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Optimal Error Estimates for a Fully Discrete Euler Scheme for Decoupled Forward Backward Stochastic Differential Equations

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Abstract. In error estimates of various numerical approaches for solving decoupled forward backward stochastic differential equations (FBSDEs), the rate of convergence for one variable is usually less than for the other. Under slightly strengthened smoothness assumptions, we show that the fully discrete Euler scheme admits a first-order rate of convergence for both variables.

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1. Introduction

On a filtered complete probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{0\leqslant t\leqslant T}, P)$ where $\mathscr{F}_t$ is generated by the standard Brownian motion W_s, $0 \leqslant s \leqslant t$ and T is a fixed time horizon, we consider the decoupled forward-backward stochastic differential equations (FBSDEs):

$$dX_t = b(t,X_t)\,dt + \sigma(t,X_t)\,dW_t, \quad X_0 = x_0\,, \tag{1.1}$$
$$-dy_t = f(t,X_t,y_t,z_t)\,dt - z_t\,dW_t, \quad y_T = g(X_T), \tag{1.2}$$

where $X_0 = x_0$ is the initial condition of the forward equation, $y_T = g(X_T)$ is the terminal condition of the backward equation, and b, σ, f and g are deterministic functions in $\mathbb{R}$. A triple $(X_t, y_t, z_t) : [0,T] \times \Omega \to \mathbb{R}\times\mathbb{R}\times\mathbb{R}$ is called a solution of Eqs. (1.1) and (1.2) if

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its components are $\mathscr{F}_t$-adapted, square integrable, and satisfy the respective forward and backward integral equations

$$X_t = X_0 + \int_0^t b(s,X_s)\,ds + \int_0^t \sigma(s,X_s)\,dW_s\,, \quad t \in [0,T],$$
$$y_t = g(X_T) + \int_t^T f(s,X_s,y_s,z_s)\,ds - \int_t^T z_s\,dW_s\,, \quad t \in [0,T],$$

where the two integrals with respect to the Brownian motion W_s are Itô-type.

Pardoux & Peng [10] proved the existence and uniqueness of the solution of nonlinear backward stochastic differential equations (BSDEs) with the $\mathscr{F}_T$-measurable terminal condition ξ, and Peng [12] gave the following probabilistic representation for the nonlinear Feynman-Kac solutions of Eqs. (1.1) and (1.2):

$$y_t = u(t,X_t), \qquad z_t = \partial_x u(t,X_t)\sigma(t,X_t), \quad t \in [0,T), \tag{1.3}$$

where $u(t,x)$ is the smooth solution of the partial differential equation

$$\partial_t u(t,x) + b(t,x)\partial_x u(t,x) + \frac{1}{2}\sigma(t,x)^2\partial_{xx}u(t,x) = -f\big(t,x,u(t,x),\partial_x u(t,x)\sigma(t,x)\big),$$

with the terminal condition $u(T,x) = g(x)$. Subsequently, FBDSEs have been studied extensively and applied in many fields — e.g. mathematical finance, stochastic optimal control, nonlinear expectation, risk measure, and related problems [4, 5, 11, 13]. It is very difficult to find solutions in explicit closed form, so considerable attention has been paid to the numerical solution of FBSDEs and many numerical schemes have already been proposed [1–3, 6, 14–21]. Here we reconsider the fully discrete Euler scheme for decoupled FBSDEs proposed in Ref. [14], and under certain regular conditions on the data b, σ, f and g we prove its first-order sup-norm convergence in solving Eqs. (1.1) and (1.2). In Section 2, some preliminaries are introduced, and the fully discrete Euler scheme is discussed in Section 3. Our error estimates of the scheme are derived in Section 4, and some concluding remarks are made in Section 5.

2. Preliminaries

Let us first list some notation — viz.

- $C_b^{\ell,k,k,k}$: the set of continuously differentiable functions $\psi : [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ with uniformly bounded partial derivatives $\partial_t^{\ell_1}\partial_x^{k_1}\partial_y^{k_2}\partial_z^{k_3}\psi$ for $0 \leqslant \ell_1 \leqslant \ell$ and $0 \leqslant k_1 + k_2 + k_3 \leqslant k$. Analogous definition applies for $C_b^{\ell,k}$.
- $(X_t^{r,x}, y_t^{r,x}, z_t^{r,x})$: solution (X_t, y_t, z_t) of (1.1), (1.2) with initial condition replaced by $X_r = x$, for $r \leqslant t$. And $(X_t^x, y_t^x, z_t^x) := (X_t^{t,x}, y_t^{t,x}, z_t^{t,x})$.
- $\mathscr{F}_t^{r,x}$: the σ-field generated by $\{X_s^{r,x} \mid r \leqslant s \leqslant t\}$, for $r \leqslant t$.

- $\mathbb{E}_t^{r,x}[\cdot] := \mathbb{E}[\cdot|\mathscr{F}_t^{r,x}]$, $\mathbb{E}_t^{x}[\cdot] := \mathbb{E}[\cdot|\mathscr{F}_t^{t,x}]$.

Given $x \in \mathbb{R}$, let $X_t^{r,x}$, $y_t^{r,x}$ and $z_t^{r,x}$ denote the solutions of the FBSDEs (1.1) and (1.2) for $t \in [r,T]$ with the initial condition $X_r = x$ and terminal condition $y_T = g(X_T^{r,x})$ — i.e. for $t \in [r,T]$,

$$X_t^{r,x} = x + \int_r^t b(s, X_s^{r,x})\,ds + \int_r^t \sigma(s, X_s^{r,x})\,dW_s\,, \tag{2.1}$$

$$y_t^{r,x} = g(X_T^{r,x}) + \int_t^T f(s, X_s^{r,x}, y_s^{r,x}, z_s^{r,x})\,ds - \int_t^T z_s^{r,x}\,dW_s\,. \tag{2.2}$$

We introduce ∇_h to denote the difference quotient operator with respect to the space point x at time r with space step h — e.g. the difference quotient

$$\nabla_h X_t^{r,x} = \frac{X_t^{r,x+h} - X_t^{r,x}}{h}\,,$$

and for a smooth function $\psi : \mathbb{R} \to \mathbb{R}$ the difference quotient

$$\nabla_h \psi(X_t^{r,x}) = \frac{\psi(X_t^{r,x+h}) - \psi(X_t^{r,x})}{h}\,. \tag{2.3}$$

From the mean value theorem, we have the identity

$$\nabla_h \psi(X_t^{r,x}) = \psi'\big(\theta X_t^{r,x+h} + (1-\theta)X_t^{r,x}\big)\nabla_h X_t^{r,x}$$

for some $\theta \in [0,1]$. Applying ∇_h to Eq. (2.1) and assuming smoothness conditions on the coefficients b and σ, we have

$$\nabla_h X_t^{r,x} = 1 + \int_r^t \partial_x b(s, \Theta_s^{r,x})\,\nabla_h X_s^{r,x}\,ds + \int_r^t \partial_x \sigma(s, \Lambda_s^{r,x})\,\nabla_h X_s^{r,x}\,dW_s\,,$$

where $\Theta_s^{r,x} = \theta X_s^{r,x+h} + (1-\theta)X_s^{r,x}$ and $\Lambda_s^{r,x} = \lambda X_s^{r,x+h} + (1-\lambda)X_s^{r,x}$ with θ and λ in the interval $[0,1]$. Assuming $b, \sigma \in C_b^{0,1}$, for $p \geqslant 1$ we have that $\mathbb{E}[(\nabla_h X_t^{r,x})^p]$ is bounded by a constant that is independent of r, t, h, x, and $X_t^{r,x}$.

Let D_t denote the Malliavin derivative, and assume b and σ are smooth functions. Then the Malliavin derivative $D_s X_t^{r,x}$ of $X_t^{r,x}$ satisfies the equation

$$D_s X_t^{r,x} = \sigma(s, X_s^{r,x}) + \int_s^t \partial_x b(\nu, X_\nu^{r,x}) D_s X_\nu^{r,x}\,d\nu + \int_s^t \partial_x \sigma(\nu, X_\nu^{r,x}) D_s X_\nu^{r,x}\,dW_\nu$$

for $r \leqslant s \leqslant t$. Assuming that $b, \sigma \in C_b^{0,1}$, for $p \geqslant 1$ we therefore have that $\mathbb{E}[(D_s X_t^{r,x})^p]$ is bounded by a constant independent of r, t, h, x, and $X_t^{r,x}$. There is similar result for the second-order Malliavin derivative of $X_t^{r,x}$. Thus we have the following lemma:

Lemma 2.1. *Assuming that $b, \sigma \in C_b^{0,2}$, we have*

$$\mathbb{E}\big[(\nabla_h X_t^{r,x})^p\big] + \mathbb{E}[(D_s X_t^{r,x})^p] + \mathbb{E}\big[(D_{s_1} D_{s_2} X_t^{r,x})^p\big] \leqslant C\,,$$

where $r \leqslant s \leqslant t$, $r \leqslant s_1 \leqslant s_2 \leqslant t$, and C is a that is constant independent of r, s, s_1, s_2, t, x, h, and $X_t^{r,x}$.

3. Fully Discrete Euler Scheme

We now introduce the fully discrete Euler method for the FBSDEs [14]. Let N denote a positive integer and $\mathscr{Z}$ the set of all integers, and define the time-space partition

$$T_p \times S_p = \left\{(t_n, x_k) : k \in \mathscr{Z}, n = 0, 1, \cdots, N\right\},$$

where $T_p : \{t_n | \, n = 0, 1, \cdots, N, \ 0 = t_0 < \cdots < t_N = T\}$, $t_{n+1} - t_n = \Delta t$, $n = 0, 1, \cdots, N-1$ is the time partition and $S_p : \{x_k | \, k \in \mathscr{Z}\}$, $x_{k+1} - x_k = \Delta x$, $k \in \mathscr{Z}$ is the partition of space $\mathbb{R}$. Then given the Feynman-Kac solutions (1.3), we solve for y_t and z_t on the time-space partition $T_p \times S_p$ with the initial value x_0 in Eq. (1.1) a grid point.

