A GENERAL CLASS OF ONE-STEP APPROXIMATION FOR INDEX-1 STOCHASTIC DELAY-DIFFERENTIAL-ALGEBRAIC EQUATIONS∗

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Abstract

This paper develops a class of general one-step discretization methods for solving the index-1 stochastic delay differential-algebraic equations. The existence and uniqueness theorem of strong solutions of index-1 equations is given. A strong convergence criterion of the methods is derived, which is applicable to a series of one-step stochastic numerical methods. Some specific numerical methods, such as the Euler-Maruyama method, stochastic θ-methods, split-step θ-methods are proposed, and their strong convergence results are given. Numerical experiments further illustrate the theoretical results.

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Key words: Stochastic delay differential-algebraic equations, One-step discretization schemes, Strong convergence.

1. Introduction

Differential algebraic equations (DAEs) are often used to model some actual problems in science and technology, such as automatic control, electric circuits, multibody dynamics, computer aid design and so on. The numerical algorithms and their analysis play the key roles in the research of this kind of equations, and the related work refers to the monographs of Ascher & Petzold and Hairer & Wanner (cf. [3, 6]). Sometimes, however, the actual problems could be influenced on delay factor or stochastic perturbation (cf. [11, 25]). Hence, it is necessary to consider these impacts when one establishes a real model of DAEs. In this way, when a delay argument is presented, DAEs can be classed into the deterministic delay differential algebraic equations (DDDAEs) and stochastic delay differential-algebraic equations (SDDAEs).

A few methods have been proposed to solve DDDAEs numerically. For instance, Ascher and Petzold [2] studied retarded and neutral equations and presented a series of convergence results for linear multistep and Runge-Kutta methods. Hauber [7] analyzed convergence of the collocation methods for DDDAEs with state-dependent delay. Luzyanina and Roose [10] investigated the periodic solutions of semi-explicit DDDAEs and their collocation methods. Zhu and Petzold [26] presented some analytical and numerical stability criteria of Hessenberg-type DDDAEs, where multistep methods and Runge-Kutta methods were concerned. Moreover, the stability properties of numerical methods applied to delay differential equations without the algebraic restriction also refer to the paper [19, 23, 24] and the references therein.

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For stochastic differential algebraic equations without delay (SDDAEs), some research results have been presented. For example, Schein and Denk applied a two-step scheme to solve linear implicit SDAEs with additive noise in [16]. Penski [15] developed a numerical method with strong order one to compute a circuit simulation model of SDDAEs and analyzed the method’s mean-square stability. In [8,9], the authors proposed a class of stiffly accurate stochastic Runge-Kutta methods for nonlinear index-1 SDAEs with scalar noise and investigated their mean-square stability. Furthermore, some closely related works can be seen in the papers [1,17,20].

Compared with the studies for DDDAEs and SDAEs, the research on numerical methods for SDDAEs is still in their infancy. To our knowledge, for index-1 SDDAEs, Xiao and Zhang [21,22] derived some existence and uniqueness results and the Euler-Maruyama methods. Whereas, most of the existing stochastic numerical methods are only for stochastic delay differential equations (SDDEs) or stochastic differential equations (SDEs) without algebraic constrain, see, e.g., [4,5,12–14,18] and their references.

In this paper, we will deal with the index-1 SDDAEs with delay \( \tau > 0 \):

\[
\begin{align}
\frac{dx(t)}{dt} &= f(t, x(t), x(t - \tau), y(t), y(t - \tau))dt \\
+ &g(t, x(t), x(t - \tau), y(t), y(t - \tau))dW(t), \quad t \in [t_0, T], \tag{1.1a}
\end{align}
\]

\[
0 = u(t, x(t), x(t - \tau), y(t)), \quad t \in [t_0, T] \tag{1.1b}
\]

whose initial values \( x(t) = a(t) \) and \( y(t) = b(t) \) for \( t \leq t_0 \). The paper is organized as follows. In section 2, we make some basic definitions, and investigate existence and uniqueness of the strong solutions of SDDAEs (1.1). In section 3, we develop a class of general one-step discretization methods for solving SDDAEs (1.1) and derive the methods’ strong convergence criteria. In section 4, some specific numerical methods are proposed for SDDAEs and SDAEs, such as the Euler-Maruyama method, stochastic \( \theta \)-methods and split-step \( \theta \)-methods. We apply the obtained results to specific numerical methods and hence some new convergence results of the methods are given. Connection and comparison between the obtained results and the existed ones are given. Finally, with several numerical experiments, we further illustrate the theoretical results.

2. Existence and Uniqueness of Strong Solutions of Index-1 SDDAEs

To give a clear statement to the index-1 SDDAEs, we first introduce some notations. Let \((\Omega, \mathcal{F}, P)\) denote a complete probability space with a right-continuous filtration \(\{\mathcal{F}_t\}_{t \geq 0}\), in which each \(\mathcal{F}_t\) (\(t \geq 0\)) contains all P-null sets in \(\mathcal{F}\), and \(W(t) = (W_1(t), \ldots, W_d(t))^T\) be the \(d\)-dimensional standard Wiener process defined on space \((\Omega, \mathcal{F}, P)\). Throughout the paper, \(|\cdot|\) denotes the Euclid norm for a vector and the trace norm for a matrix. For an integrable random variable \(\xi\), we define

\[
E(\xi) := \int_{\Omega} \xi dP, \quad E_\mathcal{F}(\xi) := E(\xi|\mathcal{F}_t), \quad \|\xi\|_{L^p} := (E|\xi|^p)^{\frac{1}{p}}.
\]

Moreover, \(C(\mathcal{J}; \mathbb{R}^d)\) is the Banach space consisting of all continuous \(\mathbb{R}^d\)-valued functions \(\varphi\) defined on \(\mathcal{J}\) with the norm \(\|\varphi\| = \sup_{t \leq t_0} |\varphi(t)|\), \(L^p(\mathcal{J}; \mathbb{R}^d)\) the family of \(\mathbb{R}^d\)-valued \(\mathcal{F}_t\)-adapted processes \(\{f(t)\}_{t \in \mathcal{J}}\) such that \(\int_{t_0}^T |f(t)|^p dt < \infty\) holds almost surely (a.s. for short), and \(M^p(\mathcal{J}; \mathbb{R}^d)\) the family of processes \(\{f(t)\}_{t \in \mathcal{J}}\) in \(L^p(\mathcal{J}; \mathbb{R}^d)\) such that \(E \int_{t_0}^T |f(t)|^p dt < \infty\).
Consider the integral form of SDDAE (1.1) with constant delay $\tau > 0$:

$$
x(t) = x(t_0) + \int_{t_0}^{t} f(s, x(s), x(s - \tau), y(s), y(s - \tau))ds + \int_{t_0}^{t} g(s, x(s), x(s - \tau), y(s), y(s - \tau))dW(s), \quad t \in [t_0, T],
$$

(2.1a)

$$
0 = u(t, x(t), x(t - \tau), y(t)), \quad t \in [t_0, T],
$$

(2.1b)

whose initial values are given by

$$
x(t) = a(t), \quad y(t) = b(t), \quad \text{for} \quad t \leq t_0.
$$

(2.2)

Here functions $f : [t_0, T] \times R^m \times R^m \times R^a \times R^a \rightarrow R^m$, $g : [t_0, T] \times R^m \times R^m \times R^n \times R^n \rightarrow R^{m \times d}$ and $u : [t_0, T] \times R^m \times R^m \times R^a \rightarrow R^a$ are continuous, and called the drift, diffusion and constrain function, respectively. The second integral in (2.1a) is an Itô stochastic integral with respect to $W(t)$. The initial functions $a(t)$ and $b(t)$ are $\mathcal{F}_t$-measurable $R^m$- and $R^a$-valued continuous random variables, respectively, H"older continuous of order $\frac{1}{2}$, and satisfy that $E\|a(t)\|^2 \vee E\|b(t)\|^2 < \infty$ and the consistent condition

$$
u(t_0, a(t_0), a(t_0 - \tau), b(t_0)) = 0.
$$

(2.3)

**Definition 2.1.** The SDDAE (2.1) is said to be index-1 if the constrain equation (2.1b) can be globally uniquely solved for the variable $y(t)$.

