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C^0 DISCONTINUOUS GALERKIN METHODS FOR A PLATE FRICTIONAL CONTACT PROBLEM*

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Abstract

Numerous C^0 discontinuous Galerkin (DG) schemes for the Kirchhoff plate bending problem are extended to solve a plate frictional contact problem, which is a fourth-order elliptic variational inequality of the second kind. This variational inequality contains a non-differentiable term due to the frictional contact. We prove that these C^0 DG methods are consistent and stable, and derive optimal order error estimates for the quadratic element. A numerical example is presented to show the performance of the C^0 DG methods; and the numerical convergence orders confirm the theoretical prediction.

Mathematics subject classification: 65N30, 49J40
Key words: Variational inequality of fourth-order, Discontinuous Galerkin method, Plate frictional contact problem, Optimal order error estimate.

1. Introduction

Many problems in physical and engineering sciences are modeled by partial differential equations (PDEs). However, various more complex physical processes are described by variational inequalities (VIs). VIs form an important class of nonlinear problems arising in a wide range of application areas of physical, engineering, financial, and management sciences. In an early reference [6], many problems in mechanics and physics are formulated and studied in the framework of variational inequalities. More detailed study on VIs and numerical methods for solving them can be found in numerous monographs, e.g. [8–11,14]. In this paper, we study a frictional contact problem for Kirchhoff plates, which is modeled by a fourth-order elliptic variational inequality of the second kind. To solve fourth order elliptic PDEs, the conforming finite element (FE) method uses C^1 finite elements, which requires a large number of degrees of freedom and in addition, the method is not easy to implement. To resolve this problem, nonconforming FE methods have been developed, and an early reference on the mathematical analysis of nonconforming FE methods for the plate bending problem is [15]. In [12], nonconforming finite element methods for solving a plate frictional contact problem are analyzed, and optimal order error

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estimates are derived for both continuous and discontinuous nonconforming finite elements. However, the nonconforming FE space needs to be carefully chosen so that the inconsistency error can be controlled. Various discontinuous Galerkin (DG) methods have also been developed to solve fourth order PDEs. DG methods are an important family of nonstandard finite element methods for solving partial differential equations. Discontinuous Galerkin methods use piecewise smooth yet globally less smooth functions to approximate problem solutions, and relate the information between two neighboring elements by numerical traces. The practical interest in DG methods is due to their flexibility in mesh design and adaptivity, in that they allow elements of arbitrary shapes, irregular meshes with hanging nodes, and the discretionary local shape function spaces. In addition, the increase of the locality in discretization enhances the degree of parallelizability. We refer to [5] for a historical account about DG methods.

Due to the inequality form of the problems, the Galerkin orthogonality is lost when DG methods are applied to solve variational inequalities, resulting in substantial difficulty in analyzing DG methods for VIs. Moreover, the bilinear forms of DG schemes are coercive only in the finite element spaces, not in the original function space. Therefore, the standard analytical techniques of finite element methods for VIs are not applicable for DG cases. In [21], numerous DG methods are extended for solving elliptic variational inequalities of 2nd-order, and a priori error estimates are established, which are of optimal order for the linear element. In [22], some discontinuous Galerkin schemes with the linear element for solving the Signorini problem are studied, and an optimal convergence order is proved. The ideas presented in [22] are extended to solve a quasistatic contact problem in [23]. In this paper, we study DG methods to solve a fourth-order elliptic variational inequality of the second kind for the Kirchhoff plates. There are two kinds of DG methods for the biharmonic equations: fully discontinuous ones and C^0 continuous ones. The fully discontinuous IP methods are investigated systematically in [16–18, 20] for biharmonic problems. These type DG methods allow meshes with hanging nodes and arbitrary locally varying polynomial degrees on each element, and thus are ideally suited for *hp*-adaptivity, but they still suffer from a large number of degrees of freedom. A C^0 IP formulation is introduced for Kirchhoff plates in [7] and quasi-optimal error estimates are obtained for smooth solutions. Unlike fully discontinuous Galerkin methods, C^0 type DG methods do not "double" the degrees of freedom on element boundaries. The FE spaces belong to C^0, not C^1; penalty terms on inter-element boundaries are added to force the derivative to be nearly continuous. Therefore, C^0 DG schemes have good accuracy with fewer number of degrees of freedom, leading to the time saving in solving the discretized problems. A rigorous error analysis is presented in [3] for the C^0 IP method under weak regularity assumption on the solution. A weakness of this method is that the penalty parameter can not be precisely quantified a priori, and it must be chosen suitably large to guarantee stability. However, a large penalty parameter has a negative impact on accuracy. Based on this observation, another C^0 DG method is introduced in [25], where the stability condition can be precisely quantified. In [13], a consistent and stable C^0 DG method, called the local C^0 DG (LCDG) method, is derived for the Kirchhoff plate bending problem.

In this paper, we consider C^0 DG methods to solve the Kirchhoff frictional contact plate problem, which is a fourth-order elliptic variational inequality of the second kind. This model variational inequality arises in the study of a frictional contact problem for Kirchhoff plates. It is difficult to construct stable DG methods for such problems because of the higher order and the inequality form. For fourth-order elliptic variational inequalities of the first kind, some DG methods are developed in [24]; however, no error estimates are given. In [4], a quadratic

C^0 IP method for the Kirchhoff plates problem with the displacement obstacle is studied, and errors in the energy norm and the L^∞ norm are given by $O(h^\alpha)$, where $0.5 < \alpha \leq 1$. Note that in these papers the variational inequalities being approximated are of the first kind; i.e., they are imposed over convex sets, and no non-differentiable terms are involved. To analyze C^0 DG methods for fourth order variational inequalities of the second kind, we need to employ new techniques. We will extend numerous C^0 DG methods in [24] to solve the frictional contact problem for Kirchhoff plates, and prove that the quadratic C^0 DG schemes achieve the optimal convergence order. To our knowledge, it is the first time in the literature that such optimal order error estimates are presented for C^0 DG methods to solve fourth order variational inequalities of the second kind. The ideas and results reported in this paper can be extended to C^0 DG methods for other fourth order elliptic variational inequalities of the second kind.

The rest of the paper is organized as follows. In Section 2, we introduce Kirchhoff plate bending problem and its variational formulation. In Section 3, we present the notation and introduce some C^0 discontinuous Galerkin methods for solving the Kirchhoff frictional contact plate problem. In Section 4, consistency of the C^0 DG methods, boundedness and stability of the bilinear forms are presented. A priori error analysis for these C^0 DG methods is established in Section 5. In the final section, we report simulation results from a numerical example.

2. Kirchhoff Plate Frictional Contact Problem

2.1. Kirchhoff plate bending problem

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain with boundary Γ. The boundary value problem of a clamped Kirchhoff plate under a given scaled vertical load $f \in L^2(\Omega)$ is (cf. [19])

$$\begin{cases} \boldsymbol{\sigma} = -(1-\kappa)\nabla^2 u - \kappa\,\mathrm{tr}(\nabla^2 u)\boldsymbol{I} & \text{in } \Omega, \\ -\nabla\cdot(\nabla\cdot\boldsymbol{\sigma}) = f & \text{in } \Omega, \\ u = \partial_{\boldsymbol{\nu}} u = 0 & \text{on } \Gamma, \end{cases} \tag{2.1}$$

where $\kappa \in (0, 0.5)$ denotes the Poisson ratio of an elastic thin plate occupying the region Ω and $\boldsymbol{\nu}$ stands for the unit outward normal vector on Γ. $\boldsymbol{I}$ is the identity matrix of order 2 and tr is the trace operation on matrices. Here, ∇ is the usual nabla operator, and we denote the Hessian of v by $\nabla^2 v$, i.e.,

$$\nabla^2 v := \nabla(\nabla v) = \nabla((\partial_1 v, \partial_2 v)^t) = \begin{pmatrix} \partial_{11} v & \partial_{12} v \\ \partial_{21} v & \partial_{22} v \end{pmatrix}.$$

Note that the first equation in (2.1) can be rewritten as

$$\frac{1}{1-\kappa}\boldsymbol{\sigma} - \frac{\kappa}{1-\kappa^2}(\mathrm{tr}\boldsymbol{\sigma})\boldsymbol{I} = -\nabla^2 u. \tag{2.2}$$

For a vector-valued function $\boldsymbol{v} = (v_1, v_2)^t$ and a matrix-valued function $\boldsymbol{\sigma} = (\sigma_{ij})_{2\times 2}$, their divergences are defined by

$$\nabla\cdot\boldsymbol{v} := v_{1,1} + v_{2,2}, \quad \nabla\cdot\boldsymbol{\sigma} := (\sigma_{11,1} + \sigma_{21,2}, \sigma_{12,1} + \sigma_{22,2})^t.$$

The normal and tangential components of a vector $\boldsymbol{v}$ on the boundary are $v_\nu = \boldsymbol{v}\cdot\boldsymbol{\nu}$ and $\boldsymbol{v}_\tau = \boldsymbol{v} - v_\nu\boldsymbol{\nu}$. Similarly, for a tensor $\boldsymbol{\sigma}$, its normal component is defined by $\sigma_\nu = \boldsymbol{\sigma}\boldsymbol{\nu}\cdot\boldsymbol{\nu}$ and

then its tangential component is $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu \boldsymbol{\nu}$. Furthermore, we have the decomposition formula

$$(\boldsymbol{\sigma}\boldsymbol{\nu}) \cdot \boldsymbol{v} = (\sigma_\nu \boldsymbol{\nu} + \boldsymbol{\sigma}_\tau) \cdot (v_\nu \boldsymbol{\nu} + \boldsymbol{v}_\tau) = \sigma_\nu v_\nu + \boldsymbol{\sigma}_\tau \cdot \boldsymbol{v}_\tau.$$

For two matrices $\boldsymbol{\tau}$ and $\boldsymbol{\sigma}$, we define their double dot inner product and corresponding norm by $\boldsymbol{\sigma} : \boldsymbol{\tau} = \sum_{i,j=1}^{2} \sigma_{ij}\tau_{ij}$ and $|\boldsymbol{\tau}| = (\boldsymbol{\tau} : \boldsymbol{\tau})^{1/2}$.

