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New Residual Based Stabilization Method for the Elasticity Problem

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Abstract. In this paper, we consider the mixed finite element method (MFEM) of the elasticity problem in two and three dimensions (2D and 3D). We develop a new residual based stabilization method to overcome the inf-sup difficulty, and use Langrange elements to approximate the stress and displacement. The new method is unconditionally stable, and its stability can be obtained directly from Céa's lemma. Optimal error estimates for the H^1-norm of the displacement and $H(\text{div})$-norm of the stress can be obtained at the same time. Numerical results show the excellent stability and accuracy of the new method.

AMS subject classifications: 65N15, 65N30

Key words: Elasticity, MFEM, residuals, stabilization.

1 Introduction

In this paper, we consider the MFEM of the elasticity problem based on the Hellinger-Reissner variational principle. As is known to all, this method requires the pairs of the finite element space satisfy the so-called inf-sup condition. Since the stress tensor requires symmetry, it is difficult to construct the stable mixed finite elements (MFEs). Some early works employed composite elements [1], or imposed the symmetry of stress tensor weakly [2–6]. Until 2002, Arnold and Winther proposed the first family of stable MFEs with respect to triangular meshes which used polynomial shape functions to approximate the stress and displacement [7], of which the simplified lowest order element has 21 degrees of freedom for the stress and 3 for the displacement (21 plus 3 DOFs), and

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optimal order error estimates were obtained for all variables. An analogous family of conforming MFEs based on rectangular meshes were proposed in [8], involving 36 plus 3 DOFs for the simplified lowest order element. In [11], using the similar method of [7], Arnold et al. presented some stable elements in 3D with respect to simplicial meshes, and even the simplified lowest element has 156 plus 6 DOFs. In addition, some efficient nonconforming MFEs for this problem also have been proposed. For example, two triangular elements were presented in [9], and the simplified element has 12 plus 3 DOFs; and a group of rectangular elements were introduced in [10], with the $\mathcal{O}(h)$ convergence order in L^2-norm for both the stress and the displacement, and the simplest element employed 12 plus 4 DOFs. Although many other stable elements were also constructed based on the ideas of [7], (see [12–17]), these elements still have too many DOFs, and the implementations are expensive [18], especially for the 3D case.

Recently, some new methods were proposed to construct stable elements for elasticity problem. In [19], a family of nonconforming rectangular and cubic elements were constructed, and an explicit constructional proof of the discrete inf-sup condition was given. The DOFs are 2 plus 1 in 1D, 7 plus 2 in 2D, and 15 plus 3 in 3D, and the error estimates for all variables are optimal. In [20–22], some conforming rectangular and cubic elements were presented, of which the lowest order elements have 8 plus 2 DOFs in 2D, and 18 plus 3 DOFs in 3D. In [23–25], some conforming elements on simplicial meshes were developed, and the lowest order elements only involve 18 plus 3 DOFs in 2D and 48 plus 6 in 3D. Compared with Arnold-Winther elements, these elements are more compact, and have less DOFs.

On the other hand, some stabilized methods were also studied for the elasticity problem to overcome the inf-sup difficulty, such as Galerkin least-squares method [26], Brezzi-Pitkäranta stabilization [27], variational multiscale method [28], projection stabilization method [29, 30], edge stabilization method [31, 32], and least-squares method [33–36]. In this paper, we propose a new residual based stabilization method for the elasticity problem. The equilibrium term is used to augment the coercivity, and the term derived from the pure displacement equation is used to control the H^1-norm of the displacement. The method is consistent and unconditionally stable. The bilinear form is strongly coercive, and its stability can be obtained directly from Céa's lemma. The Language elements of any order can be used to approximate stress and displacement, so the lowest elements on simplicial meshes have 9 plus 6 DOFs in 2D, and 24 plus 12 DOFs in 3D, and the numerical implementations are more easily. In addition, Optimal error estimates for the H^1-norm of the displacement and $H(\mathrm{div})$-norm of the stress can be obtained at the same time.