Since the noise source does not appear in the algebraic equation (2.1b), the above concept of index-1 is consistent with the same-name concept of SDAEs in Winkler [20, p. 439]. In the following, we give the definition of strong solution of SDDAEs (2.1).

**Definition 2.2.** Let $x(t), y(t)$ be $R^m$-valued and $R^n$-valued stochastic processes, respectively. The pair $(x(t), y(t))$ is called a strong solution of initial value problem (IVP) (2.1)-(2.2) of SDDAE if it has the following properties:

1. $x(t)$ and $y(t)$ are continuous and $\mathcal{F}_t$-adapted;

2. $f \in L^1([t_0, T]; R^m)$ and $g \in L^2([t_0, T]; R^{m \times d})$;

3. $x(t)$ and $y(t)$ satisfy the consistent initial conditions (2.3), equation (2.1) holds a.s. for every $t \in [t_0, T]$, and equation (2.2) holds a.s. for every $t \leq t_0$.

Moreover, the solution $(x(t), y(t))$ is said to be unique if, for any other solution $(\tilde{x}(t), \tilde{y}(t))$ of (2.1)-(2.2), it holds that

$$
P\{x(t) = \tilde{x}(t) \text{ and } y(t) = \tilde{y}(t), \quad \text{for all } t \leq T\} = 1.
$$

For the subsequent analysis, we make some assumptions on algebraic constraint (2.1b).

**Assumption 2.1.** For all $t \in [t_0, T]$, $x, x_t \in R^m$ and $y \in R^n$, the matrix $\partial u(t, x, x_t, y)/\partial y$ is nonsingular, and there exist constants $M_1$, $M_2$ and $M_3$ such that

$$
\left|\frac{\partial u}{\partial y}\right|(t, x, x_t, y) \begin{bmatrix} 1 \end{bmatrix} \leq M_1, \quad \left|\frac{\partial u}{\partial x}\right|(t, x, x_t, y) \leq M_2, \quad \left|\frac{\partial u}{\partial x_t}\right|(t, x, x_t, y) \leq M_3.
$$
Under Assumption 2.1, according to the implicit function theorem, the algebraic constraint (2.1b) can be globally and uniquely solved for \( y \). This implies that the SDDAE is index-1. We write the solution
\[
y(t) = \hat{y}(t, x(t), x(t - \tau)).
\]

Furthermore, the following estimations hold:
\[
|\dot{y}_x(t, x, x_t)| = \left| -\frac{\partial u}{\partial y}(t, x, x_t, y) \right| \leq M_1 M_2,
\]
\[
|\dot{y}_{x_t}(t, x, x_t)| = \left| -\frac{\partial u}{\partial y}(t, x, x_t, y) \right| \leq M_1 M_3,
\]
where \( \dot{y}_x := \partial y/\partial x \) and \( \dot{y}_{x_t} := \partial y/\partial x_t \).

Substituting (2.4) into (2.1a) brings the following stochastic delay differential equations
\[
x(t) = x(t_0) + \int_{t_0}^{t} \tilde{f}(s, x(s), x(s - \tau), x(s - 2\tau))ds
+ \int_{t_0}^{t} \tilde{g}(s, x(s), x(s - \tau), x(s - 2\tau))dW(s), \quad t \in [t_0, T],
\]
where
\[
\tilde{f} := f(s, x(s), x(s - \tau), \hat{y}(s, x(s), x(s - \tau)), \dot{y}(s - \tau, x(s - \tau), x(s - 2\tau))),
\]
\[
\tilde{g} := g(s, x(s), x(s - \tau), \hat{y}(s, x(s), x(s - \tau)), \dot{y}(s - \tau, x(s - \tau), x(s - 2\tau))).
\]

With Eq. (2.7), an existence and uniqueness theorem of problem (2.1)-(2.2) can be obtained.

**Theorem 2.1.** Assume that there exist positive constants \( L_1 \) and \( L_2 \) such that
\[(a) \text{ for all } t \in [t_0, T], x, x_t, \bar{x}, \bar{x}_t \in \mathbb{R}^m \text{ and } y, y_t, \bar{y}, \bar{y}_t \in \mathbb{R}^n,
|f(t, x, x_t, y, y_t) - f(t, \bar{x}, \bar{x}_t, \bar{y}, \bar{y}_t)| \leq L_1 \left( |x - \bar{x}| + |x_t - \bar{x}_t| + |y - \bar{y}| + |y_t - \bar{y}_t| \right); \tag{2.8}
\]
\[(b) \text{ for all } t \in [t_0, T], x, x_t \in \mathbb{R}^m \text{ and } y, y_t \in \mathbb{R}^n,
|f(t, x, x_t, y, y_t)| \leq L_2 \left( 1 + |x| + |x_t| + |y| + |y_t| \right). \tag{2.9}
\]

Then, under Assumption 2.1, a unique strong solution \((x(t), y(t))\) of problem (2.1)-(2.2) exists, and \( x(t) \) and \( y(t) \) belong to \( \mathcal{M}^2([t_0, T]; \mathbb{R}^m) \) and \( \mathcal{M}^2([t_0, T]; \mathbb{R}^n) \), respectively. Moreover, the solution processes \( x(t) \) and \( y(t) \) are square-integrable and fulfill
\[
E \left( \sup_{t_0 - \tau \leq t \leq T} |x(t)|^2 \right) \leq \tilde{c}_1,
\]
\[
E |x(t) - x(s)|^2 \leq \tilde{c}_2(t - s), \quad \forall \ t_0 \leq s \leq t \leq T, \tag{2.11}
\]
with bounded constants \( \tilde{c}_1 \) and \( \tilde{c}_2 \).
Proof. Since SDDAE (2.1) is equivalent to SDDE (2.7) under the Assumption 2.1, according to the theory of existence, uniqueness and estimate the moments for strong solution of SDDE (see Theorem 5.2.2 and Lemma 5.5.2 in [11]), it suffices to prove for equation (2.7) that there exist positive constants $c_1$ and $c_2$ such that the following uniform Lipschitz condition (a') and linear growth condition (b') are fulfilled:

(a') for all $t \in [t_0, T]$ and $\xi, \xi', \zeta, \eta, \bar{\eta} \in R^m$,

$$|\tilde{f}(t, \xi, \zeta, \eta) - \tilde{f}(t, \xi', \zeta', \eta')| \leq c_1 \left(|\xi - \xi'| + |\zeta - \zeta'| + |\eta - \eta'|\right); \quad (2.12)$$

(b') for all $t \in [t_0, T]$ and $\xi, \zeta, \eta \in R^m$,

$$|\tilde{f}(t, \xi, \zeta, \eta)| \leq c_2 \left(1 + |\xi| + |\zeta| + |\eta|\right). \quad (2.13)$$

Firstly, we give a property of algebraic variable $g(t)$, which will play a key role in the proof of (a') and (b') and the subsequent analysis. Under the Assumption 2.1, it follows from the well-known mean value theorem for derivative and (2.4)-(2.6) that

$$|\dot{g}(t, \xi, \zeta, \eta)| \leq M_1 M_2 |\xi - \tilde{\xi}| + M_1 M_3 |\zeta - \tilde{\zeta}|. \quad (2.14)$$

Then, by (2.8), (2.14) and the definition of function $\tilde{f}$, we have

$$|\tilde{f}(t, \xi, \zeta, \eta)| = |\tilde{f}(t, \xi, \zeta, \eta) - \tilde{f}(t, \xi, \zeta, \eta)|$$

$$\leq L_1 \left(|\xi - \tilde{\xi}| + |\zeta - \tilde{\zeta}| + |\eta - \bar{\eta}|\right) \leq c_1 \left(|\xi - \tilde{\xi}| + |\zeta - \tilde{\zeta}| + |\eta - \bar{\eta}|\right), \quad (2.15)$$

where $c_1 = L_1(1 + M_1 M_2 + M_1 M_3)$. Since $\tilde{f}$ is continuous with respect to $t$, (2.15) implies

$$|\tilde{f}(t, \xi, \zeta, \eta)| \leq |\tilde{f}(t, \xi, \zeta, \eta) - \tilde{f}(t, 0, 0, 0)| + |\tilde{f}(t, 0, 0, 0)|$$

$$\leq c_1 \left(|\xi| + |\zeta| + |\eta|\right) + c_2 \leq c_3 \left(1 + |\xi| + |\zeta| + |\eta|\right),$$

where $c_2$ is a bounded constant and $c_3 = \max(c_1, c_2)$. This completes the proof for $\tilde{f}$ part in condition (a') and (b'). Similarly, that for $\tilde{g}$ part in condition (a') and (b') can be proved. Hence the proof is now complete. \hfill $\Box$

Remark 2.1. Applying Theorem 2.1 to SDAE without delay

$$\begin{cases} dx(t) = f(t, x(t), g(t))dt + g(t, x(t), y(t))dW(t), \\ 0 = u(t, x(t), y(t)), \end{cases} \quad (2.16)$$

with initial values $x(t_0) = x_0$ and $y(t_0) = y_0$, a special existence and uniqueness result for the strong solution of the semi-explicit index-1 SDAE (2.16) can be followed, which is in accordance with the corresponding conclusion in Winkler [20].