The following integration by parts formulas will be used later.

Lemma 2.1. *Let D be a bounded Lipschitz domain, then for a scalar function v and a symmetric matrix-valued function $\boldsymbol{\tau}$, we have*

$$\int_D v\, \nabla \cdot (\nabla \cdot \boldsymbol{\tau})\, dx = \int_D \nabla^2 v : \boldsymbol{\tau}\, dx - \int_{\partial D} \nabla v \cdot (\boldsymbol{\tau}\boldsymbol{n})\, ds + \int_{\partial D} v\, \boldsymbol{n} \cdot (\nabla \cdot \boldsymbol{\tau})\, ds,$$
$$\int_D \nabla^2 v : \boldsymbol{\tau}\, dx = -\int_D \nabla v \cdot (\nabla \cdot \boldsymbol{\tau})\, dx + \int_{\partial D} \nabla v \cdot (\boldsymbol{\tau}\boldsymbol{n})\, ds,$$

whenever the terms appearing on both sides of the above identities make sense. Here, $\boldsymbol{n}$ is the unit outward normal to ∂D.

The weak formulation of problem (2.1) can be written as

$$\text{Find} \quad u \in H_0^2(\Omega): \quad a(u,v) = (f,v) \quad \forall\, v \in H_0^2(\Omega), \tag{2.3}$$

where the bilinear form is

$$a(u,v) = \int_\Omega \left[\Delta u\, \Delta v + (1-\kappa)\,(2\,\partial_{12}u\,\partial_{12}v - \partial_{11}u\,\partial_{22}v - \partial_{22}u\,\partial_{11}v)\right] dx, \tag{2.4}$$

and the linear form is

$$(f,v) = \int_\Omega f\, v\, dx.$$

Actually, multiplying the second equation in (2.1) by a test function $v \in H_0^2(\Omega)$ and noticing $v = \partial_{\boldsymbol{\nu}} v = 0$, by Lemma 2.1, we get the following equation

$$-\int_\Omega \boldsymbol{\sigma} : \nabla^2 v\, dx = \int_\Omega f v\, dx. \tag{2.5}$$

Then we can obtain (2.3) from (2.5) and the definition of $\boldsymbol{\sigma}$.

2.2. A frictional contact problem for Kirchhoff plate

In this paper, we consider a plate frictional contact problem, which is a 4th-order elliptic variational inequality (EVI) of second kind ([6]). The Lipschitz continuous boundary Γ of the domain Ω is decomposed into three parts: $\overline{\Gamma_1}$, $\overline{\Gamma_2}$ and $\overline{\Gamma_3}$ with Γ_1, Γ_2 and Γ_3 relatively open and mutually disjoint such that $\text{meas}(\Gamma_1) > 0$. Then the plate frictional contact problem we consider is:

$$\text{Find } u \in W: \quad a(u, v-u) + j(v) - j(u) \geq (f, v-u) \quad \forall\, v \in W. \tag{2.6}$$

Here,

$$W = \Big\{ v \in H^2(\Omega) :\, v = \partial_{\boldsymbol{\nu}} v = 0 \text{ on } \Gamma_1 \Big\}, \quad j(v) = \int_{\Gamma_3} g\, |v|\, ds.$$

This variational inequality describes a simply supported plate. The plate is clamped on the boundary Γ_1:

$$v = \partial_{\boldsymbol{\nu}} v = 0 \quad \text{on} \quad \Gamma_1, \tag{2.7}$$

is free on Γ_2, and is in frictional contact on Γ_3 with a rigid foundation; g can be interpreted as a frictional bound. Applying the standard theory on elliptic variational inequalities (e.g., [1, 8]), we know that the problem (2.6) has a unique solution $u \in W$.

Let

$$\Lambda = \Big\{ \lambda \in L^\infty(\Gamma_3) : |\lambda| \le 1 \text{ a.e. on } \Gamma_3 \Big\}.$$

We have the following result ([12]).

Theorem 2.1. *A function $u \in W$ is a solution of* (2.6) *if and only if there is a $\lambda \in \Lambda$ such that*

$$a(u, v) + \int_{\Gamma_3} g\ \lambda\ v\, ds = (f, v) \quad \forall\, v \in W, \tag{2.8}$$

$$\lambda\ u = |u| \quad \text{a.e. on } \Gamma_3. \tag{2.9}$$

Throughout the paper, we assume the solution of the problem (2.6) has the regularity $u \in H^3(\Omega)$. The regularity result $u \in H^3(\Omega)$ is shown for solutions of some variational inequalities of fourth-order ([9, pp. 323–327]).

2.3. Pointwise relations and characterization of λ

In error analysis of numerical solutions for the problem (2.6), we need to take advantage of pointwise relations satisfied by the solution u. Note that $\boldsymbol{\sigma}$ is defined by the first equation of (2.1). Then $\boldsymbol{\sigma} \in H^1(\Omega)^{2\times 2}$. We rewrite (2.6) as

$$\int_\Omega \big[-\boldsymbol{\sigma} : \nabla^2 (v - u) - f\,(v - u) \big]\, dx + \int_{\Gamma_3} g|v|\, ds - \int_{\Gamma_3} g|u|\, ds \ge 0 \quad \forall\, v \in W.$$

Take $v = u \pm \varphi$ for any $\varphi \in C_0^\infty(\Omega)$ to obtain

$$-\int_\Omega \boldsymbol{\sigma} : \nabla^2 \varphi\, dx = \int_\Omega f\, \varphi\, dx \quad \forall\, \varphi \in C_0^\infty(\Omega).$$

Thus,

$$-\nabla \cdot (\nabla \cdot \boldsymbol{\sigma}) = f \quad \text{in the sense of distribution.}$$

Since $f \in L^2(\Omega)$, we deduce that $\nabla \cdot (\nabla \cdot \boldsymbol{\sigma}) \in L^2(\Omega)$ and

$$-\nabla \cdot (\nabla \cdot \boldsymbol{\sigma}) = f \quad \text{a.e. in } \Omega. \tag{2.10}$$

Since $\nabla \cdot \boldsymbol{\sigma} \in L^2(\Omega)^2$ and $\nabla \cdot (\nabla \cdot \boldsymbol{\sigma}) \in L^2(\Omega)$, we can define $(\nabla \cdot \boldsymbol{\sigma}) \cdot \boldsymbol{\nu} \in H^{-1/2}(\Gamma)$ which satisfies the relation

$$\langle (\nabla \cdot \boldsymbol{\sigma}) \cdot \boldsymbol{\nu}, v \rangle_{1/2,\Gamma} = \int_\Omega \nabla \cdot (\nabla \cdot \boldsymbol{\sigma})\, v\, dx + \int_\Omega (\nabla \cdot \boldsymbol{\sigma}) \cdot \nabla v\, dx \quad \forall\, v \in H^1(\Omega). \tag{2.11}$$

Therefore, for any $v \in H^2(\Omega)$,

$$\begin{aligned} -\int_\Omega \nabla \cdot (\nabla \cdot \boldsymbol{\sigma})\, v\, dx &= \int_\Omega (\nabla \cdot \boldsymbol{\sigma}) \cdot \nabla v\, dx - \langle (\nabla \cdot \boldsymbol{\sigma}) \cdot \boldsymbol{\nu}, v \rangle_{1/2,\Gamma} \\ &= -\int_\Omega \boldsymbol{\sigma} : \nabla^2 v\, dx + \int_\Gamma (\boldsymbol{\sigma}\boldsymbol{\nu}) \cdot \nabla v\, ds - \langle (\nabla \cdot \boldsymbol{\sigma}) \cdot \boldsymbol{\nu}, v \rangle_{1/2,\Gamma}, \end{aligned}$$

i.e.,

$$a(u,v) = \int_\Omega f\, v\, dx - \int_\Gamma (\boldsymbol{\sigma\nu}) \cdot \nabla v\, ds + \langle (\nabla \cdot \boldsymbol{\sigma}) \cdot \boldsymbol{\nu}, v \rangle_{1/2,\Gamma} \quad \forall\, v \in H^2(\Omega).$$

