 2. The result obtained here also strongly supports our analysis in Section 3 that the global error bounds of ERKN integrators are independent of \( \| M \| \) when applied to wave equations. Similarly to the case in Problem 2, the global errors for RKN3s4 and RKN7s6 are not plotted either, since the unstability and nonconvergence occur for the two RKN methods with the corresponding stepsizes \( h = 1/12 \) and \( h = 1/24 \), respectively. Similarly to Problem 2, we also plot the efficiency curves in Fig. 7(a) for the fixed \( N = 400 \), from which it can be observed
that ERKN3s4 gives better performance when the accuracy is lower than $10^{-4}$, while the higher order method ERKN7s6 is preferred for high accuracy case. The two reduced RKN methods are omitted in this figure due to their instability for $N = 400$. Fig. 7(b) shows the numerical convergence order of the two ERKN methods, which are respectively 3.91 and 5.86 for ERKN3s4 and ERKN7s6.

5 Conclusions

Highly oscillatory problems pervade a very wide range of applications such as celestial mechanics, nonlinear dynamical systems, quantum chemistry and molecular modelling. The computation of highly oscillatory problems has spawned a large number of different numerical approaches and algorithms. The high accuracy and structure preservation for numerical methods are very important when applied to highly oscillatory Hamiltonian systems. In this paper, in order to gain an insight into the extension of the finite-energy condition for ERKN integrators we further investigated the error estimation for ERKN methods when applied to nonlinear wave equations. Since the solutions of (1.1) are high-frequency oscillators due to the existence of the linear term $M_{y}$, the error bounds of numerical methods depend on $\|M\|$ in general. However, we showed the error bounds for ERKN integrators are independent of $\|M\|$ when applied to conservative or dissipative wave equations. This result is of great importance for ERKN integrators, i.e., the so called generalized Gautschi-type methods. Furthermore, we conducted numerical experiments to demonstrate this point clearly. These numerical simulations soundly support our theoretical analysis.
To sum up, motivated by the seminal work on the error bound based on the analysis of the finite-energy condition (see, e.g. [7–11, 13]), the intensive study of this paper has successfully achieved the extension of the finite-energy condition for ERKN integrators when applied to nonlinear wave equations. Meanwhile we have developed a new result on the error analysis for ERKN integrators. It is believed that the new investigation of error analysis has gained further insight into ERKN integrators when applied to the initial-boundary value problems of nonlinear wave equations (3.1). Especially, it is noted that the multi-frequency oscillatory Hamiltonian system with the Hamiltonian

\[ H(q, p) = \frac{1}{2} p^T p + \frac{1}{2} q^T M q + U(q) \]

has prominently exhibited the ERKN integrators in the literature (see, e.g. [3, 10–13, 23, 25, 29, 30, 32, 35–38]) from the accuracy to the qualitative behaviour of the symplecticity preservation, energy preservation, symmetry preservation etc. in the long-term numerical simulation when applied to nonlinear wave equations (3.1).

Before the end of this paper, it should be addressed that all essential features of the analysis of this paper are present in the one-dimensional case of nonlinear wave equations, although the arguments naturally extend to the higher-dimensional case.

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Appendix

In general, the domain \( \Omega(\Delta) \), the eigenvalues and the eigenfunctions of the operator \( \Delta \) will depend on the specified boundary conditions. As an example, we prove that the operators \( \phi_j(\Delta) \) for \( j = 0, 1, \cdots \) defined by (3.3) are bounded for periodic boundary problems with \( D = [0, 2\pi] \). In this case, the operator \( \Delta \) is defined on the domain

\[ \Omega(\Delta) = \{ v \in H^2(D) : v(x) = v(x + 2\pi) \} . \]
The operator has a complete system of orthogonal eigenfunctions \( \{e^{ikx} : k \in \mathbb{Z}\} \) in the complex Hilbert space \( L^2(D) \), and the corresponding eigenvalues are \(-k^2, k \in \mathbb{Z}\). The functions \( \phi_j(z) \) for \( j=0,1,\cdots \), defined by (3.4) allow us to define the operator:

\[
\phi_j(t\Delta) : L^2(D) \rightarrow L^2(D),
\]

(A.1)

given by

\[
\phi_j(t\Delta)v(x) = \sum_{k=-\infty}^{\infty} \hat{\phi}_k(t) e^{ikx} \quad \text{for} \quad v(x) = \sum_{k=-\infty}^{\infty} \hat{v}_k e^{ikx}.
\]

Clearly, the functions \( \phi_j(z) \) are bounded for any \( z \leq 0 \), i.e. \( |\phi_j(z)| \leq 1 \) for \( j=0,1,\cdots \). Then, we will show that the functions \( \phi_j(z) \) will define the bounded operators \( \phi_j(t\Delta) \) for any \( t \geq 0 \).

**Theorem A.1.** The operators \( \phi_j(\Delta) \) defined by (3.3) and (3.4) are bounded operators under the norm \( \| \cdot \|_{L^2(D) \rightarrow L^2(D)} \), that is,

\[
\|\phi_j(t\Delta)\|_{L^2(D) \rightarrow L^2(D)} \leq \gamma_j \quad t \geq 0,
\]

(A.2)

where \( \gamma_j \) are the bounds of the functions \( \phi_j(z) \) with \( z \leq 0 \) for \( j=0,1,2,\cdots \), respectively.

**Proof.** Before going on the proof for the boundedness of the operators \( \phi_j(t\Delta) \) for \( j=0,1,2,\cdots \), we first clarify the inner product of the space \( L^2(D) \) expressed by

\[
(u,v) = \int_D u(x)\overline{v(x)} dx, \quad \forall u,v \in L^2(D).
\]

(A.3)

For any function \( v(x) \in L^2(D) \), its Fourier series can be represented as

\[
v(x) = \sum_{k=-\infty}^{\infty} \hat{v}_k e^{ikx}.
\]

The norm of the function in \( L^2(D) \) can be characterized in the frequency space by

\[
\|v\|^2 = \int_D |v(x)|^2 dx = \sum_{k=-\infty}^{\infty} |\hat{v}_k|^2.
\]

(A.4)

Therefore, we have

\[
\|\phi_j(t\Delta)v\|^2 = \sum_{k=-\infty}^{\infty} |\hat{\phi}_k(t)|^2 \leq \sup_{t \geq 0} |\phi_j(-k^2 t)|^2 \cdot \|v\|^2 \leq \gamma_j^2 \|v\|^2,
\]

(A.5)

where \( \gamma_j \) are the bounds of the functions \( \phi_j(z) \) with \( z \leq 0 \) for \( j=0,1,2,\cdots \). Thus, it follows from the definition of operator norm and (A.5) that

\[
\|\phi_j(t\Delta)v\|_{L^2(D) \rightarrow L^2(D)} = \sup_{\|v\| \neq 0} \frac{\|\phi_j(t\Delta)v\|}{\|v\|} \leq \sup_{t \geq 0} |\phi_j(-k^2 t)| \leq \gamma_j, \quad j=0,1,2,\cdots, t \geq 0.
\]

(A.6)

The conclusion of the theorem is proved.

The boundness of \( \phi_j(\Delta) \) for other boundary conditions can be proved in a similar way.
References


