East Asian Journal on Applied Mathematics
doi: 10.4208/eajam.181016.300517c

Vol. 7, No. 4, pp. 799-809
November 2017

Solution of a Nonlinear Eigenvalue Problem Using Signed Singular Values

Kouhei Ooi[1], Yoshinori Mizuno[2], Tomohiro Sogabe[1], Yusaku Yamamoto[3,*] and Shao-Liang Zhang[1]

[1] *Nagoya University, 1 Furo-cho, Chikusa, Nagoya, Aichi, 464-8603, Japan.*
[2] *Meteorological Research Institute, 1-1 Nagamine, Tsukuba, Ibaraki, 305-0052, Japan.*
[3] *The University of Electro-Communications, 1-5-1 Chofugaoka, Chofu, Tokyo, 182-8585, Japan.*

Received 18 October 2016; Accepted (in revised version) 30 May 2017.

Abstract. We propose a robust numerical algorithm for solving the nonlinear eigenvalue problem $A(\lambda)\mathbf{x}=\mathbf{0}$. Our algorithm is based on the idea of finding the value of λ for which $A(\lambda)$ is singular by computing the smallest eigenvalue or singular value of $A(\lambda)$ viewed as a constant matrix. To further enhance computational efficiency, we introduce and use the concept of signed singular value. Our method is applicable when $A(\lambda)$ is large and nonsymmetric and has strong nonlinearity. Numerical experiments on a nonlinear eigenvalue problem arising in the computation of scaling exponent in turbulent flow show robustness and effectiveness of our method.

AMS subject classifications: 15A18, 47J10, 65F15

Key words: Nonlinear eigenvalue problem, signed singular value, scaling exponent, turbulent flow.

1. Introduction

Let $A(\lambda)\in\mathbf{R}^{n\times n}$ be a matrix whose elements depend on a scalar parameter λ. In the *nonlinear eigenvalue problem*, we seek a value of λ for which there exists a nonzero vector $\mathbf{x}\in\mathbf{R}^n$ such that

$$A(\lambda)\mathbf{x}=\mathbf{0} \tag{1.1}$$

holds. In this paper, we call λ and $\mathbf{x}$ the *nonlinear eigenvalue* and *nonlinear eigenvector*, respectively. Eq. (1.1) includes many types of eigenvalue problems as a special case. When $A(\lambda)=A-\lambda B$, we have a generalized eigenvalue problem. $A(\lambda)=\lambda^2M+\lambda C+K$ leads to a quadratic eigenvalue problem. When $A(\lambda)=(e^{\lambda}-1)A_1+\lambda^2A_2+A_3$, we have a general nonlinear eigenvalue problem with exponential dependence on λ.

Nonlinear eigenvalue problems arise in a variety of applications. In structural mechanics, a decay system is described by a quadratic eigenvalue problem [16]. In electronic

*Corresponding author. *Email address:* yusaku.yamamoto@uec.ac.jp (Y. Yamamoto)

structure calculation, the augmented plane wave (APW) method [9] gives rise to a nonlinear eigenvalue problem $H(\lambda)\mathbf{x} = \lambda\mathbf{x}$, where $H(\lambda)$ is the APW Hamiltonian. In theoretical fluid dynamics, computation of the scaling exponent in turbulent flow is formulated as a nonlinear eigenvalue problem [8]. Among these problems, the last one is hard to solve numerically, because the coefficient matrix $A(\lambda)$ is large and nonsymmetric, and dependence on λ is strongly nonlinear. Also, the elements of $A(\lambda)$ are not given explicitly, but are given by a computer program. The cost of computing $A(\lambda)$ itself for a given λ is very large.

Several numerical algorithms have been proposed so far to solve nonlinear eigenvalue problems. There are algorithms based on multivariate Newton's method [10, 11], nonlinear extensions of the Arnoldi [17] and Jacobi-Davidson [4, 18] algorithms, and algorithms based on complex contour integral [2, 3, 20]. While these methods are efficient for certain types of problems, they are not suited to a problem like the one arising in the computation of the scaling exponent in turbulent flow. Multivariate Newton's method and nonlinear extensions of Arnoldi and Jacobi-Davidson methods are based on linear approximation, and require either that the target problem is only weakly nonlinear or sufficiently good initial approximations of the eigenvalue and the eigenvector are provided. Algorithms based on complex contour integral can be applied even when the problem is strongly nonlinear. But they require evaluation of $A(\lambda)$ for complex values of λ, even when the nonlinear eigenvalue to be computed is real.