Remark 2.2. Theorem 2 in [21] can be derived immediately from Theorem 2.1 when $y(t - \tau)$ does not appear in (2.1a).
3. One-Step Discretization Schemes and Their Strong Convergence

In this section, for simplicity, we will focus on the one-dimensional autonomous problems of index-1 SDDEs
\[
\begin{align*}
&dx(t) = f(x(t), x(t - \tau), y(t), y(t - \tau))dt + g(x(t), x(t - \tau), y(t), y(t - \tau))dW(t), \quad t \in [t_0, T], \quad (3.1a) \\
&0 = u(x(t), x(t - \tau), y(t)), \quad t \in [t_0, T]. \quad (3.1b)
\end{align*}
\]

Generally speaking, it is virtually impossible to find the analytical closed-form solution of such equations. Therefore, it becomes significant to seek the numerical solutions. For the numerical treatment of SDDEs, one-step methods, especially drift-implicit one-step methods are proven to be effective and easy to implement (see, e.g., [4,20] and references therein). This motivates us to extend the existing one-step methods for SDEs to solve SDDEs (3.1).

Take a uniform mesh \( \{t_k : t_k = t_0 + kh, \ h = (T - t_0)/(N) \}_{k=0}^N \) on \([t_0, T]\) and assume that there exist integer \( l \) and \( \delta \in (0,1) \) such that \( \tau = (l - \delta)h \), then a class of one-step discretization schemes for (3.1) can be defined as follows:

\[
\begin{align*}
X_{k+1} &= X_k + \phi(X_{k+1}, \bar{X}_{k+1-\tau}, Y_{k+1}, \bar{Y}_{k+1-\tau}, X_k, \bar{X}_{k-\tau}, Y_k, \bar{Y}_{k-\tau}, h) \\
&\quad + \psi(X_k, \bar{X}_{k-\tau}, Y_k, \bar{Y}_{k-\tau}, I_{t_k}^{t+1}), \quad k = 0, \ldots, N - 1, \quad (3.2a) \\
0 &= u(X_{k+1}, \bar{X}_{k+1-\tau}, Y_{k+1}), \quad k = 0, \ldots, N - 1, \quad (3.2b)
\end{align*}
\]

where \( X_k, Y_k, \bar{X}_{k-\tau} \) and \( \bar{Y}_{k-\tau} \) are approximations to \( x(t_k), y(t_k), x(t_k - \tau) \) and \( y(t_k - \tau) \), respectively, and \( I_{t_k}^{t+1} \) denotes the collection of multiple Wiener integrals of form

\[
I_{(i_0, i_1, \ldots, i_j)} = \int_{t_k}^{t_k+h} \int_{t_k}^{s_1} \cdots \int_{t_k}^{s_j} dW_{t_0}^{i_0}(s_0) \cdots dW_{t_0}^{i_{j-1}}(s_{j-1}) dW_{t_0}^{i_j}(s_j),
\]

in which \( i_p \in \{0,1\}, \ p = 0,1,\ldots,j \) and \( dW^0(t) = dt \). When \( t_k - \tau \leq 0 \), let \( \bar{X}_{k-\tau} = a(t_k - \tau) \) and \( \bar{Y}_{k-\tau} = b(t_k - \tau) \). Otherwise, The argument \( \bar{X}_{k-\tau} \) and \( \bar{Y}_{k-\tau} \) are obtained by a suitable interpolation \( \mathcal{L} \) with some known values \( X_i \) (0 \leq i \leq k) and the initial data. In general, the used interpolation is asked to satisfy

\[
\begin{align*}
&\| \mathcal{L} x(t_k - \tau) - \mathcal{L} X_{k-\tau} \|_{L_2} \leq c \max_{0 \leq i \leq k} \| x(t_i) - X_i \|_{L_2}, \\
&\| \mathcal{L} y(t_k - \tau) - \mathcal{L} Y_{k-\tau} \|_{L_2} \leq c \max_{0 \leq i \leq k} \| y(t_i) - Y_i \|_{L_2},
\end{align*}
\]

where \( c > 0 \) is a given constant. It is obvious that the linear interpolation satisfies these conditions. Moreover, when an extension of method (3.2) is used as interpolation, these conditions also can be guaranteed under the assumption that functions \( f, g, \phi, \psi \) and their derivatives satisfy the Lipschitz condition. The method (3.2) and the interpolation \( \mathcal{L} \) are required to generate iterates \( X_k \) and \( Y_k \) which are \( \mathcal{A}_{t_k} \)-measurable. Through this paper, we will always assume the above conditions hold.

In the following, we will deal with the convergence of methods (3.2). For this, we need to introduce some concepts, such as local error, consistency and strong mean square convergence, which are some extensions to the corresponding concepts of numerical methods for SDEs (see, e.g., [4,20]). Moreover, the following abbreviated notations will be used for simplicity of the presentation:

\[
x_k = x(t_k), \quad y_k = y(t_k), \quad x_{k-\tau} = x(t_{k-\tau}), \quad y_{k-\tau} = y(t_{k-\tau}).
\]
Definition 3.1. The one-step method (3.2) is called mean-square consistent of order \( p \), if its local errors
\[
\delta_k := x_{k+1} - \left[ x_k + \phi(x_{k+1}, x_{k+1}, y_{k+1}, y_{k+1}, x_k, x_k, y_k, y_k, h) + \psi(x_k, x_k, y_k, y_k, y_k, x_k, y_k, y_k, h) \right], \quad k = 0, \ldots, N - 1
\] (3.3)
satisfy the estimates:
\[
\max_{0 \leq k \leq N-1} \| E_{tk} (\delta_k) \|_{L^2} \leq C_1 h^{p+1}, \quad \text{as} \; h \to 0, \quad (3.4)
\]
\[
\max_{0 \leq k \leq N-1} \| \delta_k \|_{L^2} \leq C_2 h^{p+1}, \quad \text{as} \; h \to 0, \quad (3.5)
\]
where \( C_1, C_2 > 0 \) are constants independent of stepsize \( h \).

Definition 3.2. An interpolation \( \mathbb{L} \) has local error order \( q \), if there exists a constant \( c > 0 \) independent of stepsize \( h \) such that
\[
\| x(t_k - \tau) - \mathbb{L} x(t_k - \tau) \|_{L^2} \lor \| y(t_k - \tau) - \mathbb{L} y(t_k - \tau) \|_{L^2} \leq ch^q, \quad \text{as} \; h \to 0. \quad (3.6)
\]

Definition 3.3. The one-step method (3.2) is called mean-square convergent of order \( p \), if its global errors: \( \varepsilon_k = x_k - X_k \) and \( \varepsilon_k = y_k - Y_k \) satisfy
\[
\max_{0 \leq i \leq N} \left( \| \varepsilon_i \|_{L^2} + \| e_i \|_{L^2} \right) \leq C h^p, \quad \text{as} \; h \to 0. \quad (3.7)
\]
where \( C > 0 \) is a constant independent of the stepsize \( h \).

