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Spectral Element Methods for Stochastic Differential Equations with Additive Noise

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Abstract. In this paper, we propose numerical schemes for stochastic differential equations driven by white noise and colored noise, respectively. For this purpose, we first discretize the white noise and colored noise, and give their regularity estimates. Then we use spectral element methods to solve the corresponding stochastic differential equations numerically. The approximation errors are derived, and the numerical results demonstrate high accuracy of the proposed schemes.

AMS subject classifications: 65M70, 65L60, 41A10, 60H35

Key words: Stochastic differential equations, spectral element methods, high accuracy

1 Introduction

Many problems in science and engineering can be characterized by differential equations. However, deterministic differential equations can not reveal the essence of these problems well due to the existence of accidental phenomena and small probability events. Therefore, many scholars consider stochastic differential equations to describe these problems, see [10,22,23] and etc, and the references therein.

Since the solution of a stochastic differential equation (SDE) is a stochastic process, it is difficult to reveal visually the information the process contains. Recently, there are many works to study the numerical solution for stochastic ordinary differential equations (SDEs). Euler-Maruyama method, Milstein method and Runge-Kutta method were proposed to solve SDEs numerically, see [3,4,13,16,17,19,20,31] and the references therein.

On the other hand, many authors investigated the numerical methods for stochastic partial differential equations (SPDEs). Allen *et al.* [1], Mcdonald [21], Gyöngy [14, 15] used finite difference method to study the numerical solutions of linear SPDEs driven by additive white noise. Also, Walsh [28], Du and Zhang [11] studied finite element method

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for linear SPDEs driven by special noises. Moreover, Cao *et al.* [6, 7, 8], Theting [26, 27], Yan [29] and etc, have also made some contributions.

As we know, spectral methods take orthogonal polynomials (such as Legendre, Chebyshev, Jacobi, Laguerre and Hermite polynomials) as the basis functions to approximate the solutions of problems in mathematical physics, and tend to have higher accuracy, see [5, 12, 25]. Recently, some scholars are trying to study SPDEs by spectral methods. Shardlow [24] took the eigenvectors of $-\frac{\partial^2}{\partial x^2}$ subject to Dirichlet boundary condition as the basis functions and used spectral method to approximate the noise. Cao and Yin [9] studied spectral Galerkin method for stochastic wave equations driven by space-time white noise. In addition, Yin and Gan [30] proposed the Chebyshev spectral collocation method to solve a certain type of stochastic delay differential equations.

In the present work, we will try to solve stochastic equations by using spectral element methods. We begin with taking the SDEs driven by white noise into consideration. As we know, the regularity of the solutions of the original stochastic differential equations plays an important role in priori error estimates of the numerical solutions. Unfortunately, the regularity estimates are usually very weak because of the existence of white noise. In order to overcome this difficulty, we will approximate the white noise process by piecewise constant random process as in [1], and apply the Legendre spectral element method for the corresponding stochastic differential equation. We prove the numerical solution converges to the original solution. Moreover, the accuracy of the proposed scheme is indicated by the provided computational results. The other part is that we use spectral element methods to solve stochastic differential equations driven by colored noise numerically. Analogous to the way to deal with white noise, it is nature to use an finite dimensional noise to discretize the colored noise (see [11]), which could improve the regularity of the solutions of the original stochastic differential equations. Therefore, the relevant stochastic differential equation is capable of being approximated by the Legendre spectral element scheme. Furthermore, the numerical results show the high accuracy of spectral element methods.

This paper is organized as follows. In the next section, we study numerical solutions of stochastic differential equations driven by white noise using spectral element methods, and the error estimates as well as the numerical results are presented. In Section 3, we use spectral element methods to solve stochastic differential equations driven by colored noise numerically. The final section is for conclusion remarks.

2 Numerical solutions of SDEs driven by white noise

We consider the following stochastic problem driven by white noise

$$\begin{cases} -u''(x)+bu(x)=g(x)+\dot{W}(x), & x\in I:=(0,1), \\ u(0)=u(1)=0, \end{cases} \tag{2.1}$$

where $\dot{W}(x)$ denotes white noise, $g(x)$ is a deterministic function and b is a constant.