The rest of this paper is organized as follows. In Section 2, we introduce the mixed form of the elasticity problem and some notations used throughout the paper. In Section 3, we present the new stabilization scheme, prove the stability, and give the error analysis. In Section 4, we implement two numerical examples to test the stability and convergence rate of the new method. Throughout the paper we use C to denote a generic positive constant whose value may change from place to place but that remains independent of the mesh size h.

2 Notations and preliminaries

Firstly, we introduce some notations and function spaces. Let Ω be a bounded polygon domain in $\mathbb{R}^d$, $d=2,3$, and Γ be the boundary. The unit outward normal vector to Γ is denoted by $\boldsymbol{n}=(n_1,n_2,\cdots,n_d)^T$, where superscript T represents the transpose of vector or matrix. In what follows, all vectors are the column vectors. $\mathbb{S}$ is the space of symmetric tensors.

Let $p,\boldsymbol{v}=(v_1,v_2,\cdots,v_d)^T$ and $\boldsymbol{\tau}=(\tau_{ij})_{d\times d}$ be a scalar-valued function, vector-valued function and tensors, respectively. We define

$$
\begin{aligned}
&\nabla p=\Big(\frac{\partial p}{\partial x_1},\frac{\partial p}{\partial x_2},\cdots,\frac{\partial p}{\partial x_d}\Big)^T,\\
&\nabla \boldsymbol{v}=(\nabla v_1,\nabla v_2,\cdots,\nabla v_d)^T,\quad \mathrm{div}\boldsymbol{v}=\frac{\partial v_1}{\partial x_1}+\frac{\partial v_2}{\partial x_2}+\cdots+\frac{\partial v_d}{\partial x_d},\\
&\mathrm{div}\boldsymbol{\tau}=(\mathrm{div}\boldsymbol{\tau}_1,\mathrm{div}\boldsymbol{\tau}_2,\cdots,\mathrm{div}\boldsymbol{\tau}_d)^T,\quad \text{where}\ \ \boldsymbol{\tau}_i=(\tau_{i1},\tau_{i2},\cdots,\tau_{id})^T,\\
&\epsilon(\boldsymbol{v})=\frac{1}{2}(\nabla\boldsymbol{v}+(\nabla\boldsymbol{v})^T),\quad \mathrm{tr}\boldsymbol{\tau}=\tau_{11}+\tau_{22}+\cdots+\tau_{dd},\\
&\boldsymbol{\sigma}:\boldsymbol{\tau}=\sum_{i,j=1}^{d}\sigma_{ij}\tau_{ij},\quad \boldsymbol{\tau}\cdot\boldsymbol{n}=(\boldsymbol{\tau}_1\cdot\boldsymbol{n},\boldsymbol{\tau}_2\cdot\boldsymbol{n},\cdots,\boldsymbol{\tau}_d\cdot\boldsymbol{n}).
\end{aligned}
$$

The Sobolev space $H^k(\Omega,\mathbb{X})$ is defined as

$$
H^k(\Omega,\mathbb{X})=\big\{v\in L^2(\Omega,\mathbb{X})\,|\,D^\alpha v\in L^2(\Omega,\mathbb{X}),\ \forall|\alpha|\le k\big\}, \tag{2.1}
$$

where $\mathbb{X}$ ranges $\mathbb{R},\mathbb{R}^d$ or $\mathbb{S}$. When $\mathbb{X}=\mathbb{R}$ or $k=0$, we use $H^k(\Omega)$ or $L^2(\Omega,\mathbb{X})$ instead, respectively. Let $\|\cdot\|_k$ be the standard Sobolev norm of $H^k(\Omega,\mathbb{X})$. The divergence space $H(\mathrm{div},\Omega,\mathbb{S})$ is defined by

$$
H(\mathrm{div},\Omega,\mathbb{S})=\{\boldsymbol{\tau}\in L^2(\Omega,\mathbb{S})\,|\,\mathrm{div}\boldsymbol{\tau}\in L^2(\Omega,\mathbb{R}^d)\} \tag{2.2}
$$

equipped with the norm

$$
\|\boldsymbol{\tau}\|_{H(\mathrm{div})}=\sqrt{\|\boldsymbol{\tau}\|_0^2+\|\mathrm{div}\boldsymbol{\tau}\|_0^2}. \tag{2.3}
$$