Recalling the equation (2.8), we then have for any $v \in W$,

$$-\int_\Gamma (\boldsymbol{\sigma\nu}) \cdot \nabla v\, ds + \langle (\nabla \cdot \boldsymbol{\sigma}) \cdot \boldsymbol{\nu}, v \rangle_{1/2,\Gamma} + \int_{\Gamma_3} g\, \lambda\, v\, ds = 0. \tag{2.12}$$

Let $\sigma_{\boldsymbol{\nu}}$ and σ_τ be the normal and tangential components of the vector $\boldsymbol{\sigma\nu}$ on Γ. In (2.12), taking $v \in W$ such that $v = 0$ on Γ and $\partial_{\boldsymbol{\nu}} v$ arbitrary on $\Gamma_2 \cup \Gamma_3$, we have

$$\sigma_{\boldsymbol{\nu}} = 0 \quad \text{a.e. on } \Gamma_2 \cup \Gamma_3 \tag{2.13}$$

Then from (2.12) we get

$$-\int_{\Gamma_2 \cup \Gamma_3} \sigma_\tau \partial_\tau v\, ds + \langle (\nabla \cdot \boldsymbol{\sigma}) \cdot \boldsymbol{\nu}, v \rangle_{1/2,\Gamma} + \int_{\Gamma_3} g\, \lambda\, v\, ds = 0 \quad \forall\, v \in W. \tag{2.14}$$

Note that the closure of W in $H^1(\Omega)$ is

$$H^1_{\Gamma_1}(\Omega) = \{v \in H^1(\Omega) : v = 0 \text{ a.e. on } \Gamma_1\}.$$

Denote

$$\widetilde{H}^1_{\Gamma_1}(\Omega) = \{v \in H^1_{\Gamma_1}(\Omega) : \partial_\tau v \in L^2(\Gamma)\}.$$

Then from (2.14), we conclude that

$$-\int_{\Gamma_2 \cup \Gamma_3} \sigma_\tau \partial_\tau v\, ds + \langle (\nabla \cdot \boldsymbol{\sigma}) \cdot \boldsymbol{\nu}, v \rangle_{1/2,\Gamma} + \int_{\Gamma_3} g\, \lambda\, v\, ds = 0 \quad \forall\, v \in \widetilde{H}^1_{\Gamma_1}(\Omega). \tag{2.15}$$

3. C^0 Discontinuous Galerkin Methods

In this section, we introduce several C^0 DG methods, and we start with some notation.

3.1. Notation

Given a function space F, let $(F)^{2\times 2}_s := \left\{\boldsymbol{\tau} \in (F)^{2\times 2} : \boldsymbol{\tau}^t = \boldsymbol{\tau}\right\}$ be the space of the symmetric 2×2 matrix function. Given a positive integer m, $H^m(D)$ is the usual Sobolev space with the corresponding norm $\|\cdot\|_{m,D}$ and semi-norm $|\cdot|_{m,D}$, where D is a bounded set in $\mathbb{R}^2$. Let $\|\cdot\|_D$ denote the norm of the Lebesgue space $L^2(D)$. We assume Ω is a polygonal domain and denote by $\{\mathcal{T}_h\}_h$ a family of shape-regular triangulations of $\overline{\Omega}$. Denote $h_K = \text{diam}(K)$ and $h = \max\{h_K : K \in \mathcal{T}_h\}$. For a triangulation $\mathcal{T}_h$, let $\mathcal{E}_h$ be the set of all the element edges, $\mathcal{E}_h^b$ the set of all the element edges that lie on the boundary Γ, $\mathcal{E}_h^i := \mathcal{E}_h \backslash \mathcal{E}_h^b$ the set of all interior edges, and $\mathcal{E}_h^0 \subset \mathcal{E}_h$ the set of all the edges that do not lie on $\overline{\Gamma_2}$ or $\overline{\Gamma_3}$. For any $e \in \mathcal{E}_h$, denote by h_e its length. Given a triangulation $\mathcal{T}_h$, let

$$\begin{aligned}
\boldsymbol{\Sigma} &:= \left\{\boldsymbol{\tau} \in \left(L^2(\Omega)\right)^{2\times 2}_s : \tau_{ij}|_K \in H^1(K)\ \forall\, K \in \mathcal{T}_h,\ i,j = 1,2\right\},\\
V &:= \left\{v \in H^1_{\Gamma_1}(\Omega) : v|_K \in H^2(K)\ \forall\, K \in \mathcal{T}_h\right\},
\end{aligned}$$

and the corresponding finite element spaces are

$$\begin{aligned}
\boldsymbol{\Sigma}_h &:= \left\{\boldsymbol{\tau}_h \in \left(L^2(\Omega)\right)^{2\times 2}_s : \tau_{hij}|_K \in P_l(K)\ \forall\, K \in \mathcal{T}_h,\ i,j = 1,2\right\},\\
V_h &:= \left\{v_h \in H^1_{\Gamma_1}(\Omega) : v_h|_K \in P_2(K)\ \forall\, K \in \mathcal{T}_h\right\}.
\end{aligned}$$

Here, for a triangle $K \in \mathcal{T}_h$, $P_l(K)$ $(l = 0, 1)$ and $P_2(K)$ are the polynomial spaces on K of degrees l and 2, respectively. Note that the following property holds

$$\nabla_h^2 V_h \subset \boldsymbol{\Sigma}_h, \quad \frac{1}{1-\kappa}\boldsymbol{\Sigma}_h - \frac{\kappa}{1-\kappa^2}\,(\mathrm{tr}\boldsymbol{\Sigma}_h)\,\boldsymbol{I} \subset \boldsymbol{\Sigma}_h, \tag{3.1}$$

where $\nabla_h^2 V_h|_K := \nabla^2(V_h|_K)$ for any $K \in \mathcal{T}_h$. Correspondingly, we define the broken Hessian $\nabla_h^2 v$ by the relation $\nabla_h^2 v = \nabla^2 v$ on each element $K \in \mathcal{T}_h$.

Let $e \in \mathcal{E}_h^i$ be the common edge of two neighboring elements K^+ and K^-, with $\boldsymbol{n}^+$ and $\boldsymbol{n}^-$ be their outward unit normals on e. For a vector-valued function $\boldsymbol{w}$, denote its restriction on $K^\pm$ by $\boldsymbol{w}^\pm = \boldsymbol{w}|_{K^\pm}$. Similarly, write $\boldsymbol{\tau}^\pm = \boldsymbol{\tau}|_{K^\pm}$ for a matrix-valued function $\boldsymbol{\tau}$. Then on $e \in \mathcal{E}_h^i$, we define averages and jumps as follows:

$$\begin{aligned}
\{\boldsymbol{w}\} &= \frac{1}{2}(\boldsymbol{w}^+ + \boldsymbol{w}^-), \quad [\![\boldsymbol{w}]\!] = \frac{1}{2}(\boldsymbol{w}^+ \otimes \boldsymbol{n}^+ + \boldsymbol{n}^+ \otimes \boldsymbol{w}^+ + \boldsymbol{w}^- \otimes \boldsymbol{n}^- + \boldsymbol{n}^- \otimes \boldsymbol{w}^-),\\
\{\boldsymbol{\tau}\} &= \frac{1}{2}(\boldsymbol{\tau}^+ + \boldsymbol{\tau}^-), \qquad [\boldsymbol{\tau}] = \boldsymbol{\tau}^+\boldsymbol{n}^+ + \boldsymbol{\tau}^-\boldsymbol{n}^-.
\end{aligned}$$

For $e \in \mathcal{E}_h^b$, the above definitions need to be modified:

$$\begin{aligned}
\{\boldsymbol{w}\} &= \boldsymbol{w}, \quad [\![\boldsymbol{w}]\!] = \frac{1}{2}(\boldsymbol{w} \otimes \boldsymbol{\nu} + \boldsymbol{\nu} \otimes \boldsymbol{w}),\\
\{\boldsymbol{\tau}\} &= \boldsymbol{\tau}, \qquad [\boldsymbol{\tau}] = \boldsymbol{\tau}\boldsymbol{\nu}.
\end{aligned}$$

Here, $\boldsymbol{u} \otimes \boldsymbol{v}$ is a matrix with $u_i v_j$ as its (i, j)-th element.