For $t \in [t_n, t_{n+1}]$, let $X_t^{t_n,x_k}$, $y_t^{t_n,x_k}$ and $z_t^{t_n,x_k}$ be the the solutions of the FBSDEs (2.1) and (2.2) with r and x replaced by t_n and x_k, respectively. Then we have the following equations:

$$\begin{aligned} X_{t_{n+1}}^{t_n,x_k} &= x_k + \int_{t_n}^{t_{n+1}} b(t, X_t^{t_n,x_k})\, dt + \int_{t_n}^{t_{n+1}} \sigma(t, X_t^{t_n,x_k})\, dW_t, \\ y_{t_n}^{x_k} &= y_{t_{n+1}}^{t_n,x_k} + \int_{t_n}^{t_{n+1}} f(t, X_t^{t_n,x_k}, y_t^{t_n,x_k}, z_t^{t_n,x_k})\, dt - \int_{t_n}^{t_{n+1}} z_t^{t_n,x_k} dW_t. \end{aligned} \tag{3.1}$$

Taking the expectation operation $\mathbb{E}[\cdot]$ on (3.1), we deduce that

$$y_{t_n}^{x_k} = \mathbb{E}[y_{t_{n+1}}^{t_n,x_k}] + \Delta t\, f_{t_n}^{x_k} + \bar{R}_{y,n}^k, \tag{3.2}$$

where $f_{t_n}^{x_k} = f(t_n, x_k, y_{t_n}^{x_k}, z_{t_n}^{x_k})$ and the residue is defined by

$$\bar{R}_{y,n}^k = \int_{t_n}^{t_{n+1}} \mathbb{E}\big[f(t, X_t^{t_n,x_k}, y_t^{t_n,x_k}, z_t^{t_n,x_k})\big]\, dt - \Delta t\, f_{t_n}^{x_k}. \tag{3.3}$$

Multiplying (3.1) by $\Delta W_{t_{n+1}} := W_{t_{n+1}} - W_{t_n}$ and then taking the expectation $\mathbb{E}[\cdot]$ gives

$$\Delta t\, z_{t_n}^{x_k} = \mathbb{E}[y_{t_{n+1}}^{t_n,x_k} \Delta W_{t_{n+1}}] + \bar{R}_{z,n}^k \tag{3.4}$$

with the residue

$$\bar{R}_{z,n}^k = \int_{t_n}^{t_{n+1}} \mathbb{E}\big[f(t, X_t^{t_n,x_k}, y_t^{t_n,x_k}, z_t^{t_n,x_k}) \Delta W_{t_{n+1}}\big]\, dt + \Delta t\, z_{t_n}^{x_k} - \int_{t_n}^{t_{n+1}} \mathbb{E}[z_t^{t_n,x_k}]\, dt. \tag{3.5}$$

It is notable that the expectation terms in Eqs. (3.2) and (3.4) contain the values of $y_{t_{n+1}}$ on the whole real line, and in computing them in practice we hope that only the values of $y_{t_{n+1}}$ on S_p are involved. Initially, we need approximations of the $X_{t_{n+1}}^{t_n,x_k}$, so we introduce the Euler formula for the forward process:

$$\tilde{X}_{t_{n+1}}^{t_n,x_k} = x_k + b(t_n, x_k)\Delta t + \sigma(t_n, x_k)\Delta W_{t_{n+1}} =: x_k(\Delta W), \tag{3.6}$$

where we denote the Euler approximation $\tilde{X}_{t_{n+1}}^{t_n,x_k}$ for fixed n by $x_k(\Delta W)$ to emphasise the dependence of $\tilde{X}_{t_{n+1}}^{t_n,x_k}$ on the increment $\Delta W_{t_{n+1}}$. Using Eq. (3.6), we define $\tilde{y}_{t_{n+1}}^{t_n,x_k}$ as the value of y at the time-space point $(t_{n+1},\tilde{X}_{t_{n+1}}^{t_n,x_k})$ — i.e.

$$\tilde{y}_{t_{n+1}}^{t_n,x_k} = u(t_{n+1},\tilde{X}_{t_{n+1}}^{t_n,x_k}) = u\big(t_{n+1},x_k(\Delta W)\big)\,.$$

Then writing $\tilde{R}_{y,n}^k$ and $\tilde{R}_{z,n}^k$ for the remainders arising from this approximation of the forward process, we have the expectations

$$\mathbb{E}[y_{t_{n+1}}^{t_n,x_k}] = \mathbb{E}[\tilde{y}_{t_{n+1}}^{t_n,x_k}] + \tilde{R}_{y,n}^k\,, \tag{3.7}$$

$$\mathbb{E}[y_{t_{n+1}}^{t_n,x_k}\Delta W_{t_{n+1}}] = \mathbb{E}[\tilde{y}_{t_{n+1}}^{t_n,x_k}\Delta W_{t_{n+1}}] + \tilde{R}_{z,n}^k\,, \tag{3.8}$$

which can both be written as integrals involving the Gaussian density function $\frac{1}{\sqrt{2\pi}}e^{-\xi^2/2}$ — e.g.

$$\mathbb{E}[\tilde{y}_{t_{n+1}}^{t_n,x_k}] = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} u\big(t_{n+1},x_k(\sqrt{\Delta t}\,\xi)\big)e^{-\xi^2/2}d\xi\,.$$

We can readily approximate the integral using the L-point Gauss-Hermite quadrature rule with nodes ξ_ℓ and weights ω_ℓ, $\ell = 1,2,\cdots,L$ such that we have

$$\mathbb{E}[\tilde{y}_{t_{n+1}}^{t_n,x_k}] = \sum_{\ell=1}^{L} u(t_{n+1},x_{k,\ell})\omega_\ell + R_{E,y,n}^k \tag{3.9}$$

as the discrete approximation to the above equation, where

$$x_{k,\ell} := x_k(\sqrt{\Delta t}\,\xi_\ell) = x_k + b(t_n,x_k)\Delta t + \sigma(t_n,x_k)\sqrt{\Delta t}\,\xi_\ell\,. \tag{3.10}$$

Similarly, the expectation of $\tilde{y}\Delta W$ (3.8) can be approximated by

$$\mathbb{E}[\tilde{y}_{t_{n+1}}^{t_n,x_k}\Delta W_{t_{n+1}}] = \sum_{\ell=1}^{L} u(t_{n+1},x_{k,\ell})\sqrt{\Delta t}\,\xi_\ell\,\omega_\ell + R_{E,z,n}^k\,. \tag{3.11}$$

Here $R_{E,y,n}^k$ and $R_{E,z,n}^k$ denote the residues under the quadrature rule. Generally, $x_{k,\ell} \notin S_p$ and $u(t_{n+1},x_{k,\ell})$ should be approximated by some method involving only $\{u(t_{n+1},x_j)\}_j$, and here we use the linear interpolation

$$u(t_{n+1},x_{k,\ell}) = \mathscr{I}u(t_{n+1},x_{k,\ell}) + R_{I,y,n}^{k,\ell}\,, \tag{3.12}$$

where $R_{I,y,n}^{k,\ell}$ is the residue of the interpolation, and $\mathscr{I}$ is the linear interpolation operator over the space partition S_p defined by

$$\mathscr{I}u(\cdot,x_{k,\ell}) = \beta_{k,\ell}\,u(\cdot,x_{k,\ell}^+) + (1-\beta_{k,\ell})u(\cdot,x_{k,\ell}^-)\,, \qquad \beta_{k,\ell} = \frac{x_{k,\ell}-x_{k,\ell}^-}{\Delta x}\,,$$

with $x_{k,\ell}^+$ ($x_{k,\ell}^-$) the closest grid point on the right-hand (left-hand) side of $x_{k,\ell}$.

From the previous procedure for approximating the expectations, we note the following definition. Given a sequence $\phi = \{\phi^x\}_{x\in S_p}$ (indexed by S_p), for a natural number m the discrete expectation of the product of ϕ and $(\Delta W_{t_{n+1}})^m$ is

$$\hat{\mathbb{E}}_{t_n}^{x_k}\big[\phi\,(\Delta W_{t_{n+1}})^m\big] = \sum_{\ell=1}^{L}\big(\beta_{k,\ell}\,\phi^{x_{k,\ell}^+} + (1-\beta_{k,\ell})\phi^{x_{k,\ell}^-}\big)(\sqrt{\Delta t}\,\xi_\ell)^m\omega_\ell\,. \tag{3.13}$$

Then using the above definitions and Eqs. (3.7), (3.8), (3.9), (3.11) and (3.12), from Eqs. (3.2) and (3.4), we obtain

$$y_{t_n}^{x_k} = \hat{\mathbb{E}}_{t_n}^{x_k}[y_{t_{n+1}}] + \Delta t\, f_{t_n}^{x_k} + R_{y,n}^k\,, \tag{3.14}$$

$$\Delta t\, z_{t_n}^{x_k} = \hat{\mathbb{E}}_{t_n}^{x_k}[y_{t_{n+1}}\Delta W_{t_{n+1}}] + R_{z,n}^x\,, \tag{3.15}$$

where the residues are

$$R_{y,n}^k = \bar{R}_{y,n}^k + \tilde{R}_{y,n}^k + R_{E,y,n}^k + \sum_{\ell=1}^{L} R_{I,y,n}^{k,\ell}\omega_\ell\,, \tag{3.16}$$

$$R_{z,n}^k = \bar{R}_{z,n}^k + \tilde{R}_{z,n}^k + R_{E,z,n}^k + \sum_{\ell=1}^{L} R_{I,y,n}^{k,\ell}\sqrt{\Delta t}\,\xi_\ell\,\omega_\ell\,. \tag{3.17}$$

In Eqs. (3.14) and (3.15), the discrete expectations $\hat{\mathbb{E}}_{t_n}^{x_k}[y_{t_{n+1}}]$ and $\hat{\mathbb{E}}_{t_n}^{x_k}[y_{t_{n+1}}\Delta W_{t_{n+1}}]$ refer to $\hat{\mathbb{E}}_{t_n}^{x_k}[(y_{t_{n+1}})_{S_p}]$ and $\hat{\mathbb{E}}_{t_n}^{x_k}[(y_{t_{n+1}})_{S_p}\Delta W_{t_{n+1}}]$, where $(y_{t_{n+1}})_{S_p} = \{y_{t_{n+1}}^{x_k}\}_{x_k\in S_p}$ is the solution confined to the space partition S_p.