In this paper, we propose a robust numerical method for the nonlinear eigenvalue problem. In our method, we compute the nonlinear eigenvalue by seeking the value of λ for which $A(\lambda)$ is singular. As a measure of singularity, we use the *linear eigenvalue* of $A(\lambda)$, which is simply an eigenvalue of $A(\lambda)$ viewed as a constant matrix. Thus the problem is reduced to finding the zero of the linear eigenvalue as a function of λ. The latter problem can be solved stably even if the original problem has strong nonlinearity, because it is a one-dimensional root finding problem. To further improve the computational efficiency, we introduce the notion of the *signed singular value* of $A(\lambda)$ and propose to use it instead of the linear eigenvalue. Numerical experiments show that these methods can solve the nonlinear eigenvalue problem such as the one arising in the computation of the scaling exponent in turbulent flow stably and efficiently.

The rest of this paper is organized as follows. In Section 2, we explain our target nonlinear eigenvalue problem in more detail and show the difficulties with the existing algorithms. To overcome the difficulties, we present two algorithms, namely, the one based on the linear eigenvalue and the one based on the signed singular value in Section 3. The effectiveness of these methods is examined through numerical experiments in Section 4. Section 5 is devoted to conclusions.

2. The Target Problem and Existing Algorithms

2.1. Target problems

In this paper, we consider a nonlinear eigenvalue problem that has the following characteristics.

(a) $A(\lambda)$ is large, sparse and nonsymmetric.
(b) Dependence of $A(\lambda)$ on λ is strongly nonlinear.
(c) Computation of the matrix elements of $A(\lambda)$ takes a long time, so we want to minimize the number of evaluation of $A(\lambda)$.
(d) The derivative $\frac{d}{d\lambda}A(\lambda)$ is not readily available.
(e) We are interested in computing a few real nonlinear eigenvalues in a specified interval, along with the corresponding eigenvectors.

Item (a) means that the matrix size n is so large that the LU decomposition of $A(\lambda)$ is not feasible or inefficient. By items (c) and (d), we consider the case where the elements of $A(\lambda)$ are not given explicitly, but are given by a computer program.

An example of the nonlinear eigenvalue problem having these characteristics is the one arising in the computation of the scaling exponent of a passive scalar field in a 2-dimensional turbulent flow [8]. In the next subsection, we describe this problem briefly and explain why it gives rise to a nonlinear eigenvalue problem with exponential type nonlinearity.

2.2. Computation of the scaling exponent of a passive scalar field in a 2-dimensional turbulent flow

Let $\mathbf{v}(x, y, t)$ be the velocity field of a 2-dimensional turbulent flow and consider the time evolution of a scalar field $\psi(x, y, t)$ such as temperature or impurity concentration advected by the turbulent flow. As is well known, the governing equation of $\psi(x, y, t)$ is the convection-diffusion equation

$$\frac{\partial \psi}{\partial t} + \mathbf{v} \cdot \nabla \psi = \kappa \nabla^2 \psi + f, \tag{2.1}$$

where κ is the diffusion constant and f is the source term. $\psi(x, y, t)$ is called the passive scalar field. Now, introduce the so-called N-point correlation function $\Psi_N(\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_N)$ by

$$\Psi_N(\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_N) = \langle \psi(\mathbf{x}_1)\psi(\mathbf{x}_2)\cdots\psi(\mathbf{x}_N) \rangle, \tag{2.2}$$

where $\mathbf{x}_i = (x_i, y_i)$ and $\langle \cdot \rangle$ denotes statistical average. We are interested in computing Ψ_N in the case of $N = 4$ [8]. If the turbulent flow is homogeneous and isotropic, the correlation function has several invariance properties such as the invariance under translation, rotation and permutation of variables. Also, it is known that it has the scaling property:

$$\Psi_4(R\mathbf{x}_1, R\mathbf{x}_2, R\mathbf{x}_3, R\mathbf{x}_4) = R^{\lambda_4}\Psi_4(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4). \tag{2.3}$$