Note that the global errors \( \varepsilon_k \) and \( e_k \) are \( \mathcal{A}_t \)-measurable, since both \( X_k \) and \( Y_k \) are \( \mathcal{A}_t \)-measurable. For the subsequent convergence analysis, we always let functions \( \phi \) and \( \psi \) in method (3.2) satisfy the following assumption.

Assumption 3.1. Assume that there exist positive constants \( \alpha, \beta \) and \( \gamma \) such that functions \( \phi \) and \( \psi \) in (3.2a) satisfy:

(A1) For every \( t \in [0, T] \), \( x, x_t, \xi, \eta, \xi_t, x_t, x_t, x_t, \xi_t, \xi_t, \eta_t, y_t, y_t, \eta_t \in \mathbb{R}^m \) and \( \xi, \eta, \xi_t, \xi_t, \eta_t, x_t, x_t, \xi_t, \xi_t, \eta_t, h \),
\[
\left| \phi(\xi, \xi_t, \xi_t, x_t, x_t, y_t, y_t, h) - \phi(\xi, \xi_t, \xi_t, x_t, x_t, y_t, y_t, h) \right| \leq \alpha h \left( |\xi - \xi_t| + |\xi - \eta_t| + |\eta - \xi_t| + |x_t - x_t| + |y_t - y_t| \right), \quad (3.8)
\]

(A2) For every \( t \in [0, T] \) and all \( \mathcal{A}_t \)-measurable random variables \( x, x_t, x_t, x_t, y_t, y_t, \eta_t \) and \( y_t \),
\[
\left| E_t \left( \psi(x, x_t, y_t, y_t, \eta_t^{t+h}) - \psi(x, \xi, y_t, y_t, \eta_t^{t+h}) \right) \right| \leq \beta h \left( |x_t - \xi_t| + |y_t - \eta_t| \right), \quad (3.9)
\]
\[
E_t \left( \psi(x, x_t, y_t, y_t, \eta_t^{t+h}) - \psi(x, \xi_t, y_t, y_t, \eta_t^{t+h}) \right)^2 \leq \gamma h \left( E_t |x_t - \xi_t|^2 + E_t |x_t - \xi_t|^2 + E_t |y_t - \eta_t|^2 \right). \quad (3.10)
\]

A convergence result of method (3.2) can be stated as follows.

Theorem 3.1. Suppose that the IVP (3.1) satisfies conditions (a)-(b) and Assumption 2.1, the one-step discretization method (3.2) satisfies Assumption 3.1 and is mean-square consistent of order \( p \), and the used interpolation \( \mathbb{L} \) has local error order \( q \). Then the method (3.2) is mean-square convergent of order \( \min(p, q) \) for the IVP (3.1).
Proof. It follows from Assumption 2.1, (3.1b), (3.2b), (2.4), (2.5), (2.6) and the mean value theorem for derivative that

\[ |e_k| = |y_k - Y_k| = |\hat{y}(x_k, x_{k-\tau}) - \hat{y}(X_k, \bar{X}_{k-\tau})| \leq M_1 M_2 |\delta_k| + M_1 M_3 |x_{k-\tau} - \bar{X}_{k-\tau}|. \]  
(3.11)

Also, according to the known condition that the interpolation \( \mathcal{L} \) has local error order \( q \) and the Minkowski inequality, we have

\[
\begin{align*}
\|x_{k-\tau} - \bar{X}_{k-\tau}\|_{L^2} & \leq \|x(t_k - \tau) - \mathcal{L}x(t_k - \tau)\|_{L^2} + \|\mathcal{L}x(t_k - \tau) - \mathcal{L}X_{k-\tau}\|_{L^2} \\
& \leq ch^q + c \max_{0 \leq i \leq k} \|\epsilon_i\|_{L^2},
\end{align*}
\]  
(3.12)

here and in what follows \( c > 0 \) refers to an unspecified constant, which depends only on the used interpolation \( \mathcal{L} \). Similarly, it can be verified that

\[
\|y_{k-\tau} - \hat{Y}_{k-\tau}\|_{L^2} \leq ch^q + c \max_{0 \leq i \leq k} \|\epsilon_i\|_{L^2}.
\]  
(3.13)

A combination of (3.11) and (3.12) gives

\[
\|e_k\|_{L^2} \leq \left( M_1 M_2 + M_1 M_3 c \right) \max_{0 \leq i \leq k} \|\epsilon_i\|_{L^2} + M_1 M_3 c h^q.
\]  
(3.14)

In the following, we need to estimate \( \max_{0 \leq i \leq N} \|\epsilon_i\|_{L^2} \). For this, we assume that \( 0 < h < 1 \) and denote by \( \langle \cdot, \cdot \rangle \) the standard inner product. The inequality \( 2(x, y) \leq |x|^2 + |y|^2 \), the Hölder inequality, the Minkovski inequality and some common properties of conditional expectation (see, e.g., [11]) will be used frequently. Let

\[
\begin{align*}
\delta \phi_k &= \phi(x_{k+1}, x_{k+1-\tau}, y_{k+1}, y_{k+1-\tau}, x_k, x_{k-\tau}, y_k, y_{k-\tau}, h) \\
&\quad - \phi(X_{k+1}, \bar{X}_{k+1-\tau}, Y_{k+1}, \bar{Y}_{k+1-\tau}, X_k, \bar{X}_{k-\tau}, Y_k, \bar{Y}_{k-\tau}, h), \\
\delta \psi_k &= \psi(x_k, x_{k-\tau}, y_k, y_{k-\tau}, \mu_{t_k}^{\tilde{h}_k+1}) - \psi(X_k, \bar{X}_{k-\tau}, Y_k, \bar{Y}_{k-\tau}, \mu_{t_k}^{\tilde{h}_k+1}), \quad \Omega_k := \delta \phi_k + \delta \psi_k.
\end{align*}
\]

With these notations, we have

\[ e_{k+1} = x_{k+1} - X_{k+1} = \epsilon_k + \delta_k + \Omega_k, \]

which implies

\[
|e_{k+1}|^2 = |\epsilon_k|^2 + |\delta_k|^2 + |\Omega_k|^2 + 2\langle \epsilon_k, \delta_k \rangle + 2\langle \epsilon_k, \Omega_k \rangle + 2\langle \delta_k, \Omega_k \rangle \leq |\epsilon_k|^2 + 2|\delta_k|^2 + 2|\Omega_k|^2 + 2\langle \epsilon_k, \delta_k \rangle + 2\langle \epsilon_k, \Omega_k \rangle + 2\langle \delta_k, \Omega_k \rangle.
\]  
(3.15)

Taking expectation and taking the modules on (3.15) yields

\[
E|e_{k+1}|^2 \leq E|\epsilon_k|^2 + 2E|\delta_k|^2 + 2E|\Omega_k|^2 + 2E\langle \epsilon_k, \delta_k \rangle + 2E\langle \epsilon_k, \Omega_k \rangle + 2E\langle \delta_k, \Omega_k \rangle.
\]  
(3.16)