The integral form of (2.1) has the form

$$u(x)+b\int_0^1 k(x,y)u(y)dy=\int_0^1 k(x,y)g(y)dy+\int_0^1 k(x,y)dW(y), \tag{2.2}$$

where $k(x,y)=x\wedge y-xy$ is Green's function associated with the elliptic equation

$$\begin{cases} -v''(x)=\phi(x), & x\in I, \\ v(0)=v(1)=0. \end{cases}$$

Multiplying the equation (2.1) by $\phi(x)$ and integrating by parts on I, we derive that a weak formulation of (2.1) is

$$\int_0^1 u'(x)\phi'(x)dx+b\int_0^1 u(x)\phi(x)dx=\int_0^1 g(x)\phi(x)dx+\int_0^1 \phi(x)dW(x), \tag{2.3}$$

where $\phi\in C^2(0,1)\cap C^0[0,1]$.

Remark 2.1. Buckdahn and Pardoux [2] proved existence and uniqueness of solutions to (2.2) and (2.3). Meanwhile, (2.2) and (2.3) are equivalent.

2.1 Approximation of white noise

Partition the interval I into $0=x_0<x_1<\cdots<x_K=1$ with $x_i=ih, h=x_i-x_{i-1}, i=0,1,\cdots,K$. Then a reasonable approximation to white noise on the interval $[0,1]$ is (see [1])

$$\frac{d\hat{W}(x)}{dx}=\frac{1}{\sqrt{h}}\sum_{i=1}^{K}\eta_i\chi_i(x), \tag{2.4}$$

where

$$\chi_i(x)=\begin{cases} 1, & x\in I_i:=[x_{i-1},x_i], \\ 0, & \text{otherwise}, \end{cases}$$

$\eta_1,\eta_2,\cdots,\eta_K$ are a sequence of random variables that are independent and identically distributed (*i.i.d.*) and subject to the standard normal distribution $N(0,1)$,

$$\eta_i=\frac{1}{\sqrt{h}}\int_{x_{i-1}}^{x_i} dW(x), \quad i=1,2,\cdots,K.$$

Substituting $d\hat{W}(y)$ for $dW(y)$ in (2.2), we can get

$$\hat{u}(x)+b\int_0^1 k(x,y)\hat{u}(y)dy=\int_0^1 k(x,y)g(y)dy+\int_0^1 k(x,y)d\hat{W}(y). \tag{2.5}$$

According to Theorem 2.2 in [1], we have the following regularity result.

Lemma 2.1. *Let $\hat{u}$ be the solution of* (2.5) *with $g\in L^2(0,1)$ and $b\geq 0$. Then $\hat{u}\in H^2(0,1)\cap H_0^1(0,1)$ with $E(\|\hat{u}\|_{H^2})\leq Ch^{-\frac{1}{2}}$ where constant C depends only on g.*

From Theorem 2.1 in [1], we have the estimation between u and $\hat{u}$.

Lemma 2.2. *Let $\hat{u}$ be the solution of* (2.5), *and u be the solution of* (2.2). *There holds*

$$E(\int_0^1 (u(x)-\hat{u}(x))^2 dx)\leq \frac{2h^2}{(1-\lambda)^2}, \tag{2.6}$$

where $\lambda^2 = b^2 \int_0^1 \int_0^1 k^2(x,y)dxdy < 1.$

Remark 2.2. Lemma 2.2 implies that if the mesh is sufficiently fine, then $\hat{u}(x)$ is a good approximation to $u(x)$, and $\hat{u}(x)$ is smoother than $u(x)$.

2.2 Spectral element methods for stochastic differential problem (2.1)

Denote $H_0^1(I)=\{u|u,u'\in L^2(I),u(0)=u(1)=0\}$, we substitute $\hat{W}(x)$ for $W(x)$, then the weak form (2.3) of the problem (2.1) can be written as

$$\begin{aligned}&\int_0^1 \hat{u}'(x)\phi'(x)dx+b\int_0^1 \hat{u}(x)\phi(x)dx\\ =&\int_0^1 g(x)\phi(x)dx+\int_0^1 \phi(x)d\hat{W}(x), \quad \forall \phi\in H_0^1(I).\end{aligned} \tag{2.7}$$