Here, R is a positive scalar and λ_4 is called the *scaling exponent*. It is an important physical quantity characterizing the scalar field. To obtain λ_4, we use the fact that Ψ_4 satisfies the so-called *closure equation*:

$$\mathscr{L}\Psi_4(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4) = 0, \tag{2.4}$$

where $\mathscr{L}$ is a second order linear differential operator with respect to $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ and $\mathbf{x}_4$. Now, using translation invariance, we can assume that $\mathbf{x}_4 = \mathbf{0}$ without loss of generality. Moreover, from rotational invariance, we can assume that $y_3 = 0$ and write $\mathbf{x}_3$ as $\mathbf{x}_3 = (R, 0)$. Finally, using this R, we express $\mathbf{x}_1$ and $\mathbf{x}_2$ as $\mathbf{x}_1 = (R\xi_1, R\eta_1)$ and $\mathbf{x}_2 = (R\xi_2, R\eta_2)$. Hence we have changed the variables to $\xi_1, \eta_1, \xi_2, \eta_2$ and R. We write the 4-point correlation function with these new variables as $\bar{\Phi}_4(\xi_1, \eta_1, \xi_2, \eta_2, R)$. Since $\mathscr{L}$ is a second order linear differential operator, we can rewrite Eq. (2.4) as

$$\left(\mathscr{L}_0 \frac{\partial^2}{\partial R^2} + \mathscr{L}_1 \frac{\partial}{\partial R} + \mathscr{L}_2\right) \bar{\Phi}_4 = 0 \tag{2.5}$$

where $\mathscr{L}_j$ is a jth order linear differential operator with respect to ξ_1, η_1, ξ_2 and η_2. By further noting that $\bar{\Phi}_4(\xi_1, \eta_1, \xi_2, \eta_2, R)$ can be expressed as

$$\bar{\Phi}_4(\xi_1, \eta_1, \xi_2, \eta_2, R) = R^{\lambda_4} \Phi_4(\xi_1, \eta_1, \xi_2, \eta_2) \tag{2.6}$$

due to the scaling property (2.3), we obtain the partial differential equation for Φ_4:

$$\left\{\lambda_4(\lambda_4 - 1)\mathscr{L}_0 + \lambda_4 \mathscr{L}_1 + \mathscr{L}_2\right\} \Phi_4 = 0. \tag{2.7}$$

This shows that the scaling exponent λ_4 can be obtained by solving a *quadratic eigenvalue problem*.

The exponential dependence on λ_4 arises as follows. The quadratic eigenvalue problem (2.7) is defined in the entire 4-dimensional space of the variables ξ_1, η_1, ξ_2 and η_2. However, to solve the problem numerically, we have to truncate the domain and work with a compact domain, say Ω, which contains the origin. In that case, the value of Φ at a point $(\xi_1, \eta_1, \xi_2, \eta_2)$ outside Ω, which is needed to compute the finite difference formula at the domain boundary, must be computed from the value of Φ inside Ω. This can be done using the scaling property (2.3) because $(R\xi_1, R\eta_1, R\xi_2, R\eta_2)$ is in Ω for a sufficiently small value of R. However, then we introduce the factor R^{λ_4} to express $\Phi_4(\xi_1, \eta_1, \xi_2, \eta_2)$ in terms of $\Phi_4(R\xi_1, R\eta_1, R\xi_2, R\eta_2)$. This is the origin of *exponential nonlinearity*.

In this problem, the matrix size n is usually at least 10^5, and the elements of $A(\lambda)$ resulting from the discretisation of (2.7) have both exponential and polynomial dependence on λ. The elements of $A(\lambda)$ are given by a computer program and evaluating them for each λ takes a long time. Also, $\frac{d}{d\lambda}A(\lambda)$ is not readily given. In this problem, we are mainly interested in computing the smallest positive nonlinear eigenvalue and the corresponding eigenvector. A nonlinear eigenvalue problem with similar characteristics also arises in the APW method [9] for electronic structure calculation.

2.3. Existing methods and their limitations

For this type of problem, existing algorithms based on linearization, such as the multivariate Newton's method [10, 11] and nonlinear extensions of the Arnoldi [17] and Jacobi-Davidson [4, 18] methods will not work. They require sufficiently good initial approximations in the case of strongly nonlinear problems, but in our target problem, a good

initial approximation of the nonlinear eigenvector $\mathbf{x}$ is usually hard to obtain. In addition, Newton-based methods require $\frac{d}{d\lambda}A(\lambda)$, which is not readily available. Another disadvantage of these methods is that it is difficult to guarantee that all the nonlinear eigenvalues in a specified interval have been obtained.