We define a global lifting operator $\boldsymbol{r}_0 : \left(L^2(\mathcal{E}_h^0)\right)_s^{2\times 2} \to \boldsymbol{\Sigma}_h$ by

$$\int_\Omega \boldsymbol{r}_0(\boldsymbol{\phi}) : \boldsymbol{\tau}\, dx = -\int_{\mathcal{E}_h^0} \boldsymbol{\phi} : \{\boldsymbol{\tau}\}\, ds \quad \forall\, \boldsymbol{\tau} \in \boldsymbol{\Sigma}_h,\ \boldsymbol{\phi} \in \left(L^2(\mathcal{E}_h^0)\right)_s^{2\times 2}. \tag{3.2}$$

In addition, for each $e \in \mathcal{E}_h$, we introduce a local lifting operator $\boldsymbol{r}_e : \left(L^2(e)\right)_s^{2\times 2} \to \boldsymbol{\Sigma}_h$ by

$$\int_\Omega \boldsymbol{r}_e(\boldsymbol{\phi}) : \boldsymbol{\tau}\, dx = -\int_e \boldsymbol{\phi} : \{\boldsymbol{\tau}\}\, ds \quad \forall\, \boldsymbol{\tau} \in \boldsymbol{\Sigma}_h,\ \boldsymbol{\phi} \in \left(L^2(e)\right)_s^{2\times 2}. \tag{3.3}$$

By definition, we have the following identity

$$\boldsymbol{r}_0(\boldsymbol{\phi}) = \sum_{e \in \mathcal{E}_h^0} \boldsymbol{r}_e(\boldsymbol{\phi}|_e) \quad \forall\, \boldsymbol{\phi} \in \left(L^2(\mathcal{E}_h^0)\right)_s^{2\times 2}.$$

Consequently,

$$\|\boldsymbol{r}_0(\boldsymbol{\phi})\|_{0,\Omega}^2 = \|\sum_{e \in \mathcal{E}_h^0} \boldsymbol{r}_e(\boldsymbol{\phi}|_e)\|_{0,\Omega}^2 \le 3 \sum_{e \in \mathcal{E}_h^0} \|\boldsymbol{r}_e(\boldsymbol{\phi}|_e)\|_{0,\Omega}^2. \tag{3.4}$$

3.2. C^0 DG formulation for the Kirchhoff plate problem

In [24], a general primal formulation of C^0 DG methods is presented for a 4th-order elliptic variational inequality of the first kind. The process of deriving C^0 DG schemes for 4th-order elliptic equations can also be found in [13]. Based on the discussions in [24] and [13], we introduce five C^0 DG methods for the problem (2.6) as follows: Find $u_h \in V_h$ such that

$$B_h(u_h, v_h - u_h) + j(v_h) - j(u_h) \ge (f, v_h - u_h) \quad \forall\, v_h \in V_h, \tag{3.5}$$

where the bilinear form $B_h(w, v) = B_{1,h}^{(j)}(w, v)$ with $j = 1, \cdots, 5$, and $B_{1,h}^{(j)}(w, v)$ are given next.

The method with $j = 1$ is a C^0 interior penalty (IP) method, and the bilinear form is

$$\begin{aligned} B_{1,h}^{(1)}(u_h, v_h) = &\int_\Omega (1-\kappa)\nabla_h^2 u_h : \nabla_h^2 v_h \, dx + \int_\Omega \kappa \operatorname{tr}\left(\nabla_h^2 u_h\right) \operatorname{tr}\left(\nabla_h^2 v_h\right) dx \\ &- \int_{\mathcal{E}_h^0} [\![\nabla u_h]\!] : \left((1-\kappa)\{\nabla_h^2 v_h\} + \kappa \operatorname{tr}\left(\{\nabla_h^2 v_h\}\right) \boldsymbol{I}\right) ds \\ &- \int_{\mathcal{E}_h^0} [\![\nabla v_h]\!] : \left((1-\kappa)\{\nabla_h^2 u_h\} + \kappa \operatorname{tr}\left(\{\nabla_h^2 u_h\}\right) \boldsymbol{I}\right) ds \\ &+ \int_{\mathcal{E}_h^0} \eta h_e^{-1} [\![\nabla u_h]\!] : [\![\nabla v_h]\!] \, ds, \end{aligned} \tag{3.6}$$

where η is a function, defined to be a constant η_e on each $e \in \mathcal{E}_h^0$, with $\{\eta_e\}_{e\in\mathcal{E}_h^0}$ having a uniform positive bound from above and below. For a compact formulation, we can use lifting operator $\boldsymbol{r}_0$ (cf. (3.2)) to get

$$\begin{aligned} B_{2,h}^{(1)}(u_h, v_h) = &\int_\Omega (1-\kappa)\nabla_h^2 u_h : \left(\nabla_h^2 v_h + \boldsymbol{r}_0([\![\nabla v_h]\!])\right) dx \\ &+ \int_\Omega \kappa \operatorname{tr}\left(\nabla_h^2 u_h\right) \operatorname{tr}\left(\nabla_h^2 v_h + \boldsymbol{r}_0([\![\nabla v_h]\!])\right) dx \\ &+ \int_\Omega \boldsymbol{r}_0\left([\![\nabla u_h]\!]\right) : \left((1-\kappa)\nabla_h^2 v_h + \kappa \operatorname{tr}\left(\nabla_h^2 v_h\right) \boldsymbol{I}\right) dx \\ &+ \int_{\mathcal{E}_h^0} \eta h_e^{-1} [\![\nabla u_h]\!] : [\![\nabla v_h]\!] \, ds. \end{aligned} \tag{3.7}$$

A similar C^0 IP method was studied in [3].

Motivated by related DG methods for the second order elliptic problem, we can define the C^0 non-symmetric interior penalty (NIPG) formulation,

$$\begin{aligned} B_{1,h}^{(2)}(u_h, v_h) = &\int_\Omega (1-\kappa)\nabla_h^2 u_h : \nabla_h^2 v_h \, dx + \int_\Omega \kappa \operatorname{tr}\left(\nabla_h^2 u_h\right) \operatorname{tr}\left(\nabla_h^2 v_h\right) dx \\ &+ \int_{\mathcal{E}_h^0} [\![\nabla u_h]\!] : \left((1-\kappa)\{\nabla_h^2 v_h\} + \kappa \operatorname{tr}\left(\{\nabla_h^2 v_h\}\right) \boldsymbol{I}\right) ds \\ &- \int_{\mathcal{E}_h^0} [\![\nabla v_h]\!] : \left((1-\kappa)\{\nabla_h^2 u_h\} + \kappa \operatorname{tr}\left(\{\nabla_h^2 u_h\}\right) \boldsymbol{I}\right) ds \\ &+ \int_{\mathcal{E}_h^0} \eta h_e^{-1} [\![\nabla u_h]\!] : [\![\nabla v_h]\!] \, ds, \end{aligned} \tag{3.8}$$

or equivalently,

$$\begin{aligned} B_{2,h}^{(2)}(u_h, v_h) = &\int_\Omega (1-\kappa)\nabla_h^2 u_h : \left(\nabla_h^2 v_h + \boldsymbol{r}_0([\![\nabla v_h]\!])\right) dx \\ &+ \int_\Omega \kappa \operatorname{tr}\left(\nabla_h^2 u_h\right) \operatorname{tr}\left(\nabla_h^2 v_h + \boldsymbol{r}_0([\![\nabla v_h]\!])\right) dx \\ &- \int_\Omega \boldsymbol{r}_0\left([\![\nabla u_h]\!]\right) : \left((1-\kappa)\nabla_h^2 v_h + \kappa \operatorname{tr}\left(\nabla_h^2 v_h\right) \boldsymbol{I}\right) dx \\ &+ \int_{\mathcal{E}_h^0} \eta h_e^{-1} [\![\nabla u_h]\!] : [\![\nabla v_h]\!] \, ds. \end{aligned} \tag{3.9}$$