Next, we consider the elasticity problem: given a body force $\boldsymbol{f}$, find a symmetric stress tensor $\boldsymbol{\sigma}$ and a displacement $\boldsymbol{u}$ such that

$$
\begin{cases}
\mathbf{A}\boldsymbol{\sigma}=\epsilon(\boldsymbol{u}) & \text{in}\ \ \Omega,\\
-\mathrm{div}\boldsymbol{\sigma}=\boldsymbol{f} & \text{in}\ \ \Omega,\\
\boldsymbol{u}=\mathbf{0} & \text{on}\ \ \Gamma_D,\\
\boldsymbol{\sigma}\cdot\boldsymbol{n}=\boldsymbol{g} & \text{on}\ \ \Gamma_N,
\end{cases} \tag{2.4}
$$

where Γ_D and Γ_N are Dirichlet and Neumann boundary, respectively, $\Gamma_D \cup \Gamma_N = \Gamma$, and $|\Gamma_D| > 0$. The compliance tensor $\mathbf{A}$ and the elasticity tensor $\mathbf{A}^{-1}$ is defined as

$$\mathbf{A}\sigma = \frac{1}{2\mu}\sigma - \frac{\lambda}{2\mu(d\lambda+2\mu)}(\mathrm{tr}\sigma)\mathbf{I}, \tag{2.5a}$$

$$\mathbf{A}^{-1}\epsilon(\boldsymbol{u}) = \lambda \mathrm{tr}(\epsilon(\boldsymbol{u}))\mathbf{I} + 2\mu\epsilon(\boldsymbol{u}), \tag{2.5b}$$

where $\mathbf{I}$ is the identical matrix, and λ and μ are Lamé constants. Let E be Young's modulus, and ν be Poisson's ratio, then

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \quad \mu = \frac{E}{2(1+\nu)}. \tag{2.6}$$

Let

$$\boldsymbol{\Sigma} = H(\mathrm{div},\Omega,\mathbb{S}), \quad \boldsymbol{V} = \{\boldsymbol{v} \in H^1(\Omega,\mathbb{R}^d) | \boldsymbol{v}|_{\Gamma_D} = \mathbf{0}\},$$

then the pure displacement scheme of (2.4) is to find $\boldsymbol{u} \in \boldsymbol{V}$ such that

$$\int_\Omega \mathbf{A}^{-1}\epsilon(\boldsymbol{u}) : \epsilon(\boldsymbol{v}) d\boldsymbol{x} = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v} d\boldsymbol{x} + \int_{\Gamma_N} \boldsymbol{g}\cdot\boldsymbol{v} ds, \quad \forall \boldsymbol{v} \in \boldsymbol{V}. \tag{2.7}$$

By use of (2.7), our new MFE scheme of (2.4) is to find $(\boldsymbol{\sigma},\boldsymbol{u}) \in \boldsymbol{\Sigma}\times\boldsymbol{V}$ such that

$$\left\{ \begin{array}{ll} \displaystyle\int_\Omega \mathbf{A}\boldsymbol{\sigma} : \boldsymbol{\tau} \mathrm{d}\boldsymbol{x} - \int_\Omega \epsilon(\boldsymbol{u}) : \boldsymbol{\tau} d\boldsymbol{x} + \gamma_1 \int_\Omega \mathrm{div}\boldsymbol{\sigma}\cdot\mathrm{div}\boldsymbol{\tau} d\boldsymbol{x} & \\ \quad = -\gamma_1 \displaystyle\int_\Omega \boldsymbol{f}\cdot\mathrm{div}\boldsymbol{\tau} d\boldsymbol{x}, & \forall \boldsymbol{\tau} \in \boldsymbol{\Sigma}, \\ -\displaystyle\int_\Omega \epsilon(\boldsymbol{v}) : \boldsymbol{\sigma} d\boldsymbol{x} - \gamma_2 \int_\Omega \mathbf{A}^{-1}\epsilon(\boldsymbol{u}) : \epsilon(\boldsymbol{v}) d\boldsymbol{x} & \\ \quad = -(1+\gamma_2)\Big(\displaystyle\int_\Omega \boldsymbol{f}\cdot\boldsymbol{v} d\boldsymbol{x} + \int_{\Gamma_N} \boldsymbol{g}\cdot\boldsymbol{v} ds\Big), & \forall \boldsymbol{v} \in \boldsymbol{V}, \end{array} \right. \tag{2.8}$$