We now introduce $\hat{y}$ and $\hat{z}$ to denote the corresponding numerical solutions for the variables y and z on the time-space partition $T_p \times S_p$ — i.e. we write $\hat{y}_{t_n} = \{\hat{y}_{t_n}^{x_k}\}_{x_k\in S_p}$, $\hat{z}_{t_n} = \{\hat{z}_{t_n}^{x_k}\}_{x_k\in S_p}$ for $n = 0, 1, \cdots, N$. On omitting the residues $R_{y,n}^k$ and $R_{z,n}^k$ in Eqs. (3.14) and (3.15), we obtain the fully discrete Euler scheme for (1.1) and (1.2) as follows.

Given $\hat{y}_{t_N}^{x_k} = g(x_k)$ for $x_k \in S_p$, for $n = N-1, N-2, \cdots, 1, 0$, backwardly solve for $\hat{y}_{t_n}$ and $\hat{z}_{t_n}$ using

$$x_{k,\ell} = x_k + b(t_n, x_k)\Delta t + \sigma(t_n, x_k)\sqrt{\Delta t}\,\xi_\ell\,, \tag{3.18}$$

$$\hat{y}_{t_n}^{x_k} = \hat{\mathbb{E}}_{t_n}^{x_k}[\hat{y}_{t_{n+1}}] + \Delta t\,\hat{f}_{t_n}^{x_k}\,, \tag{3.19}$$

$$\Delta t\,\hat{z}_{t_n}^{x_k} = \hat{\mathbb{E}}_{t_n}^{x_k}[\hat{y}_{t_{n+1}}\Delta W_{t_{n+1}}]\,, \tag{3.20}$$

where $x_k \in S_p$ and $\hat{f}_{t_n}^{x_k} = f(t_n, x_k, \hat{y}_{t_n}^{x_k}, \hat{z}_{t_n}^{x_k})$.

4. Error Estimates

Under certain smoothness and boundedness conditions on b, σ, f and g, we now prove the first-order sup-norm convergence of the fully discrete Euler scheme (3.18)-(3.20). The arguments used in the proof are much the same as in Ref. [6], which basically consists of two parts — proving stability, and estimating the bound of the residues. Henceforth, ∇_h denotes the difference quotient defined in Eq. (2.3) with $h=\Delta x$. In particular, when applied to a sequence $\phi=\{\phi^{x_k}\}_{x_k\in S_p}$, the difference quotient is

$$\nabla_h\phi^{x_k}=\frac{\phi^{x_{k+1}}-\phi^{x_k}}{\Delta x}\,.$$

We first subtract Eqs. (3.19) and (3.20) from the respective Eqs. (3.14) and (3.15) to get two error equations, on which the difference quotient operator ∇_h is applied to obtain two more equations, so we have the following four equations:

$$\mu^{x_k}_{t_n}=\hat{\mathbb{E}}^{x_k}_{t_n}[\mu_{t_{n+1}}]+\Delta t(f^{x_k}_{t_n}-\hat{f}^{x_k}_{t_n})+R^k_{y,n}\,, \tag{4.1}$$

$$\Delta t\,\nu^{x_k}_{t_n}=\hat{\mathbb{E}}^{x_k}_{t_n}[\mu_{t_{n+1}}\Delta W_{t_{n+1}}]+R^k_{z,n}\,, \tag{4.2}$$

$$\nabla_h\mu^{x_k}_{t_n}=\nabla_h\hat{\mathbb{E}}^{x_k}_{t_n}[\mu_{t_{n+1}}]+\Delta t(\nabla_h f^{x_k}_{t_n}-\nabla_h\hat{f}^{x_k}_{t_n})+\nabla_h R^k_{y,n}\,, \tag{4.3}$$

$$\Delta t\,\nabla_h\,\nu^{x_k}_{t_n}=\nabla_h\hat{\mathbb{E}}^{x_k}_{t_n}\big[\mu_{t_{n+1}}\Delta W_{t_{n+1}}\big]+\nabla_h R^k_{z,n}\,, \tag{4.4}$$

where $\mu^{x_k}_{t_n}=y^{x_k}_{t_n}-\hat{y}^{x_k}_{t_n}$ and $\nu^{x_k}_{t_n}=z^{x_k}_{t_n}-\hat{z}^{x_k}_{t_n}$ are the errors. To prove the stability of the numerical scheme, we also note some basic properties of the operator $\hat{\mathbb{E}}^{x_k}_{t_n}[\cdot]$, and its difference quotient $\nabla_h\hat{\mathbb{E}}^{x_k}_{t_n}[\cdot]$. Let $\|\cdot\|_\infty$ denote the supremum norm of functions defined on S_p — i.e., $\|\phi\|_\infty=\sup_{x_k\in S_p}|\phi^{x_k}|$.

Proposition 4.1. *Let* $\hat{\mathbb{E}}^{x_k}_{t_n}[\cdot]$ *be the operator defined in* (3.13). *For fixed n and k, there are non-negative numbers* α_j, $j=0,\pm1,\pm2,\cdots$ *where only a finite number are positive, such that for the sequence* ϕ *indexed by* S_p *we have that*

$$\sum_j\alpha_j=1\,,$$
$$\hat{\mathbb{E}}^{x_k}_{t_n}[\phi]=\sum_j\alpha_j\,\phi^{x_j}\,.$$

Consequently,

$$\big|\hat{\mathbb{E}}^{x_k}_{t_n}[\phi]\big|\leqslant\|\phi\|_\infty,\qquad\big(\hat{\mathbb{E}}^{x_k}_{t_n}[\phi]\big)^2\leqslant\hat{\mathbb{E}}^{x_k}_{t_n}\big[(\phi)^2\big],$$

and moreover

$$\big(\hat{\mathbb{E}}^{x_k}_{t_n}[\phi]\big)^2+\frac{1}{\Delta t}\big(\hat{\mathbb{E}}^{x_k}_{t_n}[\phi\,\Delta W_{t_{n+1}}]\big)^2\leqslant\hat{\mathbb{E}}^{x_k}_{t_n}\big[(\phi)^2\big].$$

The proof of Proposition 4.1 readily follows, as the "non-negative with sum one" property holds for the coefficients of both the quadrature rule and the linear interpolation. Further, the quadrature rule outputs the exact value for polynomials with degree less than or equal to $2L-1$.

Under additional smoothness and boundedness conditions, we obtain the two estimates in the following proposition.

Proposition 4.2. *Assuming that* $b,\sigma \in C_b^{0,1}$*, we have the estimate*

$$\left(\nabla_h \hat{\mathbb{E}}_{t_n}^{x_k}[\phi]\right)^2 + \frac{1}{4\Delta t}\left(\nabla_h \hat{\mathbb{E}}_{t_n}^{x_k}[\phi\,\Delta W_{t_{n+1}}]\right)^2 \leqslant (1+C\Delta t)\|\nabla_h \phi\|_\infty^2 . \tag{4.5}$$

Furthermore,

$$\frac{1}{\Delta t}\left(\hat{\mathbb{E}}_{t_n}^{x_k}[\phi\,\Delta W_{n+1}]\right)^2 \leqslant C\Delta t\|\nabla_h \phi\|_\infty^2 . \tag{4.6}$$

Here C is some constant independent of n, k, Δt, Δx and the solutions of the fully discrete scheme.