Let

$$X^{K,N}=\Big\{u\in C^0(I)\Big|u|_{I_i}\in P_N\Big\}, \quad X_0^{K,N}=\Big\{u\in X^{K,N}\Big|u(0)=u(1)=0\Big\},$$

where P_N be the space of the algebraic polynomials with degree at most N. Then the corresponding Galerkin scheme of (2.7) is to find $u_N(\in X_0^{K,N})$, such that

$$\begin{aligned}&\int_0^1 u_N'(x)\phi'(x)dx+b\int_0^1 u_N(x)\phi(x)dx\\ =&\int_0^1 g(x)\phi(x)dx+\int_0^1 \phi(x)d\hat{W}(x), \quad \forall \phi\in X_0^{K,N}.\end{aligned} \tag{2.8}$$

Define the Sobolev space:

$$B_{0,0}^m(I)=\Big\{u: \int_0^1 (\partial_x^k u(x))^2 x^k(1-x)^k dx<\infty, 0\leq k\leq m\Big\}, \quad m\in\mathbb{N}.$$

By using Theorem 3.39 in [25], we obtain the following result on I directly.

Lemma 2.3. *If* $\hat{u}\in H_0^1(I),\partial_x\hat{u}\in B_{0,0}^{m-1}(I)$, *and* $\partial_x^m\hat{u}\in L^2(I)$, *then for* $1\le m\le N+1$ *and* $\mu=0,1$,

$$\|\hat{u}-u_N\|_{\mu,I}\le Ch^{m-\mu}N^{\mu-m}\|\partial_x^m\hat{u}\|_I,$$

where C *is a positive constant independent of* $\hat{u}$, h, m *and* N.

Remark 2.3. In particular, for $m=2$, let $\hat{u}$ and u_N be the solutions of (2.7) and (2.8), we have

$$\|\hat{u}-u_N\|_I\le Ch^2N^{-2}\|\partial_x^2\hat{u}\|_I. \tag{2.9}$$

Therefore, by the triangle inequality, Lemma 2.2, (2.9) and Lemma 2.1, successively, we have the following error estimates.

Theorem 2.1. *Let* u *and* u_N *be the solutions of* (2.1) *and* (2.8), *respectively. Then we have*

$$E\big(\|u-u_N\|_{L^2,I}\big)\le C\left(\frac{h}{1-\lambda}+h^{\frac{3}{2}}N^{-2}\right), \tag{2.10}$$

where C *is a positive constant independent of* h *and* N.

2.3 Numerical results

Let

$$\begin{aligned}&\phi_0(\xi)=\frac{1-\xi}{2},\quad \phi_N(\xi)=\frac{1+\xi}{2},\\&\phi_m(\xi)=L_{m-1}(\xi)-L_{m+1}(\xi),\quad 1\le m\le N-1,\quad \xi\in(-1,1).\end{aligned}$$

Here, $L_m(\xi)$ is the Legendre polynomial on $(-1,1)$ with degree m, which satisfy that $L_m(\pm1)=(\pm1)^m$, and fulfill the following relations.

$$\int_{-1}^1 L_m(\xi)L_l(\xi)dx=\gamma_m\delta_{ml},\quad \gamma_m=\frac{2}{2m+1}, \tag{2.11}$$

$$(2m+1)L_m(\xi)=L'_{m+1}(\xi)-L'_{m-1}(\xi),\quad m\ge1. \tag{2.12}$$

In actual computation, we introduce the boundary-inner decomposition basis

$$\{\phi_m(\xi)\}_{m=0}^N=\{\phi_0(\xi),\phi_N(\xi)\}\cup\{\phi_m(\xi),\phi_m(\pm1)=0\}_{m=1}^{N-1}. \tag{2.13}$$

We expand them into global variable by

$$\psi_m^{(i)}(x)=\begin{cases}\phi_m(\xi), & if\ x\in I_i,\quad \xi=m_i^{-1}(x),\\ 0, & if\ x\notin I_i,\end{cases} \tag{2.14}$$

where $m_i(\xi)=x_{i-1}\phi_0(\xi)+x_i\phi_N(\xi), i=1,\cdots,K$, $m=1,\cdots,N-1$. For ensuring the continuity of the solution of the problem, we must propose the basis functions at interfaces. Therefore, we introduce

$$\psi_i^*(x)=\begin{cases}\psi_N^{(i)}(x), & x\in I_i,\\ \psi_0^{(i+1)}(x), & x\in I_{i+1},\\ 0, & \text{otherwise},\end{cases} \tag{2.15}$$

where $i=1,\cdots,K-1$. At the left and right endpoints, we define

$$\psi_0^*(x)=\begin{cases}\psi_0^{(1)}(x), & x\in I_1,\\ 0, & \text{otherwise},\end{cases} \tag{2.16}$$

$$\psi_K^*(x)=\begin{cases}\psi_N^{(K)}(x), & x\in I_K,\\ 0, & \text{otherwise}.\end{cases} \tag{2.17}$$