The C^0 DG method with $j = 3$ has the bilinear form

$$\begin{aligned}
B_{1,h}^{(3)}(u_h, v_h) = &\int_\Omega (1-\kappa)\nabla_h^2 u_h : \nabla_h^2 v_h \, dx + \int_\Omega \kappa \operatorname{tr}\left(\nabla_h^2 u_h\right) \operatorname{tr}\left(\nabla_h^2 v_h\right) dx \\
&- \int_{\mathcal{E}_h^0} [\![\nabla u_h]\!] : \left((1-\kappa)\{\nabla_h^2 v_h\} + \kappa \operatorname{tr}\left(\{\nabla_h^2 v_h\}\right) \boldsymbol{I}\right) ds \\
&- \int_{\mathcal{E}_h^0} [\![\nabla v_h]\!] : \left((1-\kappa)\{\nabla_h^2 u_h\} + \kappa \operatorname{tr}\left(\{\nabla_h^2 u_h\}\right) \boldsymbol{I}\right) ds \\
&+ \int_\Omega \boldsymbol{r}_0([\![\nabla v_h]\!]) : \left((1-\kappa)\boldsymbol{r}_0([\![\nabla u_h]\!]) + \kappa \operatorname{tr}(\boldsymbol{r}_0([\![\nabla u_h]\!]))\boldsymbol{I}\right) dx \\
&+ \sum_{e \in \mathcal{E}_h^0} \int_\Omega \eta \left((1-\kappa)\boldsymbol{r}_e([\![\nabla u_h]\!]) : \boldsymbol{r}_e([\![\nabla v_h]\!]) + \kappa \operatorname{tr}(\boldsymbol{r}_e([\![\nabla u_h]\!]))\operatorname{tr}(\boldsymbol{r}_e([\![\nabla v_h]\!]))\right) dx,
\end{aligned}$$

or equivalently,

$$\begin{aligned}
B_{2,h}^{(3)}(u_h, v_h) = &\int_\Omega (1-\kappa) \left(\nabla_h^2 u_h + \boldsymbol{r}_0([\![\nabla u_h]\!])\right) : \left(\nabla_h^2 v_h + \boldsymbol{r}_0([\![\nabla v_h]\!])\right) dx \\
&+ \int_\Omega \kappa \operatorname{tr}\left(\nabla_h^2 u_h + \boldsymbol{r}_0([\![\nabla u_h]\!])\right) \operatorname{tr}\left(\nabla_h^2 v_h + \boldsymbol{r}_0([\![\nabla v_h]\!])\right) dx \\
&+ \sum_{e \in \mathcal{E}_h^0} \int_\Omega \eta \left((1-\kappa)\boldsymbol{r}_e([\![\nabla u_h]\!]) : \boldsymbol{r}_e([\![\nabla v_h]\!]) + \kappa \operatorname{tr}(\boldsymbol{r}_e([\![\nabla u_h]\!]))\operatorname{tr}(\boldsymbol{r}_e([\![\nabla v_h]\!]))\right) dx,
\end{aligned} \tag{3.10}$$

which is the C^0 DG formulation proposed in [25].

The bilinear form of the C^0 DG scheme with $j = 4$ is

$$\begin{aligned}
B_{1,h}^{(4)}(u_h, v_h) = &\int_\Omega (1-\kappa)\nabla_h^2 u_h : \nabla_h^2 v_h \, dx + \int_\Omega \kappa \operatorname{tr}\left(\nabla_h^2 u_h\right) \operatorname{tr}\left(\nabla_h^2 v_h\right) dx \\
&- \int_{\mathcal{E}_h^0} [\![\nabla u_h]\!] : \left((1-\kappa)\{\nabla_h^2 v_h\} + \kappa \operatorname{tr}\left(\{\nabla_h^2 v_h\}\right) \boldsymbol{I}\right) ds \\
&- \int_{\mathcal{E}_h^0} [\![\nabla v_h]\!] : \left((1-\kappa)\{\nabla_h^2 u_h\} + \kappa \operatorname{tr}\left(\{\nabla_h^2 u_h\}\right) \boldsymbol{I}\right) ds \\
&+ \sum_{e \in \mathcal{E}_h^0} \int_\Omega \eta \left((1-\kappa)\boldsymbol{r}_e([\![\nabla u_h]\!]) : \boldsymbol{r}_e([\![\nabla v_h]\!]) + \kappa \operatorname{tr}(\boldsymbol{r}_e([\![\nabla u_h]\!]))\operatorname{tr}(\boldsymbol{r}_e([\![\nabla v_h]\!]))\right) dx,
\end{aligned} \tag{3.11}$$

or equivalently,

$$\begin{aligned}
B_{2,h}^{(4)}(u_h, v_h) = &\int_\Omega (1-\kappa)\nabla_h^2 u_h : \left(\nabla_h^2 v_h + \boldsymbol{r}_0([\![\nabla v_h]\!])\right) dx \\
&+ \int_\Omega \kappa \operatorname{tr}\left(\nabla_h^2 u_h\right) \operatorname{tr}\left(\nabla_h^2 v_h + \boldsymbol{r}_0([\![\nabla v_h]\!])\right) dx \\
&+ \int_\Omega \boldsymbol{r}_0\left([\![\nabla u_h]\!]\right) : \left((1-\kappa)\nabla_h^2 v_h + \kappa \operatorname{tr}\left(\nabla_h^2 v_h\right) \boldsymbol{I}\right) dx \\
&+ \sum_{e \in \mathcal{E}_h^0} \int_\Omega \eta \left((1-\kappa)\boldsymbol{r}_e([\![\nabla u_h]\!]) : \boldsymbol{r}_e([\![\nabla v_h]\!]) + \kappa \operatorname{tr}(\boldsymbol{r}_e([\![\nabla u_h]\!]))\operatorname{tr}(\boldsymbol{r}_e([\![\nabla v_h]\!]))\right) dx,
\end{aligned} \tag{3.12}$$

which is the C^0 DG formulation extended from the DG method of [2] for elliptic problems of second order.

For the LCDG method ([13]), the bilinear form is

$$\begin{aligned} B_{1,h}^{(5)}(u_h, v_h) := & \int_\Omega (1-\kappa)\nabla_h^2 u_h : \nabla_h^2 v_h \, dx + \int_\Omega \kappa \operatorname{tr}\left(\nabla_h^2 u_h\right) \operatorname{tr}\left(\nabla_h^2 v_h\right) dx \\ & - \int_{\mathcal{E}_h^0} [\![\nabla u_h]\!] : \left((1-\kappa)\{\nabla_h^2 v_h\} + \kappa \operatorname{tr}\left(\{\nabla_h^2 v_h\}\right) \boldsymbol{I}\right) ds \\ & - \int_{\mathcal{E}_h^0} [\![\nabla v_h]\!] : \left((1-\kappa)\{\nabla_h^2 u_h\} + \kappa \operatorname{tr}\left(\{\nabla_h^2 u_h\}\right) \boldsymbol{I}\right) ds \\ & + \int_\Omega \boldsymbol{r}_0([\![\nabla v_h]\!]) : \left((1-\kappa)\boldsymbol{r}_0([\![\nabla u_h]\!]) + \kappa \operatorname{tr}(\boldsymbol{r}_0([\![\nabla u_h]\!]))\boldsymbol{I}\right) dx \\ & + \int_{\mathcal{E}_h^0} \eta h_e^{-1} [\![\nabla u_h]\!] : [\![\nabla v_h]\!] \, ds, \end{aligned} \tag{3.13}$$

or equivalently,

$$\begin{aligned} B_{2,h}^{(5)}(u_h, v_h) := & \int_\Omega (1-\kappa) \left(\nabla_h^2 u_h + \boldsymbol{r}_0([\![\nabla u_h]\!])\right) : \left(\nabla_h^2 v_h + \boldsymbol{r}_0([\![\nabla v_h]\!])\right) dx \\ & + \int_\Omega \kappa \operatorname{tr}\left(\nabla_h^2 u_h + \boldsymbol{r}_0([\![\nabla u_h]\!])\right) \operatorname{tr}\left(\nabla_h^2 v_h + \boldsymbol{r}_0([\![\nabla v_h]\!])\right) dx \\ & + \int_{\mathcal{E}_h^0} \eta h_e^{-1} [\![\nabla u_h]\!] : [\![\nabla v_h]\!] \, ds. \end{aligned} \tag{3.14}$$

Remark 3.1. The bilinear forms $B_{1,h}^{(j)}$ and $B_{2,h}^{(j)}$ $(j = 1, \cdots, 5)$ coincide on the finite element spaces V_h, so either form can be used to compute the numerical solution u_h. In this paper, a priori error estimates are proved based on the first formulas $B_{1,h}^{(j)}$. Due to the equivalence of the two formulations on V_h, the stability only needs to be proved for the second formulas $B_{2,h}^{(j)}$ on V_h, which ensures the stability of the first formulas $B_{1,h}^{(j)}$ on V_h as well.

4. Properties of C^0 DG Schemes

First, we address the consistency of the methods (3.5).