Proof. First, in the notation of Eq. (3.10) we have

$$\frac{x_{k+1,\ell} - x_{k,\ell}}{\Delta x} = 1 + \nabla_h b(t_n, x_k)\Delta t + \nabla_h \sigma(t_n, x_k)\sqrt{\Delta t}\,\xi_\ell ,$$

and it is easy to check that $x_{k+1,\ell} - x_{k,\ell} \geqslant 0$ holds for sufficiently small Δt. Hence on denoting the index of $x_{k,\ell}^-$ as the natural number such that $x_{i_{k,\ell}} = x_{k,\ell}^-$, we have $i_{k,\ell} \leqslant i_{k+1,\ell}$ so that the difference quotient of $\hat{\mathbb{E}}_{t_n}^{x_k}[\phi]$ is

$$\begin{aligned}\nabla_h \hat{\mathbb{E}}_{t_n}^{x_k}[\phi] &= \sum_{\ell=1}^{L} \frac{1}{\Delta x}\Big(\beta_{k+1,\ell}\,\phi^{x_{k+1,\ell}^+} + (1-\beta_{k+1,\ell})\phi^{x_{k+1,\ell}^-} - \beta_{k,\ell}\,\phi^{x_{k,\ell}^+} - (1-\beta_{k,\ell})\phi^{x_{k,\ell}^-}\Big)\omega_\ell \\ &= \sum_{\ell=1}^{L}\sum_j \lambda_{k,\ell,j}\,\nabla_h \phi^{x_j}\,\omega_\ell,\end{aligned}$$

where

$$\begin{aligned}\lambda_{k,\ell,j} =& \chi_{\{i_{k,\ell}=j<i_{k+1,\ell}\}}(1-\beta_{k,\ell}) + \chi_{\{i_{k,\ell}<j<i_{k+1,\ell}\}} + \chi_{\{i_{k,\ell}<j=i_{k+1,\ell}\}}\beta_{k+1,\ell} \\ &+ \chi_{\{i_{k,\ell}=j=i_{k+1,\ell}\}}(\beta_{k+1,\ell} - \beta_{k,\ell}),\end{aligned}$$

and we note that

$$\sum_j \lambda_{k,\ell,j} = \beta_{k+1,\ell} - \beta_{k,\ell} + i_{k+1,\ell} - i_{k,\ell} = \frac{x_{k+1,\ell} - x_{k,\ell}}{\Delta x} .$$

Consequently, for the natural number m, we define the new discrete expectation

$$\dot{\mathbb{E}}_{t_n}^{x_k}\left[\phi\,(\Delta W_{t_{n+1}})^m\right] := \sum_{\ell=1}^{L}\left(\sum_j \bar{\lambda}_{k,\ell,j}\,\phi^{x_j}\right)(\sqrt{\Delta t}\,\xi_\ell)^m\omega_\ell , \tag{4.7}$$

where the coefficients

$$\bar{\lambda}_{k,\ell,j} = \frac{\Delta x}{x_{k+1,\ell} - x_{k,\ell}}\lambda_{k,\ell,j}$$

satisfy the "non-negative with sum one" property

$$\bar{\lambda}_{k,\ell,j} \geqslant 0 , \qquad \sum_j \bar{\lambda}_{k,\ell,j} = 1 . \tag{4.8}$$

Consequently, we have the two representations

$$\nabla_h \hat{\mathbb{E}}_{t_n}^{x_k}[\phi] = \big(1 + \nabla_h b(t_n, x_k)\Delta t\big)\dot{\mathbb{E}}_{t_n}^{x_k}[\nabla_h \phi] + \nabla_h \sigma(t_n, x_k)\dot{\mathbb{E}}_{t_n}^{x_k}[\nabla_h \phi\, \Delta W_{t_{n+1}}], \tag{4.9}$$

$$\begin{aligned}\nabla_h \hat{\mathbb{E}}_{t_n}^{x_k}[\phi\, \Delta W_{t_{n+1}}] = &\big(1 + \nabla_h b(t_n, x_k)\Delta t\big)\dot{\mathbb{E}}_{t_n}^{x_k}[\nabla_h \phi\, \Delta W_{t_{n+1}}] \\ &+ \nabla_h \sigma(t_n, x_k)\dot{\mathbb{E}}_{t_n}^{x_k}[\nabla_h \phi\, (\Delta W_{t_{n+1}})^2],\end{aligned} \tag{4.10}$$

hence we obtain inequality (4.5) on invoking the definition (4.7) and the property (4.8).

For the second estimate (4.6), we suppose $\{\xi_\ell\}$ is sorted in ascending order (i.e. $\xi_\ell < \xi_{\ell+1}$) and define $\ell' := L+1-\ell$. We observe that $\xi_{\ell'} = -\xi_\ell$ and $\omega_{\ell'} = \omega_\ell$, and define $i_{k,\ell}$ as before. Thus again noting (3.10), if $\sigma(t_n, x_k) > 0$ we have that $i_{k,\ell} \leqslant i_{k,\ell'}$ for $1 \leqslant \ell \leqslant \lfloor L/2 \rfloor$, and if $\sigma(t_n, x_k) < 0$ then $i_{k,\ell'} \leqslant i_{k,\ell}$ for $1 \leqslant \ell \leqslant \lfloor L/2 \rfloor$, so rearranging the summation in $\hat{\mathbb{E}}_{t_n}^{x_k}[\phi\, \Delta W_{t_{n+1}}]$ we deduce that

$$\begin{aligned}\hat{\mathbb{E}}_{t_n}^{x_k}[\phi\, \Delta W_{t_{n+1}}] &= \sum_{\ell=1}^{\lfloor L/2 \rfloor} \Big(\beta_{k,\ell'}\, \phi^{x^+_{k,\ell'}} + (1-\beta_{k,\ell'})\phi^{x^-_{k,\ell'}} - \beta_{k,\ell}\, \phi^{x^+_{k,\ell}} - (1-\beta_{k,\ell})\phi^{x^-_{k,\ell}}\Big)\sqrt{\Delta t}\,|\xi_\ell|\,\omega_\ell \\ &= \Delta x \sum_{\ell=1}^{\lfloor L/2 \rfloor} \sum_j \rho_{k,\ell,j}\, \nabla_h \phi^{x_j} \sqrt{\Delta t}\,|\xi_\ell|\,\omega_\ell\,,\end{aligned}$$

where the coefficients $\rho_{k,\ell,j}$ are defined by

$$\begin{aligned}\rho_{k,\ell,j} = &\chi_{\{i_{k,\ell}=j<i_{k,\ell'}\}}(1-\beta_{k,\ell}) + \chi_{\{i_{k,\ell}<j<i_{k,\ell'}\}} + \chi_{\{i_{k,\ell}<j=i_{k,\ell'}\}}\beta_{k,\ell'} + \chi_{\{i_{k,\ell}=j=i_{k,\ell'}\}}(\beta_{k,\ell'} - \beta_{k,\ell}) \\ &- \chi_{\{i_{k,\ell'}=j<i_{k,\ell}\}}(1-\beta_{k,\ell'}) - \chi_{\{i_{k,\ell'}<j<i_{k,\ell}\}} - \chi_{\{i_{k,\ell'}<j=i_{k,\ell}\}}\beta_{k,\ell}\,.\end{aligned}$$

We note that

$$\begin{aligned}\Delta x \sum_{\ell=1}^{\lfloor L/2 \rfloor} \sum_j \rho_{k,\ell,j} \sqrt{\Delta t}\,|\xi_\ell|\,\omega_\ell &= \Delta x \sum_{\ell=1}^{\lfloor L/2 \rfloor} \big(\beta_{k,\ell'} - \beta_{k,\ell} + i_{k,\ell'} - i_{k,\ell}\big)\sqrt{\Delta t}\,|\xi_\ell|\,\omega_\ell \\ &= 2\sigma(t_n, x_k)\Delta t \sum_{\ell=1}^{\lfloor L/2 \rfloor} \xi_\ell^2\, \omega_\ell = \sigma(t_n, x_k)\Delta t\,,\end{aligned}$$

and so denote $\bar{\rho}_{k,\ell,j} = (\sigma(t_n, x_k)\Delta t)^{-1} \Delta x\, \rho_{k,\ell,j} \sqrt{\Delta t}\,|\xi_\ell|\,\omega_\ell$ such that the "non-negative with sum one" property holds for $\bar{\rho}_{k,\ell,j}$ in the sense that

$$\bar{\rho}_{k,\ell,j} \geqslant 0\,, \qquad \sum_{\ell=1}^{\lfloor L/2 \rfloor} \sum_j \bar{\rho}_{k,\ell,j} = 1\,. \tag{4.11}$$

Now we define the new discrete expectation

$$\ddot{\mathbb{E}}_{t_n}^{x_k}[\phi] = \sum_{\ell=1}^{\lfloor L/2 \rfloor} \sum_j \bar{\rho}_{k,\ell,j}\, \phi^{x_j}\,, \tag{4.12}$$

when the discrete expectation of $\phi\,\Delta W_{t_{n+1}}$ has another representation

$$\hat{\mathbb{E}}_{t_n}^{x_k}[\phi\,\Delta W_{t_{n+1}}] = \sigma(t_n, x_k)\Delta t\,\ddot{\mathbb{E}}_{t_n}^{x_k}[\nabla_h \phi]\,. \tag{4.13}$$

From Eq. (4.13), property (4.11) and definition (4.12) we have the second estimate (4.6). □

Remark 4.1. Unlike the semi-discretised scheme (only discretised in the time variable), due to the interpolation in approximating the expectation some properties may not be preserved for the scheme (3.19) and (3.20). For example, from Eqs. (4.9), (4.10) and (4.13) we fail to represent those terms on the left-hand side of the equality by the same discrete expectation $\hat{\mathbb{E}}_{t_n}^{x_k}[\cdot]$ of $\nabla_h \phi$, which is not an issue for the semi-discretised scheme. As shown in Proposition 2, we choose to add the boundedness condition and keep the estimate under the supremum norm.

Denoting

$$R_n = \sqrt{\|R_{y,n}\|_\infty^2 + \|R_{z,n}\|_\infty^2 + \|\nabla_h R_{y,n}\|_\infty^2 + \|\nabla_h R_{z,n}\|_\infty^2}\,,$$

we obtain conditions for the stability of the scheme (3.19) and (3.20) as follows.