where $\gamma_1 > 0$, $\gamma_2 > 0$ are stabilization parameters, and independent of h. In what follows, we let

$$\begin{aligned} Q((\boldsymbol{\sigma},\boldsymbol{u}),(\boldsymbol{\tau},\boldsymbol{v})) = & \int_\Omega \mathbf{A}\boldsymbol{\sigma} : \boldsymbol{\tau} d\boldsymbol{x} - \int_\Omega \boldsymbol{\tau} : \epsilon(\boldsymbol{u}) d\boldsymbol{x} + \int_\Omega \boldsymbol{\sigma} : \epsilon(\boldsymbol{v}) d\boldsymbol{x} \\ & + \gamma_1 \int_\Omega \mathrm{div}\boldsymbol{\sigma}\cdot\mathrm{div}\boldsymbol{\tau} d\boldsymbol{x} + \gamma_2 \int_\Omega \mathbf{A}^{-1}\epsilon(\boldsymbol{u}) : \epsilon(\boldsymbol{v}) d\boldsymbol{x}, \end{aligned} \tag{2.9a}$$

$$F(\boldsymbol{\tau},\boldsymbol{v}) = -\gamma_1 \int_\Omega \boldsymbol{f}\cdot\mathrm{div}\boldsymbol{\tau} d\boldsymbol{x} + (1+\gamma_2)\Big(\int_\Omega \boldsymbol{f}\cdot\boldsymbol{v} d\boldsymbol{x} + \int_{\Gamma_N} \boldsymbol{g}\cdot\boldsymbol{v} ds\Big). \tag{2.9b}$$

We rewrite the problem (2.8) as: find $(\boldsymbol{\sigma},\boldsymbol{u}) \in \boldsymbol{\Sigma}\times\boldsymbol{V}$ such that

$$Q((\boldsymbol{\sigma},\boldsymbol{u}),(\boldsymbol{\tau},\boldsymbol{v})) = F(\boldsymbol{\tau},\boldsymbol{v}), \quad \forall (\boldsymbol{\tau},\boldsymbol{v}) \in \boldsymbol{\Sigma}\times\boldsymbol{V}. \tag{2.10}$$

The norm $|||\cdot|||$ on $\boldsymbol{\Sigma}\times\boldsymbol{V}$ is defined by

$$|||(\boldsymbol{\tau},\boldsymbol{v})|||^2 = \|\boldsymbol{\tau}\|^2_{H(\mathrm{div})} + \|\boldsymbol{v}\|_1^2, \tag{2.11}$$

then $Q(\cdot,\cdot)$ is continuous with respect to $|||\cdot|||$, and there holds

$$Q((\boldsymbol{\sigma},\boldsymbol{u}),(\boldsymbol{\sigma},\boldsymbol{u}))\geq C|||(\boldsymbol{\sigma},\boldsymbol{u})|||^2, \quad \forall(\boldsymbol{\sigma},\boldsymbol{u})\in\boldsymbol{\Sigma}\times\boldsymbol{V}, \tag{2.12}$$

so (2.8) or (2.10) has a unique solution.

Remark 2.1. Our stabilized scheme (2.8) is similar to the Galerkin least squares method in [26], but our scheme is unconditionally stable.

3 The stabilized MFEMs

Let $\mathcal{T}_h$ be a family of regular meshes of Ω. For each $T\in\mathcal{T}_h$, T can be a triangle or a quadrilateral in two dimensions, or a hexahedron or a tetrahedron in three dimensions. For simplicial elements, we consider the affine finite element families

$$P_k=\{q_h\in C^0(\Omega)|q_h|_T\in P_k(T),\ \forall T\in\mathcal{T}_h\}, \tag{3.1}$$

where $P_k(T)$ is the space of k-degree polynomials on T. For quadrilateral and hexahedral elements, we consider the space

$$Q_k=\{q_h\in C^0(\Omega)|q_h|_T=\widehat{q}_h\circ F^{-1},\ \widehat{q}_h\in\hat{Q}_k(\hat{T})\}, \tag{3.2}$$