Recently, nonlinear eigenvalue solvers based on complex contour integral attract attention [2, 3, 20]. They are applicable even when the problem has strong nonlinearity and can find all the nonlinear eigenvalues in a specified closed curve on the complex plane. However, they require evaluation of $A(\lambda)$ for complex values of λ even when the nonlinear eigenvalue of interest is real. In addition, some algorithms require $\frac{d}{d\lambda}A(\lambda)$ in addition to $A(\lambda)$ [2, 20].

As another approach, we can find the nonlinear eigenvalue by seeking the value of λ for which $\det(A(\lambda)) = 0$. Once we have found such λ, which we denote by $\bar{\lambda}$, we can find the nonlinear eigenvector by computing the null space of the constant matrix $A(\bar{\lambda})$. This method is robust even when the problem has strong nonlinearity, because it is a one dimensional root finding problem. The weakness of this method is that the computation of the determinant is costly. In fact, it requires as much computational work as the computation of the LU decomposition. So this method is not efficient when $A(\lambda)$ is large and sparse.

3. The Proposed Algorithms

Based on the observations given in the previous section, we develop new robust algorithms for our target nonlinear eigenvalue problem.

3.1. Linear eigenvalues and linear singular values of $A(\lambda)$

We begin with the definitions of a linear eigenvalue and a linear singular value, which play a central role in our algorithms. For a fixed λ, we call μ a *linear eigenvalue* of $A(\lambda)$ if there exists a nonzero vector $\mathbf{y} \in \mathbf{R}^n$ such that $A(\lambda)\mathbf{y} = \mu\mathbf{y}$. Also, for a fixed λ, we call $\sigma \geq 0$ a *linear singular value* of $A(\lambda)$ if σ^2 is a linear eigenvalue of $A(\lambda)^\top A(\lambda)$. A linear eigenvalue and a linear singular value are simply an eigenvalue and a singular value of $A(\lambda)$ viewed as a constant matrix. We are mainly interested in the linear eigenvalue of the smallest modulus and the smallest linear singular value. Since they are functions of λ, we denote them by $\mu(\lambda)$ and $\sigma(\lambda)$, respectively.

It is clear that the following equivalence holds.

$$\begin{aligned} \exists \mathbf{x} \neq \mathbf{0},\ A(\lambda)\mathbf{x} = \mathbf{0} &\iff \det(A(\lambda)) = 0 \\ &\iff \mu(\lambda) = 0 \\ &\iff \sigma(\lambda) = 0. \end{aligned} \tag{3.1}$$

3.2. Algorithm based on the linear eigenvalue

Suppose that we want to compute a nonlinear eigenvalue in an interval $[\alpha, \beta]$. In the following, we assume that $\mu(\lambda)$ is real in this interval. Then, our problem is to find the

solution of $\mu(\lambda) = 0$ in this interval. The idea of using $\mu(\lambda) = 0$ has been discussed in, for example, the Ph. D thesis of Abdel-Aziz [1]. Here, we propose to use the secant method for solving this scalar equation. The algorithm is given below. Note that several conditions must be satisfied to guarantee the convergence of the secant method. Let $\bar{\lambda}$ be the nonlinear eigenvalue in the interval $[\alpha, \beta]$. Then the method converges if, for example, $\mu'(\bar{\lambda}) \neq 0$, $\mu''(\bar{\lambda}) \neq 0$, $\mu''(\lambda)$ is continuous in $[\alpha, \beta]$ and the width of the interval is sufficiently small [6].

Algorithm 1 Algorithm based on the linear eigenvalue.

Set two initial values $\lambda_0, \lambda_1 \in [\alpha, \beta]$.
$\ell = 1$
repeat
  $\ell := \ell + 1$
  $\lambda_\ell = \lambda_{\ell-1} - \frac{\mu(\lambda_{\ell-1})}{\mu(\lambda_{\ell-2}) - \mu(\lambda_{\ell-1})}(\lambda_{\ell-2} - \lambda_{\ell-1})$
until $|\mu(\lambda_\ell)|$ is sufficiently small.
Find the nonlinear eigenvector $\bar{\mathbf{x}}$ as the null space of $A(\lambda_\ell)$.