In what follows, we will estimate the various terms in the right side of inequality (3.16). The derived estimations will hold almost everywhere. From (3.5) we have

\[
2E|\delta_k|^2 = 2E(E_{t_k}|\delta_k|^2) \leq 2C_2^2h^{2p+1}.
\]  
(3.17)
Using Assumption 3.1 gives
\[2E[\Omega_k^2] = 2E[\delta \phi_k + \delta \psi_k]^2 \leq 4E[\delta \phi_k]^2 + 4E[\delta \psi_k]^2\]
\[\leq 32\alpha^2 L^2 E\left(\sum_{i=1}^{\infty} \varepsilon_{k+1}^2 + x_{k+1}^2 - \bar{X}_{k+1}^2 + \varepsilon_{k+1}^2 + |y_{k+1} - \bar{Y}_{k+1}|^2\right)\]
\[+ 4\gamma h E\left(\sum_{i=1}^{\infty} \varepsilon_{k}^2 + x_{k}^2 - \bar{X}_{k}^2 + \varepsilon_{k}^2 + |y_{k} - \bar{Y}_{k}|^2\right)\]
\[\leq 2 E[\Omega_k^2] \leq 2c_1 E[\varepsilon_{k+1}^2] + c_2 h^{2q}, \quad (3.18)\]
Moreover, from (3.12) and (3.13)-(3.14) we obtain, respectively, that
\[E[x_{k+1} - \bar{X}_{k+1}]^2 \leq 2c_1 h^{2q} + 2c_2^2 \max_{0 \leq \tau \leq k} E[\varepsilon_k]^2, \quad (3.19)\]
\[E[y_{k+1} - \bar{Y}_{k+1}]^2 \leq 2(\gamma h)^2 + 2c_2^2 \max_{0 \leq \tau \leq k} E[\varepsilon_k]^2 \]
\[\leq 2(\gamma h)^2 + 4c_1^2 \left(\sum_{i=1}^{\infty} M_i M_2 + M_1 M_3 \alpha \varepsilon_{k+1} \right) \max_{0 \leq \tau \leq k} E[\varepsilon_k]^2 + (M_1 M_3 c h^2)^2\]
\[\leq c_4 \max_{0 \leq \tau \leq k} E[\varepsilon_k]^2 + c_5 h^{2q}, \quad (3.20)\]
where \( c_4 > 0 \) is a constant dependent only on \( M_1, M_2, M_3 \) and the interpolation \( \mathcal{L} \). Substituting (3.19) and (3.20) into (3.18) follows that
\[2E[\Omega_k^2] \leq c_6 h E[\varepsilon_{k+1}^2] + c_7 h^{2q+1}, \quad (3.21)\]
where \( c_5 > 0 \) is a constant depending only on \( M_1, M_2, M_3, \alpha, \gamma \) and the interpolation \( \mathcal{L} \). Also, in the light of the condition that the method (3.2) has mean-square consistent order \( p \), it holds that
\[2E[\varepsilon_k, \delta_k] = 2E(E_t \langle \varepsilon_k, \delta_k \rangle) = 2E(E_t \langle \varepsilon_k, \varepsilon_k \rangle) \leq 2E[\langle \varepsilon_k, \varepsilon_k \rangle] + 2E[\langle \varepsilon_k, \varepsilon_k \rangle] \]
\[\leq 2E[\varepsilon_k, \varepsilon_k] h E[\varepsilon_k] + \frac{1}{2} E[\varepsilon_k, \varepsilon_k] h E[\varepsilon_k] \leq h E[\varepsilon_k]^2 + C_1^2 h^{2p+1}, \quad (3.22)\]
By Assumption 3.1 we have
\[2E[\varepsilon_k, \Omega_k] = 2E(E_t \langle \varepsilon_k, \Omega_k \rangle) = 2E(E_t \langle \varepsilon_k, \varepsilon_k \rangle + \langle \varepsilon_k, \varepsilon_k \rangle) \]
\[\leq 2E(\langle \varepsilon_k, \varepsilon_k \rangle + \langle \varepsilon_k, \varepsilon_k \rangle) + 2E[\langle \varepsilon_k, \varepsilon_k \rangle] + 2E[\langle \varepsilon_k, \varepsilon_k \rangle] \]
\[\leq 2(\alpha + \beta) h E[\varepsilon_k, \varepsilon_k] + x_{k+1} - \bar{X}_{k+1} + |\varepsilon_k| + |y_{k+1} - \bar{Y}_{k+1}| \]
\[+ |y_{k+1} - \bar{Y}_{k+1}| + \varepsilon_{k+1} + x_{k+1} - \bar{X}_{k+1} + |\varepsilon_k| + |y_{k+1} - \bar{Y}_{k+1}|] \]
\[\leq (\alpha + \beta) h E\left(\sum_{i=1}^{\infty} \varepsilon_k^2 + x_{k} - \bar{X}_{k} + \varepsilon_k^2 + |y_{k} - \bar{Y}_{k}|^2 \right) \]
\[\leq (\alpha + \beta) h E\left(\sum_{i=1}^{\infty} \varepsilon_k^2 + x_{k} - \bar{X}_{k} + \varepsilon_k^2 + |y_{k} - \bar{Y}_{k}|^2 \right) \]
This, together with (3.14), (3.19) and (3.20), gives
\[2E[\varepsilon_k, \Omega_k] \leq c_9 h E[\varepsilon_{k+1}^2] + c_9 h \max_{0 \leq \tau \leq k} E[\varepsilon_k]^2 + c_9 h^{2q+1}, \quad (3.23)\]
where \( c_9 > 0 \) is a constant depending only on \( M_1, M_2, M_3, \alpha, \beta \) and the interpolation \( \mathcal{L} \).
A combination of (3.16), (3.17) and (3.21)–(3.23) derives the following estimate
\[
E[|\xi_{k+1}|^2] \leq (c_5 + c_6) h E[|\xi_k|^2] + E[|\xi_k|^2] + (1 + c_5 + c_6) h \max_{0 \leq i \leq k} E[|\xi_i|^2] + (C_1^2 + 2C_2^2) h^{2p+1} + (c_5 + c_6) h^{2q+1}
\]
\[
\leq c_7 h E[|\xi_{k+1}|^2] + (1 + c_8) h \max_{0 \leq i \leq k} E[|\xi_i|^2] + c_9 h^{2\min(p,q)+1},
\]
where \(c_7 = c_5 + c_6, c_8 = 1 + c_7\) and \(c_9 = C_1^2 + 2C_2^2 + c_7\). Selecting a positive number \(h_0: 0 < h_0 < 1/c_7\), when \(0 < h < h_0\), we infer from (3.24) that
\[
E[|\xi_{k+1}|^2] \leq \frac{1 + c_8 h}{1 - c_7 h} \max_{0 \leq i \leq k} E[|\xi_i|^2] + \frac{c_9}{1 - c_7 h} h^{2\min(p,q)+1}.
\]
Denoting \(S_0 = 0\) and \(S_k := \max_{0 \leq i \leq k} E[|\xi_i|^2]\), and noticing that
\[
\frac{1 + c_8 h}{1 - c_7 h} \leq 1 + c_{10} h \quad \text{and} \quad \frac{c_9}{1 - c_7 h} \leq c_{11},
\]
where \(c_{10} = \frac{c_7 + c_8}{1 - c_7 h_0}\) and \(c_{11} = \frac{c_9}{1 - c_7 h_0}\), an induction to (3.25) gives that
\[
S_{k+1} \leq (1 + c_{10} h) S_k + c_{11} h^{2\min(p,q)+1}
\]
\[
\leq (1 + c_{10} h) \left[ (1 + c_{10} h) S_{k-1} + c_{11} h^{2\min(p,q)+1} \right] + c_{11} h^{2\min(p,q)+1}
\]
\[
\leq \ldots
\]
\[
\leq (1 + c_{10} h)^{k+1} S_0 + c_{11} h^{2\min(p,q)+1} \sum_{j=0}^{k} (1 + c_{10} h)^j
\]
\[
= c_{11} h^{2\min(p,q)+1} \left( \frac{1 + c_{10} h}{c_{10} h} \right)^{k+1} - 1 \right) h^{2\min(p,q)}
\]
\[
\leq \frac{c_{11}}{c_{10}} \left( e^{c_{10}(T-t_0)} - 1 \right) h^{2\min(p,q)}, \quad k = 0, \ldots, N.
\]
Thus, noting the fact that \((k + 1) h \leq T - t_0\), it follows that
\[
S_{k+1} \leq \frac{c_{11}}{c_{10}} \left( e^{c_{10}(T-t_0)} - 1 \right) h^{2\min(p,q)}, \quad k = 0, \ldots, N.
\]
Further, by taking the square-root, we obtain
\[
\max_{0 \leq i \leq N+1} \|\xi_i\|_{L^2} \leq \sqrt{\frac{c_{11}}{c_{10}} \left( e^{c_{10}(T-t_0)} - 1 \right) h^{2\min(p,q)}}.
\]
Combining (3.26) and (3.14), the conclusion is obtained immediately. This completes the proof. □

4. An Analysis for Some Special Numerical Methods

In this section, we extend some numerical methods of SDEs or SDDEs to solve SDDAEs, and then analyze their error behavior with Theorem 3.1.
Example 4.1. Semi-implicit Euler methods for SDDAEs