where $\hat{T}$ is a reference element, $\hat{Q}_k(\hat{T})$ is the space of polynomials on $\hat{T}$ whose degrees do not exceed k in each coordinate direction, and $F:\hat{T}\to T$ is a bilinear or a trilinear mapping. For convenience, in what follows we will use the symbol R_k to represent both kinds of finite element spaces P_k and Q_k. Then we define finite element spaces for the stress and displacement as

$$\begin{cases} \boldsymbol{\Sigma}_h=\{\boldsymbol{\tau}\in\boldsymbol{\Sigma}|\boldsymbol{\tau}|_T\in R_k(T,\mathbb{S})\},\\ \boldsymbol{V}_h=\{\boldsymbol{v}\in\boldsymbol{V}|\boldsymbol{v}|_T\in R_k(T,\mathbb{R}^d)\}. \end{cases} \tag{3.3}$$

Let I_h and J_h be the associated L^2-projection operators on $\boldsymbol{V}_h$ and $\boldsymbol{\Sigma}_h$, respectively. Then the mixed finite element approximation of (2.10) is given as: find $(\boldsymbol{\sigma}_h,\boldsymbol{u}_h)\in\boldsymbol{\Sigma}_h\times\boldsymbol{V}_h$ such that

$$Q((\boldsymbol{\sigma}_h,\boldsymbol{u}_h),(\boldsymbol{\tau}_h,\boldsymbol{v}_h))=F(\boldsymbol{\tau}_h,\boldsymbol{v}_h), \quad \forall(\boldsymbol{\tau}_h,\boldsymbol{v}_h)\in\boldsymbol{\Sigma}_h\times\boldsymbol{V}_h. \tag{3.4}$$

Since $\boldsymbol{\Sigma}_h\times\boldsymbol{V}_h\subset\boldsymbol{\Sigma}\times\boldsymbol{V}$, from (2.12), we have

$$Q((\boldsymbol{\sigma}_h,\boldsymbol{u}_h),(\boldsymbol{\sigma}_h,\boldsymbol{u}_h))\geq C|||(\boldsymbol{\sigma}_h,\boldsymbol{u}_h)|||^2, \quad \forall(\boldsymbol{\sigma}_h,\boldsymbol{u}_h)\in\boldsymbol{\Sigma}_h\times\boldsymbol{V}_h, \tag{3.5}$$

then the scheme (3.4) has a unique solution.

From Céa's lemma, we have the following error estimate.

Theorem 3.1. *Let $(\boldsymbol{\sigma},\boldsymbol{u})$ and $(\boldsymbol{\sigma}_h,\boldsymbol{u}_h)$ be the solutions of* (2.10) *and* (3.4)*, respectively, $(\boldsymbol{\sigma},\boldsymbol{u})\in H^{k+1}(\Omega,\mathbb{S})\times(H^{k+1}(\Omega,\mathbb{R}^d)\cap\boldsymbol{V})$, then we have*

$$\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{H(\mathrm{div})}+\|\boldsymbol{u}-\boldsymbol{u}_h\|_1\leq Ch^k(\|\boldsymbol{\sigma}\|_{k+1}+\|\boldsymbol{u}\|_{k+1}). \tag{3.6}$$

Proof. Firstly, subtracting (3.4) from (2.10), we can get the following error equation

$$Q((\boldsymbol{\sigma}-\boldsymbol{\sigma}_h,\boldsymbol{u}-\boldsymbol{u}_h),(\boldsymbol{\tau}_h,\boldsymbol{v}_h))=0,\quad \forall(\boldsymbol{\tau}_h,\boldsymbol{v}_h)\in\boldsymbol{\Sigma}_h\times\boldsymbol{V}_h. \tag{3.7}$$