This algorithm has the advantage that only initial values of λ are required. The linear eigenvalue $\mu(\lambda)$ for a given λ can be computed much more cheaply than $\det(A(\lambda))$ using linear eigensolvers such as the Lanczos, Arnoldi and Jacobi-Davidson methods. Also, $\frac{d}{d\lambda}A(\lambda)$ is not necessary if the secant methods are used to solve $\mu(\lambda) = 0$. Often the convergence of the secant method is very fast if appropriate initial values of λ are provided.

3.3. Algorithm based on the signed singular value

In general, nonsymmetric eigenvalue problems are more difficult to solve than symmetric eigenvalue problems. Thus it is sometimes easier to find the linear singular value $\sigma(\lambda)$ of $A(\lambda)$ than finding the linear eigenvalue $\mu(\lambda)$, since computing $\sigma(\lambda)$ amounts to finding the linear eigenvalue of a symmetric matrix $A(\lambda)^\top A(\lambda)$. In this subsection, we consider an algorithm based on the linear singular value.

However, it is not straightforward to use $\sigma(\lambda)$ instead of $\mu(\lambda)$ in Algorithm 1. This is because $\sigma(\mu)$ is defined as a positive square root of the linear eigenvalue of $A(\lambda)^\top A(\lambda)$ and $\sigma(\lambda)$ viewed as a function of λ is not smooth at $\sigma(\lambda) = 0$. Thus the secant method cannot be applied. Also, the bisection method cannot be used because $\sigma(\lambda)$ is always nonnegative.

To construct an algorithm like Algorithm 1, we have to extend the notion of linear singular values and make it a smooth function of λ by allowing it to become negative. One of such extensions is given by Bunse-Gerstner et al. [5] under the name of analytical singular value decomposition.

Theorem 3.1 (Bunse-Gerstner et al., 1991 [5])**.** *Let the elements of $A(\lambda)$ be analytical functions of λ. Then there exist orthogonal matrices $U'(\lambda), V'(\lambda) \in \mathbf{R}^{n\times n}$ and a diagonal matrix $\Sigma'(\lambda) = \mathrm{diag}(\sigma'_1(\lambda), \cdots, \sigma'_n(\lambda))$ whose elements are analytical functions of λ and which satisfy*

$$A(\lambda) = U'(\lambda)\Sigma'(\lambda)V'(\lambda)^\top. \tag{3.2}$$

This is called the analytical singular value decomposition of $A(\lambda)$.

Note that analytical singular values may be negative. Also, $\sigma'_1(\lambda) \geq \sigma'_2(\lambda) \geq \cdots \geq \sigma'_n(\lambda)$ does not hold in general.

While the mathematical properties of the analytical singular values are ideal for use instead of $\mu(\lambda)$, they are expensive to compute. In fact, computation of the analytical singular values requires numerical solution of a set of ordinary differential equations with $O(n^2)$ unknowns [19]. To overcome this problem, we propose another extension called the signed singular value.

Definition 3.1. Let $\mathbf{v}(\lambda)$ and $\mathbf{u}(\lambda)$ be the right and left singular vectors of $A(\lambda)$ corresponding to the smallest linear singular value $\sigma(\lambda)$. Then we call $\bar{\sigma}(\lambda) = \sigma(\lambda)\mathrm{sgn}(\mathbf{v}(\lambda)^\top \mathbf{u}(\lambda))$ the *signed singular value* of $A(\lambda)$. Here, $\mathrm{sgn}(x)$ is the sign function that takes the value of 1 for $x \geq 0$ and -1 otherwise.

The following theorem states that the signed singular value has a similar property to that of the analytical singular value.

Theorem 3.2. *Assume that $\sigma(\lambda)$ is simple and $(\mathbf{v}(\lambda))^\top \mathbf{u}(\lambda) \neq 0$ in an interval $\lambda_0 \leq \lambda \leq \lambda_1$. Then the signed singular value $\bar{\sigma}(\lambda) = \sigma(\lambda)\mathrm{sgn}(\mathbf{v}(\lambda)^\top \mathbf{u}(\lambda))$ is an analytical function of λ in this interval.*