Lemma 4.1 (Consistency) *Assume $u \in H^3(\Omega)$. Then for all the five C^0 DG methods with $B_h(w, v) = B_{1,h}^{(j)}(w, v)$, $1 \le j \le 5$, we have*

$$B_h(u, v_h - u) + j(v_h) - j(u) \ge (f, v_h - u) \quad \forall\, v_h \in V_h.$$

Proof. Noting $[\![\nabla u]\!] = 0$ on each edge $e \in \mathcal{E}_h^i$, we use (2.2) to get

$$\begin{aligned} B_h(u, v_h - u) = & \int_\Omega (1-\kappa)\nabla^2 u : \nabla_h^2 (v_h - u) \, dx + \int_\Omega \kappa \operatorname{tr}\left(\nabla^2 u\right) \operatorname{tr}\left(\nabla_h^2 (v_h - u)\right) dx \\ & - \int_{\mathcal{E}_h^0} [\![\nabla (v_h - u)]\!] : \left((1-\kappa)\nabla^2 u + \kappa \operatorname{tr}\left(\nabla^2 u\right) \boldsymbol{I}\right) ds \\ = & - \sum_{K \in \mathcal{T}_h} \int_K \boldsymbol{\sigma} : \nabla^2 (v_h - u) \, dx + \int_{\mathcal{E}_h^0} [\![\nabla (v_h - u)]\!] : \boldsymbol{\sigma} \, ds. \end{aligned}$$

Using Lemma 2.1 and noticing $[\boldsymbol{\sigma}] = \mathbf{0}$ on each edge $e \in \mathcal{E}_h^i$, we have

$$\begin{aligned}\sum_{K\in\mathcal{T}_h}\int_K \boldsymbol{\sigma} : \nabla^2(v_h - u)\,dx &= -\sum_{K\in\mathcal{T}_h}\int_K \nabla(v_h-u)\cdot(\nabla\cdot\boldsymbol{\sigma})\,dx \\ &\quad + \sum_{K\in\mathcal{T}_h}\int_{\partial K}\nabla(v_h-u)\cdot(\boldsymbol{\sigma}\boldsymbol{n}_K)\,ds \\ &= -\int_\Omega \nabla(v_h-u)\cdot(\nabla\cdot\boldsymbol{\sigma})\,dx + \int_{\mathcal{E}_h}[\![\nabla(v_h-u)]\!] : \boldsymbol{\sigma}\,ds.\end{aligned}$$

Combining the above two equations, we obtain

$$\begin{aligned}B_h(u, v_h-u) &= \int_\Omega \nabla(v_h-u)\cdot(\nabla\cdot\boldsymbol{\sigma})\,dx - \int_{\Gamma_2\cup\Gamma_3}[\![\nabla(v_h-u)]\!] : \boldsymbol{\sigma}\,ds \\ &= \int_\Omega \nabla(v_h-u)\cdot(\nabla\cdot\boldsymbol{\sigma})\,dx - \int_{\Gamma_2\cup\Gamma_3}\sigma_\tau\ \partial_\tau(v_h-u)\,ds \\ &= -\int_\Omega \nabla\cdot(\nabla\cdot\boldsymbol{\sigma})(v_h-u)\,dx + \langle(\nabla\cdot\boldsymbol{\sigma})\cdot\boldsymbol{\nu}, v_h-u\rangle_{1/2,\Gamma} \\ &\quad - \int_{\Gamma_2\cup\Gamma_3}\sigma_\tau\,\partial_\tau(v_h-u)\,ds.\end{aligned}$$

Here, the second equation comes from the relation (2.13), and the last equation holds by (2.11). We apply the relation (2.15), Lemma 2.1, (2.9) and (2.10) to obtain

$$\begin{aligned}B_h(u, v_h-u) &= -\int_\Omega \nabla\cdot(\nabla\cdot\boldsymbol{\sigma})(v_h-u)\,dx - \int_{\Gamma_3} g\ \lambda\ v_h\,ds + \int_{\Gamma_3} g\ \lambda\ u\,ds \\ &= \int_\Omega f(v_h-u)\,dx - \int_{\Gamma_3} g\ \lambda\ v_h\,ds + \int_{\Gamma_3} g\ |u|\,ds \\ &\geq \int_\Omega f(v_h-u)\,dx - \int_{\Gamma_3} g\ |v_h|\,ds + \int_{\Gamma_3} g\ |u|\,ds.\end{aligned}$$

So the stated result holds. □

For a function $v \in L^2(\Omega)$ with $v|_K \in H^m(K)$ for all $K \in \mathcal{T}_h$, define the broken norm and seminorm by

$$\|v\|_{m,h} = \Big(\sum_{K\in\mathcal{T}_h}\|v\|_{m,K}^2\Big)^{1/2},\quad |v|_{m,h} = \Big(\sum_{K\in\mathcal{T}_h}|v|_{m,K}^2\Big)^{1/2}.$$

The above symbols are used in a similar manner when v is a vector or matrix-valued function. In error analysis, C denotes a generic positive constant independent of h, which may take different values on different occurrences. To avoid writing these constants repeatedly, we use "$x \lesssim y$" to mean that "$x \leq Cy$". For two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, $\boldsymbol{u}\otimes\boldsymbol{v}$ is a matrix with u_iv_j as its (i,j)-th component. Let $V(h) := V_h + V\cap H^3(\Omega)$ and define two mesh-dependent energy norms by

$$|v|_*^2 := |v|_{2,h}^2 + \sum_{e\in\mathcal{E}_h^0} h_e^{-1}\|[\![\nabla v]\!]\|_{0,e}^2,\quad |||v|||^2 := |v|_*^2 + \sum_{K\in\mathcal{T}_h} h_K^2|v|_{3,K}^2,\quad v\in V(h).$$

The above two formulas define norms, i.e., if $|v|_* = 0$ and $v \in V(h)$, then $v = 0$. Actually, from $|v|_{2,h} = 0$, we have $v|_K \in P_1(K)$ and so ∇v is piecewise constant. Let e be the common edge of

two neighboring elements K^+ and K^-. From $\|[\![\nabla v]\!]\|_{0,e} = 0$, we get $(\nabla v)^+ = (\nabla v)^-$. Hence, ∇v is constant in Ω and so $v \in P_1(\Omega)$. Since $v = 0$ and $\nabla v = 0$ on Γ_1, we conclude that $v = 0$ in Ω.

Results similar to the following three lemmas are stated and proved in [24] in the context of fourth order VIs of the first kind. We present the results for completeness, but omit the proofs.

Lemma 4.2. *There exist two positive constants $C_1 \le C_2$ such that for any $v \in V(h)$ and $e \in \mathcal{E}_h^0$,*

$$C_1 h_e^{-1} \|[\![\nabla v]\!]\|_{0,e}^2 \le \|\boldsymbol{r}_e([\![\nabla v]\!])\|_{0,h}^2 \le C_2 h_e^{-1} \|[\![\nabla v]\!]\|_{0,e}^2. \tag{4.1}$$

From (4.1) and (3.4), we have

$$\|\boldsymbol{r}_0([\![\nabla v]\!])\|_{0,h}^2 = \|\sum_{e\in\mathcal{E}_h^0} \boldsymbol{r}_e([\![\nabla v]\!])\|_{0,h}^2 \le 3C_2 \sum_{e\in\mathcal{E}_h^0} h_e^{-1} \|[\![\nabla v]\!]\|_{0,e}^2.$$

Lemma 4.3 (Boundedness) *Let $B_h = B_{1,h}^{(j)}$ with $j = 1, \cdots, 5$. Then*

$$B_h(w, v) \lesssim |||\, w \,|||\, |||v\,||| \qquad \forall\, (w, v) \in V(h) \times V(h). \tag{4.2}$$

Lemma 4.4 (Stability) *Let $B_h = B_{1,h}^{(j)}$ with $j = 1, \cdots, 5$. Assume*

$$\min_{e\in\mathcal{E}_h^0} \eta_e > 3\,(1+\kappa)\, C_2 \qquad \text{for } j = 1,$$
$$\min_{e\in\mathcal{E}_h^0} \eta_e > 3\, C_2/C_1 \qquad \text{for } j = 4,$$

with C_1 and C_2 the constants in the inequality (4.1)*. Then*

$$|||v|||^2 \lesssim B_h(v, v), \qquad \forall\, v \in V_h. \tag{4.3}$$

5. Optimal Order Error Estimation

We now derive an optimal order error estimate for the C^0 DG methods. Write the error as

$$e = u - u_h = (u - u_I) + (u_I - u_h),$$

where $u_I \in V_h$ is the usual continuous piecewise quadratic interpolant of the exact solution u. Using the scaling argument and the trace theorem, we have the following result.

Lemma 5.1. *For all $v \in H^3(K)$ on $K \in \mathcal{T}_h$,*

$$\|v - v_I\|_K + h_K\, |v - v_I|_{1,K} + h_K^2\, |v - v_I|_{2,K} \lesssim h_K^3 |v|_{3,K},$$
$$\|\nabla\,(v - v_I)\|_{0,\partial K} \lesssim h_K^{3/2} |v|_{3,K}.$$

As a consequence of Lemma 5.1, we obtain the estimate

$$|||u - u_I||| \lesssim h|u|_{3,\Omega} \tag{5.1}$$

Now, we are ready to derive a priori error estimates of the C^0 DG methods when they are applied to solve the fourth-order elliptic variational inequality (2.6).