From (3.5) and (3.7), we can obtain

$$\begin{aligned}
&C|||(\boldsymbol{\sigma}_h-J_h\boldsymbol{\sigma},\boldsymbol{u}_h-I_h\boldsymbol{u})|||^2\\
\le&Q((\boldsymbol{\sigma}_h-J_h\boldsymbol{\sigma},\boldsymbol{u}_h-I_h\boldsymbol{u}),(\boldsymbol{\sigma}_h-J_h\boldsymbol{\sigma},\boldsymbol{u}_h-I_h\boldsymbol{u}))\\
=&Q((\boldsymbol{\sigma}-J_h\boldsymbol{\sigma},\boldsymbol{u}-I_h\boldsymbol{u}),(\boldsymbol{\sigma}_h-J_h\boldsymbol{\sigma},\boldsymbol{u}_h-I_h\boldsymbol{u}))\\
\le&C|||(\boldsymbol{\sigma}-J_h\boldsymbol{\sigma},\boldsymbol{u}-I_h\boldsymbol{u})|||\,|||(\boldsymbol{\sigma}_h-J_h\boldsymbol{\sigma},\boldsymbol{u}_h-I_h\boldsymbol{u})|||,
\end{aligned} \tag{3.8}$$

that is

$$|||(\boldsymbol{\sigma}_h-J_h\boldsymbol{\sigma},\boldsymbol{u}_h-I_h\boldsymbol{u})|||\le C|||(\boldsymbol{\sigma}-J_h\boldsymbol{\sigma},\boldsymbol{u}-I_h\boldsymbol{u})|||. \tag{3.9}$$

Then by the triangle inequality and interpolation theory we have

$$|||(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h,\boldsymbol{u}-\boldsymbol{u}_h)|||\le C|||(\boldsymbol{\sigma}-J_h\boldsymbol{\sigma},\boldsymbol{u}-I_h\boldsymbol{u})|||\le Ch^k(\|\boldsymbol{\sigma}\|_{k+1}+\|\boldsymbol{u}\|_{k+1}). \tag{3.10}$$

The proof is completed. ☐

Remark 3.1. For any $\gamma_1>0$, $\gamma_2>0$, we can obtain the optimal error estimates for the H^1-norm of the displacement and $H(\mathrm{div})$-norm of the stress at the same time. If take $\gamma_1=0$, or $\gamma_1=\mathcal{O}(h^2)$ as that in [26], our method is still stable, and we can obtain the optimal error estimate of L^2-norm for the displacement by Aubin-Nitsche duality technique, but can not get the error estimate of the $H(\mathrm{div})$-norm for the stress.

4 Numerical implementation

In this part, we test the stability and accuracy of the stabilization method. To compare the results of different elements, we employ the following relative error

$$e_{\boldsymbol{u},0}=\frac{\|\boldsymbol{u}-\boldsymbol{u}_h\|_0}{\|\boldsymbol{u}\|_0},\qquad e_{\boldsymbol{u},1}=\frac{\|\boldsymbol{u}-\boldsymbol{u}_h\|_1}{\|\boldsymbol{u}\|_1}, \tag{4.1a}$$

$$e_{\boldsymbol{\sigma},0}=\frac{\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_0}{\|\boldsymbol{\sigma}\|_0},\qquad e_{\boldsymbol{\sigma},\mathrm{div}}=\frac{\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{H(\mathrm{div})}}{\|\boldsymbol{\sigma}\|_{H(\mathrm{div})}}. \tag{4.1b}$$

Next we compute two examples.

Example 4.1. This is a two dimensional example with $\Omega=[0,1]^2$, and the exact displacement u is given as

$$\boldsymbol{u}=\begin{pmatrix} e^{(x_1-x_2)}x_1(1-x_1)x_2(1-x_2)\\ \sin(\pi x_1)\sin(\pi x_2)\end{pmatrix}. \tag{4.2}$$

Table 1: The errors of the P_1-P_1 pair for $\gamma_1=1$, $\gamma_2=1$.

	$e_{u,0}$	order	$e_{u,1}$	order	$e_{\sigma,0}$	order	$e_{\sigma,\mathrm{div}}$	order
1	1.27E-01		3.90E-01		1.95E-01		5.30E-01	
2	3.32E-02	1.94019	1.96E-01	0.98980	1.34E-01	0.54079	2.74E-01	0.95018
3	8.36E-03	1.99100	9.83E-02	0.99828	7.61E-02	0.81705	1.43E-01	0.94233
4	2.08E-03	2.00363	4.92E-02	0.99985	3.22E-02	1.24036	7.47E-02	0.93363
5	5.21E-04	1.99954	2.46E-02	1.00004	1.14E-02	1.49419	3.85E-02	0.95728

Table 2: The errors of the P_1-P_1 pair for $\gamma_1=1$, $\gamma_2=0.1$.