Proof. Since $\sigma(\lambda)$ is simple in $\lambda_0 \leq \lambda \leq \lambda_1$, from the uniqueness of the singular value decomposition, we have $\mathbf{u}(\lambda) = \pm\mathbf{u}'(\lambda)$ and $\mathbf{v}(\lambda) = \pm\mathbf{v}'(\lambda)$, where $\mathbf{u}'(\lambda)$ and $\mathbf{v}'(\lambda)$ are left and right analytical singular vectors, respectively, corresponding to the analytical singular value of the smallest modulus $\sigma'(\lambda)$. Hence,

$$\begin{aligned}\bar{\sigma}(\lambda) =& \sigma(\lambda)\mathrm{sgn}(\mathbf{v}(\lambda)^\top \mathbf{u}(\lambda)) \\ =& \sigma(\lambda)\mathbf{v}(\lambda)^\top \mathbf{u}(\lambda)/|\mathbf{v}(\lambda)^\top \mathbf{u}(\lambda)| \\ =& \mathbf{v}(\lambda)^\top A\mathbf{v}(\lambda)/|\mathbf{v}(\lambda)^\top \mathbf{u}(\lambda)| \\ =& \mathbf{v}'(\lambda)^\top A\mathbf{v}'(\lambda)/|\mathbf{v}'(\lambda)^\top \mathbf{u}'(\lambda)|. \end{aligned} \tag{3.3}$$

The right-hand side is clearly analytical when $(\mathbf{v}(\lambda))^\top \mathbf{u}(\lambda) \neq 0$. □

The signed singular value has the following desirable properties.

- $\bar{\sigma}(\lambda) = 0 \Leftrightarrow \sigma(\lambda) = 0$.
- $\bar{\sigma}(\lambda)$ is an analytical function of λ under the assumption of Theorem 3.
- It can be computed using standard singular value computation algorithms since it requires only $\sigma(\lambda)$, $\mathbf{v}(\lambda)$ and $\mathbf{u}(\lambda)$.

Using the signed singular value, we can construct an algorithm similar to Algorithm 1 as follows.

Algorithm 2 Algorithm based on the signed singular value.

Set two initial values $\lambda_0, \lambda_1 \in [\alpha, \beta]$.
$\ell = 1$
repeat
  $\ell := \ell + 1$
  $\lambda_\ell = \lambda_{\ell-1} - \frac{\bar{\sigma}(\lambda_{\ell-1})}{\bar{\sigma}(\lambda_{\ell-2}) - \bar{\sigma}(\lambda_{\ell-1})}(\lambda_{\ell-2} - \lambda_{\ell-1})$
until $|\bar{\sigma}(\lambda_\ell)|$ is sufficiently small.
Find the nonlinear eigenvector $\bar{\mathbf{x}}$ as the null space of $A(\lambda_\ell)$.

4. Numerical Results

We tested the effectiveness of our algorithms using a nonlinear eigenvalue problem that has properties (a) through (e) specified in subsection 2.1. Our target problem is the one arising in the computation of the scaling exponent in turbulent flow. We used medium and large problems, for which the matrix size is 35,000 and 100,000, respectively. We are interested in finding the smallest positive nonlinear eigenvalue, which is known to lie in the interval $[0, 4]$ from physical observations. However, no a priori information on the corresponding eigenvector is known. In this problem, the matrix elements are analytical functions of λ, having both polynomial and exponential dependence on λ, as stated in subsection 2.2. Hence, the assumptions of Theorem 3.1 hold and the matrix has analytical singular values, which are of course continuous and differentiable as functions of λ. We tried to apply a Newton-type nonlinear eigenvalue solver [11] to this problem, but it did not converge because of strong nonlinearity and lack of good initial estimate for the nonlinear eigenvector. Our computational environment is Fujitsu PrimePower HPC2500 and we used 16 processors.

First, to see the behavior of the smallest linear eigenvalue $\mu(\lambda)$ and the signed singular value $\bar{\sigma}(\lambda)$, we plotted them for $\lambda \in [0, 4]$ with intervals of 0.1. The graphs of $\mu(\lambda)$ and $\bar{\sigma}(\lambda)$ are shown in Fig. 1 and Fig. 2, respectively. From Fig. 1, we know that $\mu(\lambda)$ has zeros at $\lambda = 0$, $2.9 < \lambda < 3.0$ and $3.8 < \lambda < 3.9$. Thus nonlinear eigenvalues exist in these intervals. In particular, the smallest positive nonlinear eigenvalue that we seek lies between 2.9 and 3.0. The same conclusion follows from Fig. 2, because $\bar{\sigma}(\lambda)$ has zeros at the same positions as $\mu(\lambda)$. Moreover, it can be seen that $\mu(\lambda)$ is a smooth function of λ near the zeros, as predicted by Theorem 3.2. Thus we can use the secant method to find the zeros of $\bar{\sigma}(\lambda)$, as shown in Algorithm 2.