Theorem 5.1. *Assume the solution of the problem* (2.6) *satisfies* $u \in H^3(\Omega)$ *and the assumptions in Lemma* 4.4 *hold. Let* $B_h = B_h^{(j)}$ *with* $j = 1, \cdots, 5$, *and* $u_h \in K_h$ *be the solution of* (3.5). *Then we have the optimal order error estimate*

$$|||u - u_h||| \lesssim h \left(\|u\|_{3,\Omega} + h^{\frac{1}{4}} \|u\|_{3,\Omega}^{\frac{1}{2}} \|g\|_{0,\Gamma_3}^{\frac{1}{2}} \right). \tag{5.2}$$

Proof. Recall the boundedness and stability of the bilinear form B_h. We have

$$|||u_I - u_h|||^2 \lesssim B_h(u_I - u_h, u_I - u_h) \equiv T_1 + T_2, \tag{5.3}$$

where

$$T_1 = B_h(u_I - u, u_I - u_h), \quad T_2 = B_h(u - u_h, u_I - u_h).$$

We bound T_1 as follows:

$$T_1 \lesssim |||u_I - u||| \, |||u_I - u_h||| \lesssim \epsilon \, ||| u_I - u_h |||^2 + \frac{1}{4\epsilon} \, ||| u_I - u|||^2, \tag{5.4}$$

where $\epsilon > 0$ is an arbitrarily small number.

Following the same argument in Lemma 4.1, we have

$$B_h(u, u_I - u_h) = \int_\Omega f(u_I - u_h)\, dx - \int_{\Gamma_3} g \, \lambda \, u_I \, ds + \int_{\Gamma_3} g \, \lambda \, u_h \, ds. \tag{5.5}$$

Let $v_h = u_I$ in (3.5),

$$B_h(u_h, u_I - u_h) + j(u_I) - j(u_h) \geq (f, u_I - u_h). \tag{5.6}$$

Combining (5.6) and (5.5), and with the use of (2.9), we can bound T_2 as follows:

$$\begin{aligned} T_2 &\leq - \int_{\Gamma_3} g \, \lambda \, u_I \, ds + \int_{\Gamma_3} g \, \lambda \, u_h \, ds + j(u_I) - j(u_h) \\ &= \int_{\Gamma_3} g(|u_I| - \lambda \, u_I) \, ds + \int_{\Gamma_3} g(\lambda \, u_h - |u_h|) \, ds \\ &\leq \int_{\Gamma_3} g(|u_I| - \lambda \, u_I) \, ds = \int_{\Gamma_3} g(|u_I| - |u| + \lambda \, u - \lambda \, u_I) \, ds \\ &\leq 2 \int_{\Gamma_3} g \, |u_I - u| \, ds \leq 2 \|g\|_{0,\Gamma_3} \|u_I - u\|_{0,\Gamma_3}. \end{aligned}$$

Hence, by trace inequality and Lemma 5.1, we obtain

$$T_2 \lesssim h^{5/2} \|u\|_{3,\Omega} \|g\|_{0,\Gamma_3}. \tag{5.7}$$

The combination of (5.1), (5.3), (5.4), and (5.7) leads to

$$|||u_I - u_h|||^2 \lesssim h^2 \|u\|_{3,\Omega}^2 + h^{5/2} \|u\|_{3,\Omega} \|g\|_{0,\Gamma_3}. \tag{5.8}$$

Finally, from the triangle inequality $|||u - u_h||| \leq |||u - u_I||| + ||| u_I - u_h|||$, (5.1) and (5.8), we obtain the error bound. □

6. Numerical Results

In this section, we present simulation results from a numerical example with the five C^0 DG schemes studied in solving the elliptic variational inequality (2.6).

Example 6.1. Let $\Omega = (-1,1)\times(-1,1)$, $\kappa = 0.3$. A generic point in $\overline{\Omega}$ is denoted as $\mathbf{x} = (x,y)^T$. The Dirichlet boundary is $\Gamma_1 = (-1,1)\times\{1\}$, and the free boundary is $\Gamma_2 = \{\{-1\}\times(-1,1)\}\cup\{\{1\}\times(-1,1)\}$. On the friction boundary $\Gamma_3 = (-1,1)\times\{-1\}$, we choose $g = 1$. The right hand side function is $f(\mathbf{x}) = 24(1-x^2)^2 + 24(1-y^2)^2 + 32(3x^2-1)(3y^2-1)$.

For a discretization of the variational inequality (2.6), we use uniform triangulations $\{\mathcal{T}_h\}_h$ of the domain $\overline{\Omega}$, and define the finite element spaces to be

$$V_h := \{v_h \in H^1_{\Gamma_1}(\Omega) : v_h|_K \in P_2(K)\ \forall\, K \in \mathcal{T}_h\},$$
$$\mathbf{\Sigma}_h := \left\{\boldsymbol{\tau}_h \in \left(L^2(\Omega)\right)^{2\times 2}_s : \tau_{h,ij}|_K \in P_1(K)\ \forall\, K \in \mathcal{T}_h,\ i,j = 1,2\right\}.$$

Any function $v^h \in V_h$ can be expressed as

$$v^h(\mathbf{x}) = \sum v_i \phi_i(\mathbf{x}),$$

where $v_i = v^h(\mathbf{x}_i)$, $\{\mathbf{x}_i\}$ are the nodal points, and $\{\phi_i\}$ are the standard nodal basis functions of the space V_h. The basis functions satisfy the relation $\phi_i(\mathbf{x}_j) = \delta_{ij}$, with δ_{ij} being the Kronecker delta. The functional $j(\cdot)$ is approximated through numerical integration:

$$j_h(v^h) = S_n^{\Gamma_3}(g\,|v^h|) = \sum w_j g(\mathbf{x}_j)|\sum v_i\phi_i(\mathbf{x}_j)| = \sum |w_j g(\mathbf{x}_j)\, v_j|,$$

where the summations extend to all the finite element nodes on $\overline{\Gamma_3}$, and $S_n^{\Gamma_3}$ denotes the composite Simpson's rule using these finite element nodes. Then the discrete problem is

$$\min_{u^h\in V^h} \frac{1}{2}\, a(u^h,u^h) + j_h(u^h) - (f,u^h). \tag{6.1}$$

The matrix/vector form of the discrete optimization problem is

$$\min_{\mathbf{u}} \frac{1}{2}\mathbf{u}^T\mathbf{A}\mathbf{u} + \|\mathbf{B}\mathbf{u}\|_{\ell_1} - \mathbf{u}^T\mathbf{f}, \tag{6.2}$$

where $\mathbf{u} = (u_i)^T$, $\mathbf{A} = (a(\phi_i,\phi_j))$, $\mathbf{B} = (w_i g(\mathbf{x}_i)\delta_{ij})$, and $\mathbf{f} = ((f,\phi_j))^T$.

Algorithm 6.1. Primal Dual Fixed Point Algorithm

Initialize $\mathbf{u}_0$ and $\mathbf{v}_0$, set parameters $\lambda \in (0, \frac{1}{\lambda_{\max}(BB^T)}]$, $\gamma \in (0, \frac{2}{\|A\|_2})$
for $i = 1,2,3,\cdots$ **do**
  $\mathbf{u}_{k+\frac{1}{2}} = \mathbf{u}_k - \gamma(\mathbf{A}\mathbf{u}_k - \mathbf{f})$,
  $\mathbf{v}_{k+1} = (I - \mathrm{prox}_{\frac{\gamma}{\lambda}\|\cdot\|_1})(\mathbf{B}\mathbf{u}_{k+\frac{1}{2}} + (I - \lambda\mathbf{B}\mathbf{B}^T)\mathbf{v}_k)$,
  $\mathbf{u}_{k+1} = \mathbf{u}_{k+\frac{1}{2}} - \lambda\mathbf{B}^T\mathbf{v}_{k+1}$
end for

To solve the discrete problem (6.2), we use the following primal-dual fixed point iteration Algorithm **??** proposed in [26]. Here for a given function F of a vector variable $\mathbf{x}$, the proximal operator prox_F is defined as

$$\mathrm{prox}_F(\mathbf{x}) = \arg\min_{\mathbf{y}} F(\mathbf{y}) + \frac{1}{2}\|\mathbf{x}-\mathbf{y}\|_2^2.$$

Table 6.1: Error for C^0 IP method (3.7).

h	$\lvert\lvert\lvert u-u_h\rvert\rvert\rvert$			$\lvert u-u_h\rvert_{H^1(\Omega)}$		
	$\eta=1$	$\eta=10$	$\eta=100$	$\eta=1$	$\eta=10$	$\eta=100$
1/2	5.1859	4.2973	3.1164	0.9367	0.8126	0.5635
1/4	3.3677	2.6726	1.5923	0.5105	0.4135	0.3133
1/8	1.8625	1.4407	0.8122	0.2691	0.2049	0.1580
1/16	0.8601	0.7652	0.4422	0.1355	0.1059	0.0801

Table 6.2: Error for NIPG method (3.9).

h	$\lvert\lvert\lvert u-u_h\rvert\rvert\rvert$			$\lvert u-u_h\rvert_{H^1(\Omega)}$		
	$\eta=1$	$\eta=10$	$\eta=100$	$\eta=1$	$\eta=10$	$\eta=100$
1/2	5.5411	4.4659	3.1178	1.0000	0.8250	0.6638
1/4	3.6029	2.7708	1.6071	0.5699	0.4233	0.3607
1/8	1.9137	1.5359	0.7961	0.2829	0.2246	0.1853
1/16	0.9485	0.7594	0.3929	0.1491	0.1144	0.0934