Theorem 4.1. *Assume $f \in C_b^{0,2,2,2}$, $u \in C_b^{0,2}$ and $b, \sigma \in C_b^{0,1}$. Then we have the estimates*

$$\begin{aligned}
\|\mu_{t_n}\|_\infty^2 &\leqslant C\|\mu_{t_N}\|_\infty^2 + \frac{C}{\Delta t}\sum_{i=n}^{N-1}\big(\|R_{y,i}\|_\infty^2 + \|R_{z,i}\|_\infty^2\big), \\
\|\nu_{t_n}\|_\infty^2 &\leqslant C\big(\|\mu_{t_N}\|_\infty^2 + \|\nabla_h \mu_{t_N}\|_\infty^2\big) + \frac{2}{(\Delta t)^2}\|R_{z,n}\|_\infty^2 + \frac{C}{\Delta t}\sum_{i=n}^{N-1}(R_i)^2\,.
\end{aligned}$$

Proof. From Eqs. (4.1)–(4.4), we obtain the inequalities

$$(\mu_{t_n}^{x_k})^2 \leqslant (1+\gamma_1\Delta t)\big(\hat{\mathbb{E}}_{t_n}^{x_k}[\mu_{t_{n+1}}]\big)^2 + (1+\gamma_1\Delta t)\frac{2}{\gamma_1}\Big(\Delta t(f_{t_n}^{x_k} - \hat{f}_{t_n}^{x_k})^2 + \frac{1}{\Delta t}(R_{y,n}^k)^2\Big), \tag{4.14}$$

$$\Delta t(\nu_{t_n}^{x_k})^2 \leqslant \frac{2}{\Delta t}\big(\hat{\mathbb{E}}_{t_n}^{x_k}[\mu_{t_{n+1}}\Delta W_{t_{n+1}}]\big)^2 + \frac{2}{\Delta t}(R_{z,n}^k)^2\,, \tag{4.15}$$

$$\begin{aligned}
(\nabla_h \mu_{t_n}^{x_k})^2 \leqslant{}& (1+\gamma_2\Delta t)\big(\nabla_h \hat{\mathbb{E}}_{t_n}^{x_k}[\mu_{t_{n+1}}]\big)^2 + (1+\gamma_2\Delta t)\frac{2}{\gamma_2}\Big(\Delta t(\nabla_h f_{t_n}^{x_k} - \nabla_h \hat{f}_{t_n}^{x_k})^2 \\
&+ \frac{1}{\Delta t}(\nabla_h R_{y,n}^k)^2\Big),
\end{aligned} \tag{4.16}$$

$$\Delta t(\nabla_h \nu_{t_n}^{x_k})^2 \leqslant \frac{2}{\Delta t}\big(\nabla_h \hat{\mathbb{E}}_{t_n}^{x_k}[\mu_{t_{n+1}}\Delta W_{t_{n+1}}]\big)^2 + \frac{2}{\Delta t}(\nabla_h R_{z,n}^k)^2\,. \tag{4.17}$$

In inequality (4.16), we have

$$\begin{aligned}\nabla_h f_{t_n}^{x_k} - \nabla_h \hat{f}_{t_n}^{x_k} =& \partial_x f_{t_n}^{x_k;\theta} + \partial_y f_{t_n}^{x_k;\theta} \nabla_h y_{t_n}^{x_k} + \partial_z f_{t_n}^{x_k;\theta} \nabla_h z_{t_n}^{x_k} \\ & - \Big(\partial_x \hat{f}_{t_n}^{x_k;\theta} + \partial_y \hat{f}_{t_n}^{x_k;\theta} \nabla_h \hat{y}_{t_n}^{x_k} + \partial_z \hat{f}_{t_n}^{x_k;\theta} \nabla_h \hat{z}_{t_n}^{x_k} \Big) \\ =& \partial_x f_{t_n}^{x_k;\theta} - \partial_x \hat{f}_{t_n}^{x_k;\theta} \\ & + (\partial_y f_{t_n}^{x_k;\theta} - \partial_y \hat{f}_{t_n}^{x_k;\theta}) \nabla_h y_{t_n}^{x_k} + \partial_y \hat{f}_{t_n}^{x_k;\theta} \nabla_h \mu_{t_n}^{x_k} \\ & + (\partial_z f_{t_n}^{x_k;\theta} - \partial_z \hat{f}_{t_n}^{x_k;\theta}) \nabla_h z_{t_n}^{x_k} + \partial_z \hat{f}_{t_n}^{x_k;\theta} \nabla_h \nu_{t_n}^{x_k},\end{aligned}$$

where $\theta \in [0,1]$, $\partial_x f_{t_n}^{x_k;\theta} = \partial_x f(t_n, x_k + \theta \Delta x, y_{t_n}^{x_k} + \theta \Delta x \nabla_h y_{t_n}^{x_k}, z_{t_n}^{x_k} + \theta \Delta x \nabla_h z_{t_n}^{x_k})$, and other terms containing the superscript θ are defined similarly. Under the smoothness conditions, we therefore have

$$(f_{t_n}^{x_k} - \hat{f}_{t_n}^{x_k})^2 \leqslant C\big((\mu_{t_n}^{x_k})^2 + (\nu_{t_n}^{x_k})^2\big), \tag{4.18}$$

$$(\nabla_h f_{t_n}^{x_k} - \nabla_h \hat{f}_{t_n}^{x_k})^2 \leqslant C\big((\mu_{t_n}^{x_k})^2 + (\nu_{t_n}^{x_k})^2 + (\nabla_h \mu_{t_n}^{x_k})^2 + (\nabla_h \nu_{t_n}^{x_k})^2\big). \tag{4.19}$$

Using Proposition 4.1, inequalities (4.14), (4.15) and (4.18), and letting $\gamma_1 = 8C$, we thus have the following estimate for sufficiently small Δt:

$$(\mu_{t_n}^{x_k})^2 + \frac{\Delta t}{2} (\nu_{t_n}^{x_k})^2 \leqslant (1 + 4C\Delta t) \|\mu_{t_{n+1}}\|_\infty^2 + \frac{\Delta t}{2}\big((\mu_{t_n}^{x_k})^2 + (\nu_{t_n}^{x_k})^2\big) + \frac{1}{2\Delta t}\big((R_{y,n}^k)^2 + (R_{z,n}^k)^2\big).$$

Consequently,

$$\begin{aligned}\|\mu_{t_n}\|_\infty^2 &\leqslant (1 + C\Delta t)\|\mu_{t_{n+1}}\|_\infty^2 + \frac{1}{\Delta t}\big(\|R_{y,n}\|_\infty^2 + \|R_{z,n}\|_\infty^2\big) \\ &\leqslant C\|\mu_{t_N}\|_\infty^2 + \frac{C}{\Delta t} \sum_{i=n}^{N-1} \big(\|R_{y,i}\|_\infty^2 + \|R_{z,i}\|_\infty^2\big),\end{aligned} \tag{4.20}$$

so letting $\gamma_2 = 32C$ and combining with Eqs. (4.16), (4.17) and (4.19), using Proposition 4.2 we obtain for sufficiently small Δt that

$$\begin{aligned}(\nabla_h \mu_{t_n}^{x_k})^2 + \frac{\Delta t}{8} (\nabla_h \nu_{t_n}^{x_k})^2 \leqslant& (1 + C\Delta t)\|\nabla_h \mu_{t_{n+1}}\|_\infty^2 + \frac{\Delta t}{8}\big((\mu_{t_n}^{x_k})^2 + (\nabla_h \mu_{t_n}^{x_k})^2 + (\nabla_h \nu_{t_n}^{x_k})^2\big) \\ & + \frac{1}{2\Delta t}\big((R_{z,n}^k)^2 + (\nabla_h R_{y,n}^k)^2 + (\nabla_h R_{z,n}^k)^2\big),\end{aligned}$$

hence

$$\begin{aligned}\|\nabla_h \mu_{t_n}\|_\infty^2 \leqslant& (1 + C\Delta t)\|\nabla_h \mu_{t_{n+1}}\|_\infty^2 + C\Delta t \, \|\mu_{t_n}\|_\infty^2 \\ & + \frac{1}{\Delta t}\big(\|R_{z,n}\|_\infty^2 + \|\nabla_h R_{y,n}\|_\infty^2 + \|\nabla_h R_{z,n}\|_\infty^2\big) \\ \leqslant& C\big(\|\mu_{t_N}\|_\infty^2 + \|\nabla_h \mu_{t_N}\|_\infty^2\big) + \frac{C}{\Delta t} \sum_{i=n}^{N-1} (R_i)^2\end{aligned} \tag{4.21}$$

on using the previous result (4.20). Then using inequalities (4.15), (4.21) and Proposition 4.2 we have

$$\|\nu_{t_n}\|_\infty^2 \leqslant C\big(\|\mu_{t_N}\|_\infty^2 + \|\nabla_h\mu_{t_N}\|_\infty^2\big) + \frac{C}{\Delta t}\sum_{i=n}^{N-1}(R_i)^2 + \frac{1}{(\Delta t)^2}\|R_{z,n}\|_\infty^2\,, \tag{4.22}$$

so invoking the estimates (4.20) and (4.22) we complete the proof. □

From Theorem 4.1, in order to get the estimates of μ_{t_n} and ν_{t_n} it remains to estimate the residues $R_{y,n}$, $R_{z,n}$, $\nabla_h R_{y,n}$ and $\nabla_h R_{z,n}$ defined in Eqs. (3.16) and (3.17), which we do after introducing the following Lemmas 4.1–4.4.