We introduce a class of semi-implicit Euler methods for solving IVP for index-1 SDDAE of retarded type (3.1):

\[
X_{k+1} = X_k + h\left[\theta f(X_{k+1}, \bar{X}_{k+1-\tau}, Y_{k+1}, \bar{Y}_{k+1-\tau}) + (1-\theta) f(X_k, \bar{X}_{k-\tau}, Y_k, \bar{Y}_{k-\tau})\right] \\
+ g(X_k, \bar{X}_{k-\tau}, Y_k, \bar{Y}_{k-\tau})\Delta W_k,
\]

\[0 = u(X_{k+1}, \bar{X}_{k+1-\tau}, Y_{k+1}),\]

where

\[
\Delta W_k := I(1) = \int_{t_k}^{t_{k+h}} dW(s) = W(t_{k+1}) - W(t_k),
\]

denoting independent \(N(0, h)\)-distributed Gaussian random variables and \(\theta \in [0, 1], k = 0, \ldots, N-1\). The arguments \(\bar{X}_{k-\tau}\) and \(\bar{Y}_{k-\tau}\) are obtained by linear interpolation at points \(t_{k-l}\) and \(t_{k-l+1}\), i.e.,

\[
\bar{X}_{k-\tau} = \mathcal{L} X(t_k) = \begin{cases} \delta X_{k-l+1} + (1-\delta)X_{k-l}, & \text{if } t_{k-l} > t_0, \\ a(t_k - \tau), & \text{if } t_{k-l} \leq t_0, \end{cases}
\]

\[
\bar{Y}_{k-\tau} = \mathcal{L} Y(t_k) = \begin{cases} \delta Y_{k-l+1} + (1-\delta)Y_{k-l}, & \text{if } t_{k-l} > t_0, \\ b(t_k - \tau), & \text{if } t_{k-l} \leq t_0. \end{cases}
\]

The starting values on mesh points \(t_k \leq t_0\) are given as \(X_k = a(t_k)\) and \(Y_k = b(t_k)\).

The linear interpolation we used is accurate to \(O(h^{\frac{1}{2}})\).

Lemma 4.1. The linear interpolations (4.2) and (4.3) have local error of order \(\frac{1}{2}\).

Proof. The result is obviously true when \(t_{k-l} \leq t_0\). If \(t_{k-l} > t_0\), since \(\tau = (l+\delta)h\) with integer \(l\) and \(\delta \in [0, 1]\), we deduce from (2.11) that

\[
E|x(t_k-\tau) - \mathcal{L} x(t_k-\tau)|^2 \\
= E|\theta f(x(t_{k-l+1}) + (1-\delta)X_{k-l}) + (1-\theta) f(x(t_{k-l}), Y_{k-l})|\]

\[\leq 2\delta^2 E|x(t_{k-l}) - x(t_{k-l+1})|^2 + 2(1-\delta)^2 E|x(t_k-\tau) - x(t_{k-l})|^2 \\
\leq 2c_2|\delta - 1|h + 2(1-\delta)^2c_2|h| = 2c(1-\delta)c_2h,
\]

which implies \(\|x(t_k-\tau) - \mathcal{L} x(t_k-\tau)\|_{L^2} \leq c^{\frac{1}{2}}\) with constant \(c = \sqrt{2c_1c_2}\) does not depend on stepsize \(h\). Similarly, it is clear that \(\|y(t_k-\tau) - \mathcal{L} y(t_k-\tau)\|_{L^2} \leq c^{\frac{1}{2}}\). This completes the proof.

\[\square\]

Theorem 4.1. Suppose that for IVP for index-1 SDDAE of retarded type (3.1), condition (a) and (b) in Theorem 2.1 and Assumption 2.1 hold with constants \(L_1, L_2, M_1, M_2\) and \(M_3\). Then semi-implicit Euler method (4.1) for solving (3.1) is mean-square convergent with order \(\frac{1}{2}\).

Proof. First, since \(\Delta W_k\) are independent \(N(0, h)\)-distributed Gaussian random variables with the mean

\[
E(\Delta W_k) = 0
\]

and the variance

\[
E[\Delta W_k]^2 = h,
\]
satisfies the estimates (3.8) and (3.10) in Assumption 3.1 hold with constants $\alpha = L_1 \cdot \max(\theta, 1 - \theta)$ and $\beta = 4L_1$. Moreover, (3.9) holds since the diffusion term is discretized explicitly.

In what follows, we will investigate the mean-square consistency of method (4.1). The local error of method (4.1)

$$
\delta_k = x_{k+1} - \left( x_k + h\theta f(x_{k+1}, x_{k+1-\tau} - \tau, y_{k+1}, y_{k+1-\tau}) + g(x_k, x_{-\tau}, y_k, y_{-\tau}) \Delta W_k \right),
$$

where $k = 0, \ldots, N - 1$. (4.7)

For $t \in [t_0, T]$, consider the inherent SDDE of SDDAE (3.1)

$$
x(t) = x(t_0) + \int_{t_0}^{t} \tilde{f}(x(s), x(s-\tau), x(s-2\tau))ds + \int_{t_0}^{t} \tilde{g}(x(s), x(s-\tau), x(s-2\tau))dW(s),
$$

where

$$
\tilde{f} := f(x(s), x(s-\tau), \tilde{y}(x(s), x(s-\tau), \tilde{y}(x(s), x(s-2\tau)))
$$

and

$$
\tilde{g} := g(x(s), x(s-\tau), \tilde{y}(x(s), x(s-\tau), \tilde{y}(x(s), x(s-2\tau))).
$$

Employing semi-implicit Euler method for SDDEs (cf. [18]) to (4.8) we have the following discretization scheme

$$
\tilde{X}_{k+1} = \tilde{X}_k + h[\tilde{f}(\tilde{X}_{k+1}, \tilde{X}_{k-1}, \tilde{X}_{k-2}) + \tilde{g}(\tilde{X}_k, \tilde{X}_{k-1}, \tilde{X}_{k-2}) \Delta W_k],
$$

where $\tau = lh$ with integer $l$. It has been clarified in the proof of Theorem 2.1 that functions $\tilde{f}$ and $\tilde{g}$ satisfy the uniform Lipschitz condition (2.12) and linear growth condition (2.13). Thus, the proof of Theorem 2 in [18] shows that method (4.11) is mean square consistent, i.e., the local error of method (4.11)

$$
\tilde{\delta}_k = x_{k+1} - \left( x_k + h[\tilde{f}(x_{k+1}, x_{k+1-\tau}, x_{k+1-2\tau}) + \tilde{g}(x_k, x_{-\tau}, x_{-2\tau}) \Delta W_k] \right)
$$

satisfies the estimates

$$
\max_{0 \leq k \leq N-1} \|E_{t_k}(\tilde{\delta}_k)\|_{L^2} \leq C_1 h^{\frac{1}{2}} \quad \text{as } h \to 0,
$$

and

$$
\max_{0 \leq k \leq N-1} \|\tilde{\delta}_k\|_{L^2} \leq C_2 h \quad \text{as } h \to 0
$$

with constants $C_1$ and $C_2$ which do not depend on stepsize $h$. Note that the mean square consistency does not restrict that delay $\tau$ must be integral multiple of stepsize $h$. Hence, from (2.4), (3.1b), (4.7), (4.9), (4.10) and (4.12) we derive $\delta_k = \delta_k$, which together with (4.13) and (4.14) implies that method (4.1) is mean square consistent with order $\frac{1}{2}$. Moreover, Lemma 4.1 shows that the linear interpolation for approximating the delay terms is accurate to $O(h^{\frac{1}{2}})$. Therefore, an application of Theorem 3.1 yields the desired result.

Remark 4.1. When $\theta = 0$, we call method (4.1) explicit Euler method. Theorem 2 in [22] can be derived immediately.
Remark 4.2. When \( \theta = 1 \) (resp. \( \theta = \frac{1}{2} \)), we call method (4.1) backward Euler method (resp. trapezoidal method). It can be viewed as the analogue of drift-implicit Euler method (resp. trapezoidal method) for SDAEs (2.16) introduced in Winkler [20].