Table 6.3: Error for Wells-Dung DG formulation (3.10).

h	$\lvert\lvert\lvert u-u_h\rvert\rvert\rvert$			$\lvert u-u_h\rvert_{H^1(\Omega)}$		
	$\eta=1$	$\eta=10$	$\eta=100$	$\eta=1$	$\eta=10$	$\eta=100$
1/2	4.4617	3.4573	2.0932	0.8062	0.7650	0.3785
1/4	2.8185	2.2331	1.3618	0.4301	0.3885	0.2486
1/8	1.4545	1.1473	0.6794	0.2131	0.2035	0.1253
1/16	0.7322	0.6270	0.3832	0.1085	0.1036	0.0645

For $F = \frac{\gamma}{\lambda}\|\cdot\|_1$, the proximal operator has the explicit form (applied to each component of the vector variable):

$$\mathrm{prox}_{\frac{\gamma}{\lambda}\|\cdot\|_1} x = \mathrm{sgn}(x)\ \max\Big(|x| - \frac{\gamma}{\lambda}, 0\Big) = \mathrm{sgn}(x)\ \Big(|x| - \frac{\gamma}{\lambda}\Big)_+ .$$

Tables 6.1–6.5 provide numerical solution errors in the energy norm $|||\cdot|||$ and $H^1(\Omega)$ seminorm for the five C^0 DG methods discussed in this paper. Since the true solution of the variational inequality (2.6) is not known, we use the numerical solution corresponding to the meshsize $h = 1/64$ as the reference solution to compute the numerical solution errors. Therefore, only the errors for numerical solutions with h larger than or equal to 1/16 are reported. We observe that the numerical convergence orders in the energy norm are around one, agreeing with the theoretical error estimate (5.2). We note that the numerical convergence orders in the $H^1(\Omega)$-seminorm are also close to one.

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Table 6.4: Error for Baker-DG formulation (3.12).

h	$\lvert\lvert\lvert u-u_h\rvert\rvert\rvert$			$\lvert u-u_h\rvert_{H^1(\Omega)}$		
	$\eta=1$	$\eta=10$	$\eta=100$	$\eta=1$	$\eta=10$	$\eta=100$
1/2	4.8538	4.1180	2.0977	0.8771	0.8360	0.3793
1/4	2.8524	2.4987	1.3632	0.4662	0.4625	0.2489
1/8	1.5067	1.2842	0.6765	0.2447	0.2409	0.1255
1/16	0.7629	0.6747	0.3842	0.1259	0.1212	0.0657

Table 6.5: Error for LCDG method (3.14).

h	$\lvert\lvert\lvert u-u_h\rvert\rvert\rvert$			$\lvert u-u_h\rvert_{H^1(\Omega)}$		
	$\eta=1$	$\eta=10$	$\eta=100$	$\eta=1$	$\eta=10$	$\eta=100$
1/2	4.6407	4.2599	2.5863	0.8384	0.7696	0.5579
1/4	2.8265	2.2147	1.6213	0.4546	0.4111	0.2863
1/8	1.5011	1.2460	0.8669	0.2311	0.2301	0.1453
1/16	0.7517	0.6341	0.4705	0.1189	0.1186	0.0812

References

[1] K. Atkinson and W. Han, *Theoretical Numerical Analysis: A Functional Analysis Framework*, third edition, Springer-Verlag, New York, Texts in Applied Mathematics, Volume 39, 2009.

[2] F. Bassi, S. Rebay, G. Mariotti, S. Pedinotti, and M. Savini, A high-order accurate discontinuous finite element method for inviscid and viscous turbomachinery flows, in *Proceedings of 2nd European Conference on Turbomachinery, Fluid Dynamics and Thermodynamics*, R. Decuypere and G. Dibelius, eds., Technologisch Instituut, Antwerpen, Belgium, 1997, 99–108.

[3] S.C. Brenner and L. Sung, C^0 interior penalty methods for fourth order elliptic boundary value problems on polygonal domains, *J. Sci. Comput.*, **22/23** (2005), 83–118.

[4] S.C. Brenner, L. Sung, H. Zhang, and Y. Zhang, A quadratic C^0 interior penalty method for the displacement obstacle problem of clamped Kirchhoff plates, *SIAM J. Numerical Analysis*, **50** (2012), 3329–3350.

[5] B. Cockburn, G.E. Karniadakis, and C.-W. Shu, eds, *Discontinuous Galerkin Methods. Theory, Computation and Applications*, Lecture Notes in Comput. Sci. Engrg., **11**, Springer-Verlag, New York, 2000.

[6] G. Duvaut and J.-L. Lions, *Inequalities in Mechanics and Physics*, Springer-Verlag, Berlin, 1976.

[7] G. Engel, K. Garikipati, T. Hughes, M. Larson, L. Mazzei, and R. Taylor, Continuous /discontinuous finite element approximations of fourth-order elliptic problems in structural and continuum mechanics with applications to thin beams and plates, and strain gradient elasticity, *Comput. Methods Appl. Mech. Engrg.*, **191** (2002), 3669–3750.

[8] R. Glowinski, *Numerical Methods for Nonlinear Variational Problems*, Springer-Verlag, New York, 1984.

[9] R. Glowinski, J.-L. Lions and R. Trémolières, *Numerical Analysis of Variational Inequalities*, North-Holland, New York, 1981.

[10] W. Han and B.D. Reddy, *Plasticity: Mathematical Theory and Numerical Analysis*, Springer-Verlag, New York, 1999.

[11] W. Han and M. Sofonea, *Quasistatic Contact Problems in Viscoelasticity and Viscoplasticity*, American Mathematical Society and International Press, 2002.

[12] W. Han and L.H. Wang, Non-conforming finite element analysis for a plate contact problem, *SIAM Journal on Numerical Analysis*, **40** (2002), 1683–1697.

[13] J. Huang, X. Huang, and W. Han, A new C^0 discontinuous Galerkin method for Kirchhoff plates,

Comput. Methods Appl. Mech. Engrg., **199** (2010), 1446–1454.

[14] N. Kikuchi and J.T. Oden, *Contact Problems in Elasticity: A Study of Variational Inequalities and Finite Element Methods*, SIAM, Philadelphia, 1988.

[15] P. Lascaux and P. Lesaint, Some nonconforming finite elements for the plate bending problem, *Rev. Française Automat. Informat. Recherche Opérationnelle Sér. Rouge Anal. Numér.*, **9** (1975), 9–53.

[16] I. Mozolevski and P.R. Bösing, Sharp expressions for the stabilization parameters in symmetric interior-penalty discontinuous Galerkin finite element approximations of fourth-order elliptic problems, *Comput. Methods Appl. Math.*, **7** (2007), 365–375.

[17] I. Mozolevski and E. Süli, A priori error analysis for the *hp*-version of the discontinuous Galerkin finite element method for the biharmonic equation, *Comput. Methods Appl. Math.*, **3** (2003), 596–607.

[18] I. Mozolevski, E. Süli, and P.R. Bösing, *hp*-version a priori error analysis of interior penalty discontinuous Galerkin finite element approximations to the biharmonic equation, *J. Sci. Comput.*, **30** (2007), 465–491.

[19] J.N. Reddy, *Theory and Analysis of Elastic Plates and Shells*, second edition, CRC Press, New York, 2007.

[20] E. Süli and I. Mozolevski, *hp*-version interior penalty DGFEMs for the biharmonic equation, *Comput. Methods Appl. Mech. Engrg.*, **196** (2007), 1851–1863.

[21] F. Wang, W. Han, and X. Cheng, Discontinuous Galerkin methods for solving elliptic variational inequalities, *SIAM Journal on Numerical Analysis*, **48** (2010), 708–733.

[22] F. Wang, W. Han, and X. Cheng, Discontinuous Galerkin methods for solving Signorini problem, *IMA J. Numer. Anal.*, **31** (2011), 1754–1772.

[23] F. Wang, W. Han, and X. Cheng, Discontinuous Galerkin methods for solving a quasistatic contact problem, *Numer. Math.*, **126** (2014), 771–800.

[24] F. Wang, W. Han, J. Huang, and T. Zhang, Discontinuous Galerkin methods for an elliptic variational inequality of 4th-order, in *Advances in Variational and Hemivariational Inequalities: Theory, Numerical Analysis, and Applications*, eds. W. Han, S. Migórski, and M. Sofonea, Springer-Verlag, New York, 2015, 199–222.

[25] G.N. Wells and N.T. Dung, A C^0 discontinuous Galerkin formulation for Kirchhoff plates, *Comput. Methods Appl. Mech. Engrg.*, **196** (2007), 3370–3380.

[26] X. Zhang, M. Burger and S. Osher, A unified primal-dual algorithm framework based on Bregman iteration, *J. Sci. Comput.*, **46** (2011), 20–46.