So the global basis set are

$$V=\{\psi_j(x)\}_{j=0}^{KN}:=\{\psi_0^*(x),\psi_K^*(x)\}\cup\{\psi_i^*(x)\}_{i=1}^{K-1}\cup\{\psi_m^{(i)}(x), m=1,\cdots,N-1\}_{i=1}^{K}. \tag{2.18}$$

Then, we expand the numerical solution of the problem (2.1) as follows:

$$u_N(x)=\sum_{j=0}^{KN}\tilde{u}_j\psi_j(x).$$

Substituting the above expression into Galerkin scheme (2.8) and taking $\phi=\psi_l(x)$, $l=0,\cdots,KN$, which yields that

$$\begin{aligned}&\sum_{j=0}^{KN}\tilde{u}_j\int_0^1\psi_j'(x)\psi_l'(x)dx+b\sum_{j=0}^{KN}\tilde{u}_j\int_0^1\psi_j(x)\psi_l(x)dx\\ =&\int_0^1 g(x)\psi_l(x)dx+\int_0^1\psi_l(x)d\hat{W}(x),\quad l=0,\cdots,KN.\end{aligned} \tag{2.19}$$

Let us denote

$$\begin{aligned}&M_{l,j}=\int_0^1\psi_j(x)\psi_l(x)dx, && \boldsymbol{M}=(M_{l,j})_{0\le j,l\le KN},\\ &S_{l,j}=\int_0^1\psi_j'(x)\psi_l'(x)dx, && \boldsymbol{S}=(S_{l,j})_{0\le j,l\le KN},\\ &\hat{f}_l=\int_0^1 g(x)\psi_l(x)dx+\int_0^1\psi_l(x)d\hat{W}(x), &&\\ &\boldsymbol{F}=(\hat{f}_0,\hat{f}_1,\cdots,\hat{f}_{KN})^t, && \boldsymbol{U}=(\tilde{u}_0,\tilde{u}_1,\cdots,\tilde{u}_{KN})^t.\end{aligned}$$

Then the linear algebraic system (2.19) becomes the following compact form

$$(\boldsymbol{S}+b\boldsymbol{M})\boldsymbol{U}=\boldsymbol{F}. \tag{2.20}$$

To compute the error $E\|u-u_N\|_{L^2}$, we set the state of random numbers to be 400, perform 10000 runs with different samples of noise for each computation, then for each sample calculate $\|u-u_N\|_{L^2}$, and finally, obtain the average value of $\|u-u_N\|_{L^2}$.

For the problem (2.1) with $b=\frac{1}{2}$, we take the test function $u(x)=x(1-x)\sin 3x$. In Table 1, the values of the error $E\|u-u_N\|_{L^2}$ with various N and K are listed. We can see that the errors decay as N and K increase. Comparison with the numerical results in [1], Allen, Novosel and Zhang took $h=0.0294$ and the smoother test function $u(x)=x(1-x)$, the accuracy of errors can only be reached to $O(10^{-4})$. While in Table 1, though the test function $u(x)=x(1-x)\sin 3x$ is oscillatory, the accuracy can be reached to $O(10^{-6})$ with $N=8$. This indicates that our new approach seems to be more accurate.

Table 1: The values of $E\|u_N-u\|_{L^2}$.

	$K=1$	$K=2$	$K=3$
$N=4$	$5.03e-05$	$4.19e-06$	$5.55e-06$
$N=6$	$3.59e-06$	$4.72e-06$	$6.30e-06$
$N=8$	$4.05e-06$	$5.24e-06$	$6.97e-06$

3 Numerical solutions of SDEs driven by colored noise

Due to the existence of white noise, the solution to (2.1) has low regularity, thus it doesn't achieve higher precision, which can be seen from Table 1. In this section, we will study numerical methods for stochastic differential equations driven by colored noise.