Example 4.2. Split-step \( \theta \)-methods for SDDAEs

We introduce a class of split-step \( \theta \)-methods for solving IVP for index-1 SDDAE of retarded type (3.1):

\[
\begin{align*}
X_{k+1} &= X_k + h \left[ \theta f(X_{k+1}, \dot{X}_{k+1}, Y_{k+1}, \dot{Y}_{k+1}) + (1 - \theta) f(X_k, \dot{X}_{k-\tau}, Y_k, \dot{Y}_{k-\tau}) \right], \\
0 &= u(X_{k+1}, \dot{X}_{k+1}, Y_{k+1}),
\end{align*}
\]

\[
X_{k+1} = X_{k+1}^* + g(X_{k+1}, \dot{X}_{k+1}, Y_{k+1}, \dot{Y}_{k+1}) \Delta W_k,
\]

\[
0 = u(X_{k+1}, \dot{X}_{k+1}, Y_{k+1}),
\]

where \( \Delta W_k = W(t_{k+1}) - W(t_k) \), the arguments \( \dot{X}_{k-\tau} \) and \( \dot{Y}_{k-\tau} \) are obtained by linear interpolation (4.2) and (4.3), \( k = 0, \ldots, N - 1 \).

In the implementation of methods (4.15), an implicit system of equations (4.15a)-(4.15b) must be solved to obtain the intermediate approximations \( X_{k+1}^* \) and \( Y_{k+1}^* \). Having obtained \( X_{k+1}^* \) and \( Y_{k+1}^* \), we can obtain the next approximation \( \{X_{k+1}, \dot{Y}_{k+1}\} \) by solving (4.15c)-(4.15d).

If the intermediate approximations \( X_{k+1}^* \) and \( Y_{k+1}^* \) can be solved from (4.15a)-(4.15b), we denote them by

\[
\begin{align*}
X_{k+1}^* &= \hat{X}(X_k, Y_k, \dot{X}_{k+1-\tau}, \dot{Y}_{k+1-\tau}, \dot{X}_{k-\tau}, \dot{Y}_{k-\tau}; h, \theta), \\
Y_{k+1}^* &= \hat{Y}(X_k, Y_k, \dot{X}_{k+1-\tau}, \dot{Y}_{k+1-\tau}, \dot{X}_{k-\tau}, \dot{Y}_{k-\tau}; h, \theta).
\end{align*}
\]

Inserting (4.16)-(4.17) into (4.15c)-(4.15d) we can derive the equivalent schemes of split-step \( \theta \)-methods in form (3.2):

\[
\begin{align*}
X_{k+1} &= \hat{X}(X_k, Y_k, \dot{X}_{k+1-\tau}, \dot{Y}_{k+1-\tau}, \dot{X}_{k-\tau}, \dot{Y}_{k-\tau}; h, \theta), \\
0 &= u(X_{k+1}, \dot{X}_{k+1}, Y_{k+1}),
\end{align*}
\]

Then using properties of function \( u \) in Assumption 2.1 and the uniform Lipschitz condition on functions \( f \) and \( g \), it is easy to verify that the equivalent schemes satisfy (3.8) and (3.10) in Assumption 3.1. Moreover, (3.9) in Assumption 3.1 holds due to the fact that \( g(X, \dot{X}_{k+1-\tau}, \dot{Y}, \dot{Y}_{k+1-\tau}) \) is \( \omega_{\delta_k} \)-measurable.

Besides, the local error of method (4.15) is the sequence of random variables \( \delta_k \) \((k = 0, \ldots, N - 1) \) satisfies

\[
\begin{align*}
x_{k+1} &= x_k + h \left[ \theta f(x_{k+1}, x_{k+1-\tau}, y_{k+1-\tau}, y_{k+1}) + (1 - \theta) f(x_k, x_{k-\tau}, y_k, y_{k-\tau}) \right], \\
0 &= u(x_{k+1}, x_{k+1-\tau}, y_{k+1}),
\end{align*}
\]

\[
\begin{align*}
x_{k+1} &= x_{k+1}^* + g(x_{k+1}, x_{k+1-\tau}, y_{k+1}, y_{k+1-\tau}) \Delta W_k + \delta_k, \\
0 &= u(x_{k+1}, x_{k+1-\tau}, y_{k+1}).
\end{align*}
\]

Similar to the proof of Theorem 4.1, the mean square consistency of the local error \( \delta_k \) of method (4.15) for SDDAE (3.1) can be deduced from the mean square consistency of the local error \( \tilde{\delta}_k \) of corresponding split-step \( \theta \)-methods for its inherent SDDEs (4.8)

\[
\begin{align*}
\left\{ \begin{array}{l}
x_{k+1} = x_k + h \left[ \theta f(x_{k+1}, x_{k+1-\tau}, x_{k+1-2\tau}) + (1 - \theta) f(x_k, x_{k-\tau}, x_{k-2\tau}) \right], \\
x_{k+1} = x_{k+1}^* + g(x_{k+1}, x_{k+1-\tau}, x_{k+1-2\tau}) \Delta W_k + \tilde{\delta}_k.
\end{array} \right.
\]

\[ (4.19) \]
Following the skills of mean square consistency analysis of \( \theta \)-Maruyama methods [18], we can prove that \( \theta_k \) satisfy estimates (4.13) and (4.14). Therefore, split-step \( \theta \)-methods (4.15) are mean square consistent with order 1/2. In conclusion, by Theorem 3.1 we deduce that split-step \( \theta \)-methods (4.15) is mean square convergent with order 1/2.

**Example 4.3.** Numerical methods for SDAEs without delay

Consider the IVP for SDAE (2.16) with initial values \( x(t_0) = x_0 \) and \( y(t_0) = y_0 \). It can be viewed as a special case of IVP for index-1 SDDAE of retarded type (3.1). Therefore from Example 4.1 we have the following strong order 1/2 semi-implicit Euler methods for SDAE (2.16)

\[
\begin{align*}
X_{k+1} &= X_k + h[\theta f(X_{k+1}, Y_{k+1}) + (1 - \theta) f(X_{k}, Y_{k})] + g(X_{k}, Y_{k})\Delta W_k, \\
0 &= u(X_{k+1}, Y_{k+1}),
\end{align*}
\]

where \( \theta \in [0, 1], k = 0, \ldots, N - 1 \). Here and in what follows, the numerical methods are performed on the equidistant mesh \( t_0 < t_1 < \ldots < t_N = T \) with stepsize \( h = T/N \).

From Example 4.2 we have the following strong order 1/2 split-step \( \theta \)-methods for SDAE (2.16)

\[
\begin{align*}
X_{k+1}^* &= X_k + h[\theta f(X_{k+1}^*, Y_{k+1}^*) + (1 - \theta) f(X_{k}, Y_{k})], \\
0 &= u(X_{k+1}^*, Y_{k+1}),
\end{align*}
\]

\[
\begin{align*}
X_{k+1} &= X_{k+1}^* + g(X_{k+1}^*, Y_{k+1})\Delta W_k, \\
0 &= u(X_{k+1}, Y_{k+1}),
\end{align*}
\]

where \( \theta \in [0, 1] \).