Lemma 4.1. *For $f \in C_b^{1,3,3,3}$, $u \in C_b^{1,4}$ and $b,\sigma \in C_b^{1,3}$ we have the estimates*

$$|\bar{R}_{y,n}^k| = O\big((\Delta t)^2\big)\,, \qquad |\bar{R}_{z,n}^k| = O\big((\Delta t)^2\big)\,,$$
$$|\nabla_h\bar{R}_{y,n}^k| = O\big((\Delta t)^2\big)\,, \qquad |\nabla_h\bar{R}_{z,n}^k| = O\big((\Delta t)^2\big)\,.$$

Proof. From the definitions in (3.3) and (3.5) for $\bar{R}_{y,n}^k$ and $\bar{R}_{z,n}^k$ and the Itô formula, we have

$$\begin{aligned}
\bar{R}_{y,n}^k &= \int_{t_n}^{t_{n+1}} \mathbb{E}[f(t,X_t^{t_n,x_k},y_t^{t_n,x_k},z_t^{t_n,x_k})]\,dt - \Delta t\, f_{t_n}^{x_k} \\
&= \int_{t_n}^{t_{n+1}} (\mathbb{E}[f_t^{t_n,x_k}] - f_{t_n}^{x_k})\,dt = \int_{t_n}^{t_{n+1}}\int_{t_n}^{t} \mathbb{E}[\mathscr{L}^{(0)} f_s^{t_n,x_k}]\,ds\,dt\,,
\end{aligned} \tag{4.23}$$

$$\begin{aligned}
\bar{R}_{z,n}^k &= \int_{t_n}^{t_{n+1}} \mathbb{E}[f(t,X_t^{t_n,x_k},y_t^{t_n,x_k},z_t^{t_n,x_k})\Delta W_{t_{n+1}}]\,dt + \Delta t\, z_{t_n}^{x_k} - \int_{t_n}^{t_{n+1}} \mathbb{E}[z_t^{t_n,x_k}]\,dt \\
&= \int_{t_n}^{t_{n+1}}\int_{t_n}^{t} \Big(\mathbb{E}\big[\mathscr{L}^{(0)} f_s^{t_n,x_k}\Delta W_{t_{n+1}} + \mathscr{L}^{(1)} f_s^{t_n,x_k}\big]\Big)\,ds\,dt - \int_{t_n}^{t_{n+1}}\int_{t_n}^{t} \mathbb{E}[\mathscr{L}^{(0)} z_s^{t_n,x_k}]\,ds\,dt\,,
\end{aligned} \tag{4.24}$$

where $f_s^{t,x} = f(s,X_s^{t,x},y_s^{t,x},z_s^{t,x})$, and for $v \in C^{1,2}$ the operators $\mathscr{L}^{(0)}$ and $\mathscr{L}^{(1)}$ are

$$\begin{aligned}
&\mathscr{L}^{(0)} v(t,x) = \partial_t v(t,x) + b(t,x)\partial_x v(t,x) + \frac{1}{2}\sigma(t,x)^2\partial_{xx} v(t,x)\,,\\
&\mathscr{L}^{(1)} v(t,x) = \sigma(t,x)\partial_x v(t,x)\,.
\end{aligned}$$

The four estimates stated then follow from Eqs. (4.23) and (4.24) with the difference operator ∇_h defined in (2.3), under the conditions of the lemma. □

Lemma 4.2. *For $u \in C_b^{0,4}$ and $b,\sigma \in C_b^{1,3}$ we have the estimates*

$$\big|\tilde{R}_{y,n}^k\big| = O\big((\Delta t)^2\big)\,, \qquad \big|\nabla_h\tilde{R}_{y,n}^k\big| = O\big((\Delta t)^2\big)\,,$$
$$\big|\tilde{R}_{z,n}^k\big| = O\big((\Delta t)^2\big)\,, \qquad \big|\nabla_h\tilde{R}_{z,n}^k\big| = O\big((\Delta t)^2\big)\,.$$

Proof. Subtracting $\tilde{y}$ from y gives

$$y_{t_{n+1}}^{t_n,x_k} - \tilde{y}_{t_{n+1}}^{t_n,x_k} = \partial_x u(t_{n+1}, \Theta_{t_{n+1}}^{t_n,x_k})(X_{t_{n+1}}^{t_n,x_k} - \tilde{X}_{t_{n+1}}^{t_n,x_k}),$$

where $\Theta_{t_{n+1}}^{t_n,x_k} = \theta X_{t_{n+1}}^{t_n,x_k} + (1-\theta)\tilde{X}_{t_{n+1}}^{t_n,x_k}$, $\theta \in [0,1]$. From the Itô-Taylor expansion, $X - \tilde{X}$ has the form

$$\begin{aligned}X_{t_{n+1}}^{t_n,x_k} - \tilde{X}_{t_{n+1}}^{t_n,x_k} = \int_{t_n}^{t_{n+1}}\int_{t_n}^{t} \Big(&\mathscr{L}^{(0)} b(s, X_s^{t_n,x_k}) ds\, dt + \mathscr{L}^{(1)} b(s, X_s^{t_n,x_k})\, dW_s\, dt \\ &+ \mathscr{L}^{(0)}\sigma(s, X_s^{t_n,x_k}) ds\, dW_t + \mathscr{L}^{(1)}\sigma(s, X_s^{t_n,x_k})\, dW_s\, dW_t\Big),\end{aligned}$$

so from Eq. (3.7) we deduce that

$$\begin{aligned}\tilde{R}_{y,n}^k &= \mathbb{E}\Big[\partial_x u(t_{n+1}, \Theta_{t_{n+1}}^{t_n,x_k})(X_{t_{n+1}}^{t_n,x_k} - \tilde{X}_{t_{n+1}}^{t_n,x_k})\Big] \\ &= \mathbb{E}\Big[\int_{t_n}^{t_{n+1}}\int_{t_n}^{t} \Big(\partial_x u(t_{n+1}, \Theta_{t_{n+1}}^{t_n,x_k})\mathscr{L}^{(0)} b(s, X_s^{t_n,x_k}) + D_s\partial_x u(t_{n+1}, \Theta_{t_{n+1}}^{t_n,x_k})\mathscr{L}^{(1)} b(s, X_s^{t_n,x_k}) \\ &\quad + D_t\partial_x u(t_{n+1}, \Theta_{t_{n+1}}^{t_n,x_k})\mathscr{L}^{(0)}\sigma(s, X_s^{t_n,x_k}) + D_s D_t \partial_x u(t_{n+1}, \Theta_{t_{n+1}}^{t_n,x_k})\mathscr{L}^{(1)}\sigma(s, X_s^{t_n,x_k})\Big) ds\, dt\Big].\end{aligned} \tag{4.25}$$

Using (4.25), Lemma 4.1 and the smoothness conditions of the lemma, we get the estimates of $\tilde{R}_{y,n}^k$ and $\nabla_h \tilde{R}_{y,n}^k$. Similarly, from Eq. (3.8) we have

$$\begin{aligned}\tilde{R}_{z,n}^k =&\mathbb{E}\left[\partial_x u(t_{n+1}, \Theta_{t_{n+1}}^{t_n,x_k})(X_{t_{n+1}}^{t_n,x_k} - \tilde{X}_{t_{n+1}}^{t_n,x_k})\int_{t_n}^{t_{n+1}} dW_t\right] \\ =&\mathbb{E}\Bigg[\int_{t_n}^{t_{n+1}} \Big(D_t\partial_x u(t_{n+1}, \Theta_{t_{n+1}}^{t_n,x_k})(X_{t_{n+1}}^{t_n,x_k} - \tilde{X}_{t_{n+1}}^{t_n,x_k}) \\ &+ \partial_x u(t_{n+1}, \Theta_{t_{n+1}}^{t_n,x_k})(D_t X_{t_{n+1}}^{t_n,x_k} - D_t\tilde{X}_{t_{n+1}}^{t_n,x_k})\Big) dt\Bigg] \\ =&\mathbb{E}\Bigg[\int_{t_n}^{t_{n+1}} \Bigg(D_t\partial_x u(t_{n+1}, \Theta_{t_{n+1}}^{t_n,x_k})(X_{t_{n+1}}^{t_n,x_k} - \tilde{X}_{t_{n+1}}^{t_n,x_k}) + \partial_x u(t_{n+1}, \Theta_{t_{n+1}}^{t_n,x_k})\int_{t_n}^{t} \mathscr{L}^{(0)}\sigma(s, X_s^{t_n,x_k}) ds \\ &+ \partial_x u(t_{n+1}, \Theta_{t_{n+1}}^{t_n,x_k})\int_{t}^{t_{n+1}} \partial_x b(s, X_s^{t_n,x_k}) D_t X_s^{t_n,x_k} ds \\ &+ \int_{t_n}^{t} D_s\partial_x u(t_{n+1}, \Theta_{t_{n+1}}^{t_n,x_k})\mathscr{L}^{(1)}\sigma(s, X_s^{t_n,x_k}) ds \\ &+ \int_{t}^{t_{n+1}} D_s\partial_x u(t_{n+1}, \Theta_{t_{n+1}}^{t_n,x_k})\partial_x\sigma(s, X_s^{t_n,x_k}) D_t X_s^{t_n,x_k} ds\Bigg) dt\Bigg].\end{aligned}$$

Under the conditions of the lemma, from Lemma 4.1 we then obtain $\tilde{R}_{z,n}^k$ and $\nabla_h \tilde{R}_{z,n}^k$. $\square$

Lemma 4.3. *For $u \in C_b^{0,5}$ and $b, \sigma \in C_b^{0,1}$ we have*

$$|R^k_{E,y,n}| = O\big((\Delta t)^2\big), \qquad |R^k_{E,z,n}| = O\big((\Delta t)^2\big),$$
$$|\nabla_h R^k_{E,y,n}| = O\big((\Delta t)^2\big), \qquad |\nabla_h R^k_{E,z,n}| = O\big((\Delta t)^2\big).$$