Based on these results, we applied the secant method to $\mu(\lambda)$ and $\bar{\sigma}(\lambda)$ with initial values $\lambda_0 = 2.9$ and $\lambda_1 = 3.0$. For the medium size problem, both iterations converged in 4 steps and resulted in the same solution, $\lambda = 2.926654$, which agrees well with the value obtained by perturbation analysis [8]. One step of the secant iteration took 35,520 seconds and 2,005 seconds, respectively, when using $\mu(\lambda)$ and $\bar{\sigma}(\lambda)$. For the large problem, one step of the secant method took 16,200 seconds when using $\bar{\sigma}(\lambda)$, while the computation using $\mu(\lambda)$ could not be finished due to time limit. Thus we can conclude that the use of signed singular value greatly improves the computational efficiency.

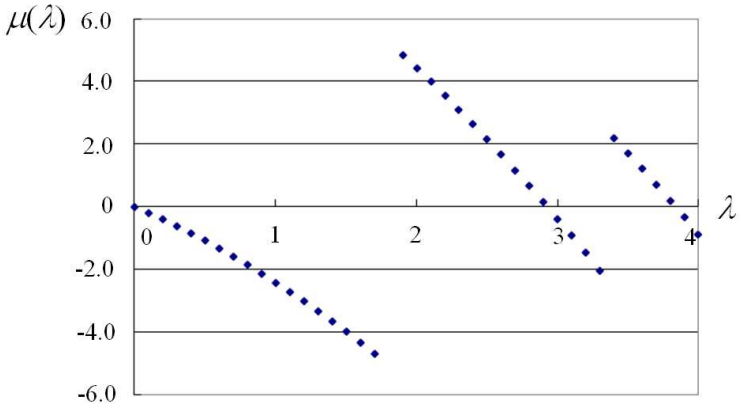

Figure 1: The smallest linear eigenvalue as a function of λ.

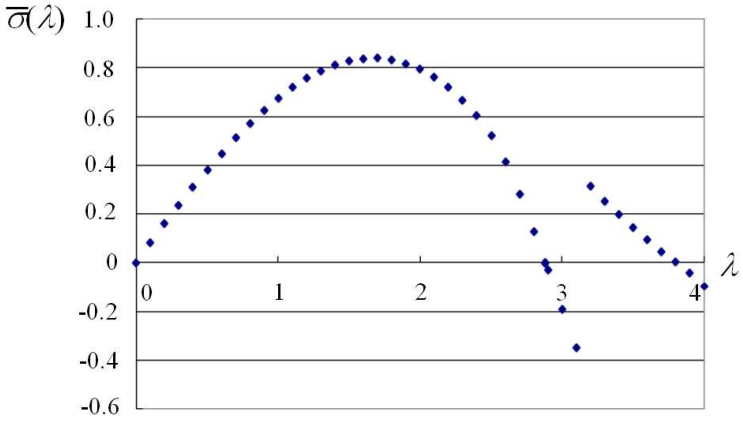

Figure 2: The signed singular value as a function of λ.

5. Conclusion

In this paper, we proposed a robust numerical algorithm for solving the nonlinear eigenvalue problem. Our algorithm is based on the smallest linear eigenvalue or the signed singular value and is applicable when the matrix $A(\lambda)$ is large and nonsymmetric and has strong nonlinearity. It can be used even when the matrix elements are not given explicitly but are given by a computer program, which takes a long time to execute.

We applied our method to a nonlinear eigenvalue problem arising in the computation of the scaling exponent in turbulent flow, for which an algorithm based on the Newton method did not converge. Our method succeeded in finding the nonlinear eigenvalue of interest stably. Furthermore, it was shown that the variant using the signed singular value proved to be much more efficient than the one using the smallest linear eigenvalue. Our future work includes applying this method to nonlinear eigenvalue problems arising in other fields.

Very recently, several new algorithms have been proposed for the nonlinear eigenvalue problem. They include a QR-based algorithm [7], a preconditioned CG type method [15]

and Newton-like methods [12–14]. It remains as our future work to apply these methods to our problem and compare their performance with that of our algorithm.