3.1 Properties of colored noise

Let $W=\{W(t),t\geq 0\}$ be a standard Wiener process and for fixed $h>0$, we define colored noise by (see [18])

$$\dot{W}(x)=\frac{W(x+h)-W(x)}{h}, \qquad \forall x\geq 0.$$

This is a wide-sense stationary Gaussian process with zero means and with covariances

$$c(x-s)=\frac{1}{h}\max\{0,1-\frac{1}{h}|x-s|\},$$

it thus has spectral density

$$S_h(\nu)=\frac{1}{h}\int_{-h}^{h}(1-\frac{|s|}{h})\cos(2\pi\nu s)ds=(\frac{\sin(2\pi\nu h)}{\pi\nu h})^2.$$

In general, we may use the following abstract formulation to simulate colored noise (see [11]):

$$\dot{W}(x)=\sum_{k=1}^{\infty}\sigma_k\eta_k\varphi_k(x), \tag{3.1}$$

where the *i.i.d.* random sequence $\eta_k \sim N(0,1)$, the deterministic functions $\{\varphi_k(x)\}$ form an orthogonal basis of $L^2(I)$ or its subspace, and the coefficients $\{\sigma_k\}$ are to be chosen to ascertain the convergence of the series in the mean square sense with respect to some suitable norms.

Let $\{\sigma_k^n\}_{k=1}^\infty$ approach $\{\sigma_k\}_{k=1}^\infty$ as $n\to\infty$ in some appropriate sense, then an approximation of $\dot{W}(x)$ is

$$\dot{W}_n(x)=\sum_{k=1}^{\infty}\sigma_k^n\eta_k\varphi_k(x), \tag{3.2}$$

where

$$\sigma_k^n=\begin{cases}\sigma_k, & k\le n,\\ 0\,, & k>n.\end{cases}$$

As was shown by Du and Zhang [11], the faster the coefficients σ_k decay, the smoother the noise trajectory $\dot{W}_n(x)$ looks. On the other hand, if the coefficients decay sufficiently slowly, then the trajectory can clearly resemble that of a white noise away from the boundary. In the end of this section, we will give the numerical results to demonstrate that different forms of coefficients σ_k^n induce different rates of convergence.

From Lemma 3.1 in [11], a bound is stated on $\dot{W}_n$ in the following lemma.

Lemma 3.1. *For* $\dot{W}_n(x)=\sum\limits_{k=1}^{\infty}\sigma_k^n\eta_k\varphi_k(x), \varphi_k(x)=\sqrt{2}\sin k\pi x$, *if s is a non-negative integer, then*

$$E\|\dot{W}_n\|_{H^s}\le C\Big(\sum_{k=1}^{\infty}(\sigma_k^n k^s)^2\Big)^{\frac{1}{2}}, \tag{3.3}$$

provided that the right-hand side is convergent.

3.2 Spectral element methods for SDEs driven by colored noise

We consider the following stochastic problem driven by colored noise

$$\begin{cases}-u''(x)+bu(x)=g(x)+\dot{W}(x), \quad x\in I,\\ u(0)=u(1)=0,\end{cases} \tag{3.4}$$

where $\dot{W}(x)$ denotes colored noise, $g(x)$ is a deterministic function and b is a constant.

The integral form of the problem (3.4) is

$$u(x)+b\int_0^1 k(x,y)u(y)dy=\int_0^1 k(x,y)g(y)dy+\int_0^1 k(x,y)dW(y), \tag{3.5}$$

and the weak form is

$$\begin{aligned}&\int_0^1 u'(x)\phi'(x)dx+b\int_0^1 u(x)\phi(x)dx\\ =&\int_0^1 g(x)\phi(x)dx+\int_0^1\phi(x)dW(x), \quad \forall\phi\in C^2(0,1)\cap C^0[0,1].\end{aligned} \tag{3.6}$$

We replace $dW(y)$ with $dW_n(y)$ in (3.5), and obtain

$$\hat{u}(x)+b\int_0^1 k(x,y)\hat{u}(y)dy=\int_0^1 k(x,y)g(y)dy+\int_0^1 k(x,y)dW_n(y). \tag{3.7}$$

According to Theorem 3.1 in [11], the following lemma holds.