Proof. In the error estimate of the Gauss quadrature rule [8], it is proved that given $f \in C^r$, $0 < \epsilon < 1$ we have

$$\left| \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(\xi) e^{-\xi^2/2} d\xi - \sum_{\ell=1}^{L} f(\xi_\ell)\omega_\ell \right| \leqslant \frac{CL^{-r/2}}{\sqrt{2\pi}} \int_{\mathbb{R}} \left| f^{(r)}(\xi) e^{-(1-\epsilon)\xi^2/2} \right| d\xi ,$$

with constant C independent of L and f but dependent on r. Thus there is $\bar{\xi}$ such that

$$|R^k_{E,y,n}| \leqslant C(\Delta t)^{r/2} \left| \sigma(t_n, x_k)^r \partial_x^r u\big(t_{n+1}, x_k(\sqrt{\Delta t}\,\bar{\xi})\big) \right| ,$$

and $\bar{\xi}_1$ and $\bar{\xi}_2$ such that

$$\begin{aligned} |R^k_{E,z,n}| \leqslant & C(\Delta t)^{(r+1)/2} \left| \sigma(t_n, x_k)^r \partial_x^r u\big(t_{n+1}, x_k(\sqrt{\Delta t}\,\bar{\xi}_1)\big) \right| \\ & + C(\Delta t)^{r/2} \left| \sigma(t_n, x_k)^{r-1} \partial_x^{r-1} u\big(t_{n+1}, x_k(\sqrt{\Delta t}\,\bar{\xi}_2)\big) \right| . \end{aligned}$$

Similar estimates hold for the difference quotients of the residues, so under the conditions of the lemma and letting $r = 4$ we complete the proof. □

Lemma 4.4. *For $u \in C_b^{0,3}$ and $b, \sigma \in C_b^{0,1}$ we have*

$$\left| \sum_{\ell=1}^{L} R^{k,\ell}_{I,y,n} \omega_\ell \right| = O\big((\Delta x)^2\big), \qquad \left| \sum_{\ell=1}^{L} R^{k,\ell}_{I,y,n} \sqrt{\Delta t}\, \xi_\ell\, \omega_\ell \right| = O\big(\sqrt{\Delta t}\,(\Delta x)^2\big),$$
$$\left| \nabla_h \sum_{\ell=1}^{L} R^{k,\ell}_{I,y,n} \omega_\ell \right| = O\big((\Delta x)^2\big) + O(\sqrt{\Delta t}\,\Delta x),$$
$$\left| \nabla_h \sum_{\ell=1}^{L} R^{k,\ell}_{I,y,n} \sqrt{\Delta t}\, \xi_\ell\, \omega_\ell \right| = O\big(\sqrt{\Delta t}\,(\Delta x)^2\big) + O(\Delta t\, \Delta x).$$

Proof. From the error estimate of the linear interpolation there is $\theta_{k,\ell} \in [x^-_{k,\ell}, x^+_{k,\ell}]$ such that

$$R^{k,\ell}_{I,y,n} = u(t_{n+1}, x_{k,\ell}) - \mathscr{I} u(t_{n+1}, x_{k,\ell}) = \frac{1}{2} u''_{xx}(t_{n+1}, \theta_{k,\ell})(x_{k,\ell} - x^-_{k,\ell})(x_{k,\ell} - x^+_{k,\ell}).$$

Taking the difference quotient ∇_h of $R^{k,\ell}_{I,y,n}$, we obtain

$$\nabla_h R^{k,\ell}_{I,y,n} = \frac{1}{\Delta x}(R^{k+1,\ell}_{I,y,n} - R^{k,\ell}_{I,y,n}) =: S^{k,\ell}_1 + S^{k,\ell}_2, \tag{4.26}$$

where we denote

$$S_1^{k,\ell} = \frac{1}{2}\frac{u''_{xx}(t_{n+1},\theta_{k+1,\ell}) - u''_{xx}(t_{n+1},\theta_{k,\ell})}{\theta_{k+1,\ell}-\theta_{k,\ell}}\ \frac{\theta_{k+1,\ell}-\theta_{k,\ell}}{\Delta x}(x_{k+1,\ell}-x^-_{k+1,\ell})(x_{k+1,\ell}-x^+_{k+1,\ell}), \tag{4.27}$$

$$S_2^{k,\ell} = \frac{1}{2\Delta x}u''_{xx}(t_{n+1},\theta_{k,\ell})\big((x_{k+1,\ell}-x^-_{k+1,\ell})(x_{k+1,\ell}-x^+_{k+1,\ell}) - (x_{k,\ell}-x^-_{k,\ell})(x_{k,\ell}-x^+_{k,\ell})\big). \tag{4.28}$$

From the definition (3.10),

$$\frac{x_{k+1,\ell}-x_{k,\ell}}{\Delta x} = 1 + \nabla_h b(t_n,x_k)\Delta t + \nabla_h \sigma(t_n,x_k)\sqrt{\Delta t}\,\xi_\ell\,,$$

so for sufficiently small Δt we have $0 \leqslant x^-_{k+1,\ell} - x^-_{k,\ell} \leqslant 2\Delta x$ such that

$$\left|\frac{\theta_{k+1,\ell}-\theta_{k,\ell}}{\Delta x}\right| \leqslant 3\,,$$

and therefore from Eq. (4.27) there is $\gamma_{k,\ell}$ in between of $\theta_{k+1,\ell}$ and $\theta_{k,\ell}$ such that

$$|S_1^{k,\ell}| \leqslant C\big|u'''_{xxx}(t_{n+1},\gamma_{k,\ell})\big|(\Delta x)^2\,. \tag{4.29}$$

For the second term $S_2^{k,\ell}$, we discuss its upper bound for the three cases $x^-_{k+1,\ell} - x^-_{k,\ell} = 0$, $x^-_{k+1,\ell} - x^-_{k,\ell} = \Delta x$ and $x^-_{k+1,\ell} - x^-_{k,\ell} = 2\Delta x$.

- If $x^-_{k+1,\ell} - x^-_{k,\ell} = \Delta x$, then there is $\theta_{k,\ell}$ in between of $x_{k+1,\ell} - \Delta x$ and $x_{k,\ell}$ such that

$$\begin{aligned}
&\left|\frac{1}{\Delta x}\big((x_{k+1,\ell}-x^-_{k+1,\ell})(x_{k+1,\ell}-x^+_{k+1,\ell}) - (x_{k,\ell}-x^-_{k,\ell})(x_{k,\ell}-x^+_{k,\ell})\big)\right| \\
=&\left|\frac{1}{\Delta x}\big((x_{k+1,\ell}-\Delta x-x^-_{k,\ell})(x_{k+1,\ell}-\Delta x-x^+_{k,\ell}) - (x_{k,\ell}-x^-_{k,\ell})(x_{k,\ell}-x^+_{k,\ell})\big)\right| \\
=&\left|\frac{1}{\Delta x}(\theta_{k,\ell}-x^-_{k,\ell}+\theta_{k,\ell}-x^+_{k,\ell})(x_{k+1,\ell}-\Delta x - x_{k,\ell})\right| \leqslant \left|x_{k+1,\ell}-\Delta x-x_{k,\ell}\right| \\
=&\,\Delta x\left|\nabla_h b(t_n,x_k)\Delta t + \nabla_h\sigma(t_n,x_k)\sqrt{\Delta t}\,\xi_\ell\right| \leqslant C\Delta x\sqrt{\Delta t}\,.
\end{aligned}$$

- If $x^-_{k+1,\ell} - x^-_{k,\ell} = 2\Delta x$, then there is $\theta_{k,\ell}$ in between of $x_{k+1,\ell} - 2\Delta x$ and $x^+_{k,\ell} + x^-_{k,\ell} - x_{k,\ell}$ such that

$$\begin{aligned}
&\left|\frac{1}{\Delta x}\big((x_{k+1,\ell}-x^-_{k+1,\ell})(x_{k+1,\ell}-x^+_{k+1,\ell}) - (x_{k,\ell}-x^-_{k,\ell})(x_{k,\ell}-x^+_{k,\ell})\big)\right| \\
=&\left|\frac{1}{\Delta x}\big((x_{k+1,\ell}-2\Delta x-x^-_{k,\ell})(x_{k+1,\ell}-2\Delta x-x^+_{k,\ell})\right. \\
&\left.-(x^+_{k,\ell}+x^-_{k,\ell}-x_{k,\ell}-x^-_{k,\ell})(x^+_{k,\ell}+x^-_{k,\ell}-x_{k,\ell}-x^+_{k,\ell})\big)\right| \\
=&\left|\frac{1}{\Delta x}(\theta_{k,\ell}-x^-_{k,\ell}+\theta_{k,\ell}-x^+_{k,\ell})(x_{k+1,\ell}-x^-_{k+1,\ell}-x^+_{k,\ell}+x_{k,\ell})\right| \\
\leqslant&\left|x_{k+1,\ell}-x^-_{k+1,\ell}-(x^+_{k,\ell}-x_{k,\ell})\right| \leqslant \left|x_{k+1,\ell}-x^-_{k+1,\ell}+x^+_{k,\ell}-x_{k,\ell}\right| \\
=&\,\Delta x\left|\nabla_h b(t_n,x_k)\Delta t + \nabla_h\sigma(t_n,x_k)\sqrt{\Delta t}\,\xi_\ell\right| \leqslant C\Delta x\sqrt{\Delta t}\,.
\end{aligned}$$