Acknowledgments

The author would like to thank Professor Yukio Kaneda for fruitful discussion. The authors are also grateful to the anonymous reviewers for valuable comments, which helped us improving the quality of this paper. This study has been supported by JSPS KAKENHI Grant Numbers JP26286087, JP15H02708, JP15H02709, JP16KT0016, 17H02828 and 17K19966.

References

[1] M. R. H. Abdel-Aziz, *Numerical safeguarded use of the implicit restarted Lanczos algorithm for solving nonlinear eigenvalue problems and its monotonicity analysis*, PhD thesis, Rice University, USA, 1993.

[2] T. Amako, Y. Yamamoto, and S.-L. Zhang, *A large-grained parallel algorithm for nonlinear eigenvalue problems and its implementation using OmniRPC*, In *Proceedings of the 2008 IEEE International Conference on Cluster Computing*, pages 42–49, 2008.

[3] J. Asakura, T. Sakurai, H. Tadano, T. Ikegami and K. Kimura, *A numerical method for nonlinear eigenvalue problems using contour integrals*, JSIAM Letters, 1 (2009), 52–55.

[4] T. Betcke and H. Voss, *A Jacobi-Davidson-type projection method for nonlinear eigenvalue problems*, Future Generation Comput. Syst., 20 (2004), 363–372.

[5] A. Bunse-Gerstner, R. Byers, V. Mehrmann and N. K. Nichols, *Numerical computation of an analytic singular value decomposition of a matrix valued function*, Numer. Math., 60 (1991), 1–39.

[6] G. Forsythe, M. Malcolm and C. Moler, *Computer Methods for Mathematical Computations*. Englewood Cliffs, New Jersey, 1977.

[7] C. Garrett, Z. Bai and R. -C. Li, *A nonlinear QR algorithm for banded nonlinear eigenvalue problems*, ACM Trans. Math. Software, 43 (2016), 4:1–4:19.

[8] Y. Mizuno, K. Ohi, T. Sogabe, Y. Yamamoto and Y. Kaneda, *Four-point correlation function of a passive scalar field in rapidly fluctuating turbulence: numerical analysis of an exact closure equation*, Phys. Rev. E, 82 (2010), 036316–036324.

[9] V. V. Nemoshkalenko and N. V. Antonov, *Computational Methods in Solid State Physics*. CRC Press, 1999.

[10] A. Neumaier, *Residual inverse iteration for the nonlinear eigenvalue problem*, SIAM J. Numer. Anal., 22 (1985), 914–923.

[11] A. Ruhe, *Algorithms for the nonlinear eigenvalue problem*, SIAM J. Numer. Anal., 10 (1973), 674–689.

[12] D. Szyld and F. Xue, *Local convergence analysis of several inexact Newton-type algorithms for general nonlinear eigenvalue problems*, Numer. Math., 123 (2013), 333–362.

[13] D. Szyld and F. Xue, *Local convergence of Newton-like methods for degenerate eigenvalues of nonlinear eigenproblems. I. Classical algorithms*, Numer. Math., 129 (2015), 353–381.

[14] D. Szyld and F. Xue, *Local convergence of Newton-like methods for degenerate eigenvalues of nonlinear eigenproblems: II. Accelerated algorithms*, Numer. Math., 129 (2015), 383–403.

[15] D. Szyld and F. Xue, *Preconditioned eigensolvers for large-scale nonlinear Hermitian eigenproblems with variational characterizations. I. Extreme eigenvalues*, Math. Comp., 85 (2016), 2887–2918.
[16] F. Tisseur and K. Meerbergen, *The quadratic eigenvalue problem*, SIAM Rev., 43 (2001), 235–286.
[17] H. Voss, *An Arnoldi method for nonlinear eigenvalue problems*, BIT Numerical Mathematics, 44 (2004), 387–401.
[18] H. Voss, *A Jacobi-Davidson method for nonlinear and nonsymmetric eigenproblems*, Computers & Structures, 85 (2007), 1284–1292.
[19] K. Wright, *Differential equations for the analytic singular value decomposion of a matrix*, Numer. Math., 63 (1992), 283–295.
[20] Y. Yamamoto, *An elementary derivation of the projection method for nonlinear eigenvalue problems based on complex contour integration*, in em Eigenvalue Problems: Algorithms, Software and Applications in Petascale Computing, Proceedings of EPASA 2015, Springer (to appear).