Lemma 3.2. *For $\dot{W}_n(x)=\sum_{k=1}^{\infty}\sigma_k^n\eta_k\varphi_k(x)$, $\varphi_k(x)=\sqrt{2}\sin k\pi x$, if u and $\hat{u}$ are the solutions of* (3.5) *and* (3.7)*, respectively, then, for some constant $C>0$,*

$$E\|u-\hat{u}\|_{L^2,I}\le\frac{C}{1-\lambda}\|\vec{\sigma}^n-\vec{\sigma}\|_{Q_{-1}}, \tag{3.8}$$

where $\lambda^2=b^2\int_0^1\int_0^1 k^2(x,y)dxdy<1$, $\vec{\sigma}^n=(\sigma_1^n,\sigma_2^n,\cdots,\sigma_k^n,\cdots)^t$, $\vec{\sigma}=(\sigma_1,\sigma_2,\cdots,\sigma_k,\cdots)^t$,

$$\|\vec{\sigma}^n-\vec{\sigma}\|_{Q_{-1}}^2=\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}(\sigma_k^n-\sigma_k)(\sigma_l^n-\sigma_l)(kl)^{-1}\delta_{kl}.$$

Substituting $W_n(x)$ for $W(x)$ in (3.6), the weak form (3.6) can be written as

$$\begin{aligned}&\int_0^1\hat{u}'(x)\phi'(x)dx+b\int_0^1\hat{u}(x)\phi(x)dx\\ =&\int_0^1 g(x)\phi(x)dx+\int_0^1\phi(x)dW_n(x),\quad\forall\phi(x)\in H_0^1(I),\end{aligned} \tag{3.9}$$

and the corresponding Galerkin scheme of (3.4) is to find $u_N(\in X_0^{K,N})$, such that

$$\begin{aligned}&\int_0^1 u_N'(x)\phi'(x)dx+b\int_0^1 u_N(x)\phi(x)dx\\ =&\int_0^1 g(x)\phi(x)dx+\int_0^1\phi(x)dW_n(x),\quad\forall\phi\in X_0^{K,N}.\end{aligned} \tag{3.10}$$

Following the same line of Theorem 2.1, together with the triangle inequality, Lemma 3.2 and Lemma 2.3, we then have the following error estimates.

Theorem 3.1. *Let u and u_N are the solutions of* (3.4) *and* (3.10)*, respectively. Then we have*

$$E(\|u-u_N\|_{L^2,I})\le C\Big(\|\vec{\sigma}^n-\vec{\sigma}\|_{Q_{-1}}+h^mN^{-m}\|\partial_x^m\hat{u}\|_I\Big), \tag{3.11}$$

where C is a positive constant independent of u, h, m and N. Furthermore, according to Lemma 3.1, for $2\le m\le N+1$, we have

$$E(\|u-u_N\|_{L^2,I})\le C\left(\|\vec{\sigma}^n-\vec{\sigma}\|_{Q_{-1}}+h^mN^{-m}\Big(\sum_{k=1}^{\infty}(\sigma_k^nk^{m-2})^2\Big)^{\frac{1}{2}}\right). \tag{3.12}$$

3.3 Numerical results

In actual computation, we expand the numerical solution of the problem (3.4) as follows:

$$u_N(x)=\sum_{j=0}^{KN}\hat{u}_j\psi_j(x),$$

where $\{\psi_j\}_{j=0}^{KN}$ are defined in (2.14)-(2.17). Substituting the above expansion into Galerkin scheme (3.10) and taking $\phi=\psi_l(x)$, $l=0,\cdots,KN$, we can obtain the following system of linear algebraic equations

$$\begin{aligned}&\sum_{j=0}^{KN}\hat{u}_j\int_0^1\psi_j'(x)\psi_l'(x)dx+b\sum_{j=0}^{KN}\hat{u}_j\int_0^1\psi_j(x)\psi_l(x)dx\\&=\int_0^1 g(x)\psi_l(x)dx+\int_0^1\psi_l(x)dW_n(x),\quad l=0,\cdots,KN.\end{aligned}\tag{3.13}$$

Let us denote

$$\begin{aligned}&\hat{f}_l=\int_0^1 g(x)\psi_l(x)dx+\int_0^1\psi_l(x)dW_n(x),\\&\boldsymbol{F}=(\hat{f}_0,\hat{f}_1,\cdots,\hat{f}_{KN})^t,\qquad \boldsymbol{U}=(\hat{u}_0,\hat{u}_1,\cdots,\hat{u}_{KN})^t.\end{aligned}$$

Then we can rewrite the linear algebraic system (3.13) as

$$(\boldsymbol{S}+b\boldsymbol{M})\boldsymbol{U}=\boldsymbol{F},\tag{3.14}$$

where the matrix $\boldsymbol{S}$, $\boldsymbol{M}$ is defined as in (2.20).