- If $x^-_{k+1,\ell} - x^-_{k,\ell} = 0$, then there is $\theta_{k,\ell}$ in between of $x_{k+1,\ell}$ and $x^+_{k,\ell} + x^-_{k,\ell} - x_{k,\ell}$ such that

$$
\begin{aligned}
&\left|\frac{1}{\Delta x}\big((x_{k+1,\ell} - x^-_{k+1,\ell})(x_{k+1,\ell} - x^+_{k+1,\ell}) - (x_{k,\ell} - x^-_{k,\ell})(x_{k,\ell} - x^+_{k,\ell})\big)\right| \\
=&\left|\frac{1}{\Delta x}\big((x_{k+1,\ell} - x^-_{k,\ell})(x_{k+1,\ell} - x^+_{k,\ell}) - (x^+_{k,\ell} + x^-_{k,\ell} - x_{k,\ell} - x^-_{k,\ell})(x^+_{k,\ell} + x^-_{k,\ell} - x_{k,\ell} - x^+_{k,\ell})\big)\right| \\
=&\left|\frac{1}{\Delta x}(\theta_{k,\ell} - x^-_{k,\ell} + \theta_{k,\ell} - x^+_{k,\ell})(x_{k+1,\ell} - x^+_{k,\ell} - x^-_{k,\ell} + x_{k,\ell})\right| \\
\leqslant&\left| -(x^+_{k,\ell} - x_{k+1,\ell}) + (x_{k,\ell} - x^-_{k,\ell})\right| \leqslant \left|x^+_{k,\ell} - x_{k+1,\ell} + x_{k,\ell} - x^-_{k,\ell}\right| \\
=&\Delta x\left| -\nabla_h b(t_n, x_k)\Delta t - \nabla_h \sigma(t_n, x_k)\sqrt{\Delta t}\,\xi_\ell\right| \leqslant C\Delta x\sqrt{\Delta t}.
\end{aligned}
$$

Thus from Eq. (4.28) we have

$$
|S_2^{k,\ell}| \leqslant C\left|u''_{xx}(t_{n+1}, \theta_{k,\ell})\right|\Delta x\sqrt{\Delta t}\,, \tag{4.30}
$$

and from Eqs. (4.26), (4.29) and (4.30), we complete the proof. □

Using Theorem 4.1 and Lemmas 4.1–4.4, we obtain error estimates for our fully discrete scheme summarised in the following theorem.

Theorem 4.2. *For $f \in C_b^{1,3,3,3}$ and $u \in C_b^{1,5}$, $b, \sigma \in C_b^{1,3}$ we have the error estimates*

$$
\begin{aligned}
&\|\mu_{t_n}\|^2_\infty = O\big((\Delta t)^2\big) + O\big((\Delta t)^{-2}(\Delta x)^4\big)\,, \\
&\|\nu_{t_n}\|^2_\infty = O\big((\Delta t)^2\big) + O\big((\Delta t)^{-2}(\Delta x)^4\big) + O\big((\Delta t)^{-1}(\Delta x)^2\big)\,.
\end{aligned}
$$

Proof. Under the specified conditions, from Eqs. (3.16), (3.17), and Lemmas 4.1–4.4, we have the following estimates:

$$
\begin{aligned}
&\left|R^k_{y,n}\right| \leqslant \left|\bar{R}^k_{y,n}\right| + \left|\tilde{R}^k_{y,n}\right| + \left|R^k_{E,y,n}\right| + \left|\sum_{\ell=1}^{L} R^{k,\ell}_{I,y,n}\omega_\ell\right| = O\big((\Delta t)^2\big) + O\big((\Delta x)^2\big)\,, \\
&\left|R^k_{z,n}\right| \leqslant \left|\bar{R}^k_{z,n}\right| + \left|\tilde{R}^k_{z,n}\right| + \left|R^k_{E,z,n}\right| + \left|\sum_{\ell=1}^{L} R^{k,\ell}_{I,y,n}\sqrt{\Delta t}\,\xi_\ell\,\omega_\ell\right| = O\big((\Delta t)^2\big) + O\big(\sqrt{\Delta t}(\Delta x)^2\big)\,, \\
&\left|\nabla_h R^k_{y,n}\right| \leqslant \left|\nabla_h \bar{R}^k_{y,n}\right| + \left|\nabla_h \tilde{R}^k_{y,n}\right| + \left|\nabla_h R^k_{E,y,n}\right| + \left|\nabla_h \sum_{\ell=1}^{L} R^{k,\ell}_{I,y,n}\omega_\ell\right| \\
&\qquad = O\big((\Delta t)^2\big) + O\big((\Delta x)^2\big) + O(\sqrt{\Delta t}\,\Delta x)\,, \\
&\left|\nabla_h R^k_{z,n}\right| \leqslant \left|\nabla_h \bar{R}^k_{z,n}\right| + \left|\nabla_h \tilde{R}^k_{z,n}\right| + \left|\nabla_h R^k_{E,z,n}\right| + \left|\nabla_h \sum_{\ell=1}^{L} R^{k,\ell}_{I,y,n}\sqrt{\Delta t}\,\xi_\ell\,\omega_\ell\right| \\
&\qquad = O\big((\Delta t)^2\big) + O\big(\sqrt{\Delta t}(\Delta x)^2\big) + O(\Delta t\,\Delta x)\,.
\end{aligned}
$$

The results then follow from Theorem 4.1. □

Furthermore, by relating Δt to Δx we obtain first-order convergence for the numerical scheme.

Corollary 4.1. *Under the assumptions of Theorem* 4.2*, if we let* $\Delta x = (\Delta t)^{3/2}$ *then we have the first-order error estimates*

$$\|\mu_{t_n}\|_\infty = O(\Delta t), \qquad \|\nu_{t_n}\|_\infty = O(\Delta t).$$

5. Conclusions

We have considered a fully discrete Euler scheme for solving decoupled forward backward stochastic differential equations. Under conditions slightly stronger than traditional assumptions, we proved the convergence rate is first-order. The technique used to obtain our error estimates is general, and may be extended to more complicated numerical schemes.

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References

[1] V. Bally, *Approximation scheme for solutions of BSDE*, in *"Backward Stochastic Differential Equations"* (Paris, 1995–1996), pp. 177–191, Pitman Res. Notes Math. Ser. **364**, Longman, Harlow (1997).

[2] C. Bender and R. Denk, *A forward scheme for backward SDEs*, Stochastic Process. Appl. **117**, 1793–1812 (2007).

[3] Y. Fu, W. Zhao and T. Zhou, *Multistep schemes for forward backward stochastic differential equations with jumps*, J. Sci. Comput. **69**, 651–672 (2016).

[4] E.R. Gianin, *Risk measures via g-expectations*, Insurance Math. Econom. **39**, 19–34 (2006).

[5] N. El Karoui, S. Peng and M.C. Quenez, *Backward stochastic differential equations in finance*, Math. Finance **7**, 1–71 (1997).

[6] Y. Li, J. Yang and W. Zhao, *Convergence error estimates of the Crank-Nicolson scheme for solving decoupled FBSDEs*, Sci. China Math. **60**, doi: 10.1007/s11425-016-0178-8 (2017).

[7] T. Kong, W. Zhao and T. Zhou, *Probabilistic high order numerical schemes for fully nonlinear parabolic PDEs*, Commun. Comput. Phys. **18**, 1482–1503 (2015).

[8] G. Mastroianni and G. Monegato, *Error estimates for Gauss-Laguerre and Gauss-Hermite quadrature formulas*, in *Approximation and Computation: A Festschrift in Honor of Walter Gautschi*. Eds. Zahar, R. V. M., pp. 421–434, Birkhäuser, Boston (1994).

[9] G.N. Milstein and M.V. Tretyakov, *Numerical algorithms for forward-backward stochastic differential equations*, SIAM J. Sci. Comput. **28**, 561–582 (2006).

[10] E. Pardoux and S. Peng, *Adapted solution of a backward stochastic differential equation*, Syst. Control Lett. **14**, 55–61 (1990).

[11] S. Peng, *A general stochastic maximum principle for optimal control problems*, SIAM J. Control Optim. **28**, 966–979 (1990).

[12] S. Peng, *Probabilistic interpretation for systems of quasilinear parabolic partial differential equations*, Stochastics Rep. **37**, 61–74 (1991).

[13] S. Peng, *Nonlinear expectations, nonlinear evaluations and risk measures*, in *Stochastic Methods in Finance: Lectures given at the C.I.M.E.-E.M.S. Summer School held in Bressanone/Brixen, Italy, July 6-12, 2003*, Springer Berlin Heidelberg, Berlin, Heidelberg, 165–253 (2004).

[14] W. Zhao, L. Chen and S. Peng, *A new kind of accurate numerical method for backward stochastic differential equations*, SIAM J. Sci. Comput. **28**, 1563–1581 (2006).

[15] W. Zhao, Y. Li and G. Zhang, *A generalized θ-scheme for solving backward stochastic differential equations*, Discrete and Continuous Dynamic Systems Series B **17**, 1585-1603 (2012).

[16] W. Zhao, J. Wang and S. Peng, *Error Estimates of the θ-scheme for backward stochastic differential equations*, Dis. Cont. Dyn. Sys. B **12**, 905–924 (2009).

[17] W. Zhao, G. Zhang and L. Ju, *A stable multistep scheme for solving backward stochastic differential equations*, SIAM J. Numer. Anal. **48**, 1369–1394 (2010).

[18] G. Zhang, M. Gunzburger and W. Zhao, *A sparse-grid method for multi-dimensional backward stochastic differential equations*, J. Comput. Math. **31**, 221–248 (2013).

[19] W. Zhao, Y. Fu and T. Zhou, *New kinds of high-order multistep schemes for coupled forward backward stochastic differential equations*, SIAM J. Sci. Comput. **36**, A1731-1751 (2014).

[20] W. Zhao, Y. Li and Y. Fu, *Second-order schemes for solving decoupled forward backward stochastic differential equations*, Science China Math. **57**, 665–686 (2014).

[21] W. Zhao, W. Zhang and L. Ju, *A numerical method and its error estimates for the decoupled forward-backward stochastic differential equations*, Commun. Comput. Phys. **15**, 618–646 (2014).