We can also consider a family of semi-implicit Milstein methods for (2.16)

\[
\begin{align*}
X_{k+1} &= X_k + h[\theta f(X_{k+1}, Y_{k+1}) + (1 - \theta) f(X_{k}, Y_{k})] + g(X_{k}, Y_{k})\Delta W_k \\
& \quad + [g_1'(X_{k}, Y_{k}) - g_2'(X_{k}, Y_{k})u_2(X_{k}, Y_{k})^{-1} u_1'(X_{k}, Y_{k})]g(X_{k}, Y_{k}) I_{(1,1)}, \\
0 &= u(X_{k+1}, Y_{k+1}),
\end{align*}
\]

where

\[
I_{(1,1)} = \int_{t_k}^{t_k+h} \int_{s_k}^{s_k+h} dW(s_2)dW(s_1) = [(\Delta W_k)^2 - h]/2.
\]

It’s obviously that (3.9) holds since the diffusion term is discretized explicitly and \( E(\Delta W_k) = E(I_{(1,1)}) = 0 \). Employing the corresponding semi-implicit Milstein method for SDEs (cf. [13]) to the following inherent SODE of SDAE (2.16)

\[
x(t) = x(t_0) + \int_{t_0}^{t} f(x(s))ds + \int_{t_0}^{t} g(x(s))dW(t),
\]

where \( \hat{f}(x(s)) := f(x(s), \hat{g}(x(s))) \) and \( \hat{g}(x(s)) := g(x(s), \hat{g}(x(s))) \), we have discretization scheme

\[
\hat{X}_{k+1} = \hat{X}_k + h[\theta \hat{f}(\hat{X}_{k+1}) + (1 - \theta) \hat{f}(\hat{X}_k)] + \hat{g}(\hat{X}_k)\Delta W_k + \hat{g}_x'(\hat{X}_k)\hat{g}(\hat{X}_k) I_{(1,1)}.
\]

Its local error

\[
\delta_k = x_{k+1} - \{x_k + h[\theta f(x_{k+1}, y_{k+1}) + (1 - \theta) f(x_k, y_k)] + g(x_k, y_k)\Delta W_k + \frac{\partial g}{\partial x}(x_k, \hat{g}(x_k))g(x_k, y_k) I_{(1,1)}\},
\]

\[
= x_{k+1} - \{x_k + h[\theta f(x_{k+1}, y_{k+1}) + (1 - \theta) f(x_k, y_k)] + g(x_k, y_k)\Delta W_k + \frac{\partial g}{\partial x}(x_k, \hat{g}(x_k))g(x_k, y_k) I_{(1,1)}\},
\]
and it has been proven to be strongly order 1.0 mean-square consistent (cf. [13]). Since

\[
\frac{\partial g}{\partial x}(x_k, \hat{y}(x_k)) = g'_1(x_k, \hat{y}(x_k)) + g'_2(x_k, \hat{y}(x_k)) \frac{\partial \hat{y}}{\partial x}(x_k)
\]

\[= g'_1(x_k, y_k) + g'_2(x_k, y_k) \cdot [\beta' u_2(x_k, y_k)^{-1}u'_1(x_k, y_k)],\]

it is obvious that the local error \( \delta_k \) of the semi-implicit Milstein method (4.22) is equivalent to \( \delta_k \), so we only need to verify if the semi-implicit Milstein method (4.22) satisfy estimates (3.8), (3.9) and (3.10) in Assumption 3.1. These estimates can be guaranteed under the additional assumption that \( g'_1 \) and \( g'_2 \) are Lipschitz continuous with respect to variables \( x \) and \( y \). Hence we know from Theorem 3.1 that the semi-implicit Milstein method (4.22) is mean square convergent with order 1.

**Remark 4.3.** Schemes (4.20)–(4.22) with parameter \( \theta = 1 \) and (4.20) with parameter \( \theta = 0.5 \) can be viewed as another forms of drift-implicit Euler scheme, split-step backward Euler scheme, drift-implicit Milstein scheme and trapezoidal rule for SDAEs introduced in [20] respectively. Generally speaking, explicit solutions are not available for nonlinear SDAEs (see [20] for details). However, from Example 4.3 we can see that, if SDAEs can be decoupled into semi-explicit form (2.16), not only fully implicit methods but also explicit schemes (\( \theta = 0 \)) or semi-implicit schemes (\( \theta \neq 0, 1 \)) can be used to discrete the nonlinear differential constraints.

### 5. Numerical Illustration

In order to illustrate the effectiveness and exactness of the one-step discretization schemes for index-1 SDDAEs (3.1), in this section, we will give two numerical examples. In the subsequent simulating experiments, we will always use 1000 sample paths for each test, and adopt the following quantity to describe the global error:

\[\epsilon := \max_{0 \leq i \leq N} \left( \frac{1}{1000} \sum_{j=1}^{1000} |X_i - x(t_i)|^2 + \max_{0 \leq i \leq N} \frac{1}{1000} \sum_{j=1}^{1000} |Y_i - y(t_i)|^2 \right),\]

where \( N = (T - t_0)/h \) and explicit solution \( x(t_i), y(t_i) \) (1 ≤ \( i \) ≤ 1000) are obtained approximately by the suitable numerical methods with stepsize \( h = 2^{-13} \).

**Example 5.1.** Consider the following linear SDDAE system

\[
\begin{align*}
\dot{x}(t) &= [\alpha_1x(t) + \alpha_2x(t - 1) + \alpha_3y(t) + \alpha_4y(t - 1)]dt \\
&\quad + [\beta_1x(t) + \beta_2x(t - 1) + \beta_3y(t) + \beta_4y(t - 1)]dW(t), \quad t \in [0, 10], \\
0 &= \gamma_1x(t) + \gamma_2x(t - 1) + \gamma_3y(t), \quad t \in [0, 10]
\end{align*}
\]

with initial values

\[x(t) = \cos(t), \quad y(t) = -[\gamma_1 \cos(t) + \gamma_2 \cos(t - 1)]/\gamma_3, \quad \forall t \leq 0,\]

where \( \alpha_i, \beta_i, \gamma_j \) are some parameters. We will consider four test problems of the form (5.1) with different parameters, which are listed in Table 5.1.

Applying the backward Euler method with stepsize \( h = 2^{-13} \) to solve the above-mentioned four test problems, the obtained solutions are plotted in Fig. 5.1. Since the solutions of SDAE
Table 5.1: Coefficients of the four test problems from system (5.1).

<table>
<thead>
<tr>
<th>test problem</th>
<th>$\alpha_1$</th>
<th>$\alpha_2$</th>
<th>$\alpha_3$</th>
<th>$\alpha_4$</th>
<th>$\beta_1$</th>
<th>$\beta_2$</th>
<th>$\beta_3$</th>
<th>$\beta_4$</th>
<th>$\gamma_1$</th>
<th>$\gamma_2$</th>
<th>$\gamma_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>SDDAE (5.1)</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>DDAE (5.1)</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>SDAE (5.1)</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>DAE (5.1)</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>2</td>
</tr>
</tbody>
</table>

Fig. 5.1. Solutions of the four test problems in Table 5.1: top left and right are sample paths of solution of SDDAE (5.1) and SDAE (5.1) respectively; bottom left and right are solutions of DDAE (5.1) and DAE (5.1) respectively.

For testing the convergence order of semi-implicit Euler method, we apply explicit Euler method, trapezoidal method and backward Euler method with stepsizes $h = 2^{-4}, 2^{-5}, 2^{-6}, 2^{-7}, 2^{-8}$, respectively, to SDDAE (5.1) on [0, 10]. In Fig. 5.2, we plot the curves of $\varepsilon$ versus $h$ for (5.1) and DAE (5.1) rapidly tended to zero, we only plot their solutions on [0, 5] to guarantee the visibility. Comparing these curves of solution we can see that index-1 SDDAE of retarded type contains both retarded and stochastic elements, and their influence on the system can not be neglected.
each method and the reference line with slope 0.5 on a log-log scale. This figure shows that the convergence orders of the numerical methods are almost in accordance with the theoretical order $1/2$.

**Example 5.2.** Consider the following nonlinear SDDAE system

$$
\begin{align*}
\frac{dx(t)}{dt} &= [-6x(t) + (2 + y(t)) \sin(x(t-1))] dt + [x(t) + y(t) \cos(x(t-1))] dW(t), \\
0 &= e^t y(t) - x(t) \cos(x(t-1)), & t \in [0, 10]
\end{align*}
$$

(5.2)

with initial values

$$x(t) = t + 1, \quad y(t) = e^{-t}(t + 1) \cos(t), \quad t \leq 0.$$

Applying explicit Euler method, trapezoidal method and backward Euler method with stepsizes $h = 2^{-4}, 2^{-5}, 2^{-6}, 2^{-7}, 2^{-8}$, respectively, to nonlinear problem (5.2) on $[0, 10]$, the global error $\varepsilon$ versus $h$ for each method are plotted in Fig. 5.3 on a log-log scale. From this figure, we can find that the used numerical methods have all convergence order $1/2$.
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