For numerical experiment, we let $b=\frac{1}{2}$, and the test function $u_d(x)=x(1-x)\sin 3x$. Then the exact solution to (3.4) is given by $u(x)=u_d(x)+u_s(x)$, where

$$u_s(x)=\sum_{k=1}^{\infty}\frac{\sqrt{2}\sigma_k}{b+(k\pi)^2}\eta_k\sin k\pi x.$$

The state of random numbers is also set to be 400. Tables 2-3 list the values of the error $E\|u-u_N\|_{L^2}$ with $\sigma_k=k^{-\frac{3}{2}},k^{-\frac{7}{2}}$, respectively. The results indicate that the errors decay rapidly as N and K increase. At the same time, they demonstrate that if the coefficients σ_k decay sufficiently fast, the numerical solution would be smoother.

Also, in [11], Du and Zhang took $h=\frac{1}{128}$ and the smoother test function $u_d(x)=x(1-x)$, the accuracy of error estimates could only be reached to $O(10^{-6})$ when $\sigma_k=k^{-\frac{3}{2}}$. While for spectral element methods, from Table 2, the accuracy of error estimates can be reached to $O(10^{-13})$ with $N=240$, even if we take the oscillatory function as $u_d(x)=x(1-x)\sin 3x$. For $\sigma_k=k^{-\frac{7}{2}}$, took the same h and test function, Du and Zhang [11] get the accuracy of estimates reached to $O(10^{-10})$. As we can see from Table 3, the accuracy can be reached to $O(10^{-14})$ only with $N=140$. This demonstrates the accuracy of the proposed method.

Table 2: The values of $E\|u_N-u\|_{L^2}$ with $\sigma_k=k^{-\frac{3}{2}}$.

	$K=1$	$K=2$	$K=3$
$N=20$	$3.97e-05$	$2.10e-05$	$1.41e-05$
$N=60$	$1.33e-05$	$6.71e-06$	$4.35e-06$
$N=100$	$7.91e-06$	$3.73e-06$	$2.37e-06$
$N=140$	$5.61e-06$	$2.56e-06$	$2.62e-09$
$N=180$	$4.26e-06$	$1.62e-06$	$1.48e-10$
$N=220$	$3.39e-06$	$9.54e-10$	$2.21e-11$
$N=240$	$2.83e-06$	$2.02e-10$	$2.32e-13$

Table 3: The values of $E\|u_N-u\|_{L^2}$ with $\sigma_k=k^{-\frac{7}{2}}$.

	$K=1$	$K=2$	$K=3$
$N=20$	$4.95e-08$	$6.13e-09$	$1.82e-09$
$N=60$	$1.87e-09$	$2.27e-10$	$6.71e-11$
$N=100$	$4.08e-10$	$4.87e-11$	$1.44e-11$
$N=140$	$1.50e-10$	$1.77e-11$	$1.12e-14$

4 Conclusion

In this paper, the Legendre spectral element schemes were proposed for stochastic differential equations driven by white noise and colored noise, respectively. For stochastic differential equations driven by white noise, we improved the regularity of the solution by employing piecewise constant random process to approximate white noise process. Thus, the Legendre spectral element scheme was able to apply to approximate the corresponding stochastic differential equation. The error analysis was provided and the accuracy of the proposed scheme was showed by the numerical experiments. As for stochastic differential equations driven by colored noise, we approximated colored noise process by a finite dimensional noise and employed the Legendre spectral element scheme to the corresponding stochastic differential equation. The error estimation was presented and the numerical results demonstrated the high accuracy of the proposed schemes.

Although the spectral element methods was only considered for second order elliptic stochastic differential equations in the present work, it is suitable to other types of stochastic differential equations.

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