	$e_{u,0}$	order	$e_{u,1}$	order	$e_{\sigma,0}$	order	$e_{\sigma,\mathrm{div}}$	order
1	3.55E-01		6.47E-01		1.92E-01		5.31E-01	
2	9.76E-02	1.86369	2.40E-01	1.43067	1.34E-01	0.51768	2.74E-01	0.95240
3	2.66E-02	1.87545	1.05E-01	1.19209	7.61E-02	0.81613	1.43E-01	0.94293
4	7.13E-03	1.89818	5.01E-02	1.06648	3.22E-02	1.24065	7.47E-02	0.93377
5	1.85E-03	1.94408	2.47E-02	1.02047	1.14E-02	1.49416	3.85E-02	0.95733

Table 3: The errors of the P_1-P_1 pair for $\gamma_1=1$, $\gamma_2=0.01$.

	$e_{u,0}$	order	$e_{u,1}$	order	$e_{\sigma,0}$	order	$e_{\sigma,\mathrm{div}}$	order
1	1.15E+00		1.73E+00		1.91E-01		5.40E-01	
2	2.99E-01	1.94364	4.66E-01	1.89038	1.34E-01	0.51019	2.76E-01	0.97112
3	7.99E-02	1.90203	1.49E-01	1.64133	7.61E-02	0.81878	1.43E-01	0.94804
4	2.13E-02	1.90504	5.81E-02	1.36036	3.22E-02	1.24183	7.47E-02	0.93509
5	5.53E-03	1.94979	2.59E-02	1.16370	1.14E-02	1.49410	3.85E-02	0.95768

Table 4: The errors of the P_1-P_1 pair for $\gamma_1=0.1$, $\gamma_2=1$.

	$e_{u,0}$	order	$e_{u,1}$	order	$e_{\sigma,0}$	order	$e_{\sigma,\mathrm{div}}$	order
1	1.20E-01		3.89E-01		1.45E-01		5.49E-01	
2	3.08E-02	1.96213	1.96E-01	0.98659	5.47E-02	1.41108	2.93E-01	0.90508
3	7.80E-03	1.98358	9.83E-02	0.99759	1.86E-02	1.55452	1.53E-01	0.94208
4	1.96E-03	1.99279	4.92E-02	0.99955	6.07E-03	1.61605	7.78E-02	0.97268
5	4.91E-04	1.99589	2.46E-02	0.99991	2.00E-03	1.60456	3.92E-02	0.98807

Let $E=1$, $\nu=0.3$. The computations are carried out on the uniform rectangular and triangular meshes, and the triangular meshes are obtained from the uniform rectangular meshes by inserting diagonal edges. The mesh size h is taken as $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, $\frac{1}{32}$, $\frac{1}{64}$. We compute the P_1-P_1 pair and Q_1-Q_1 pair with different stabilization parameters, to check the influence of stabilization parameters on numerical results, and the errors and convergence orders are shown in Tables 1-10 and Figs. 1-2.

From Tables 1-10 and Figs. 1-2, we can see that the convergence orders of the L^2-norm for the displacement and stress are 2.0 and 1.0-2.0, respectively, which are higher than our theoretical analysis. The H^1-norm error of the displacement and the $H(\mathrm{div})$-norm error of the stress can achieve one order, which accord with our theoretical analysis. The error

Table 5: The errors of the P_1-P_1 pair for $\gamma_1=0.01$, $\gamma_2=1$.

	$e_{u,0}$	order	$e_{u,1}$	order	$e_{\sigma,0}$	order	$e_{\sigma,\text{div}}$	order
1	1.18E-01		3.89E-01		1.25E-01		6.39E-01	
2	2.98E-02	1.98242	1.96E-01	0.98645	3.17E-02	1.98011	3.16E-01	1.01542
3	7.46E-03	1.99581	9.82E-02	0.99702	8.07E-03	1.97275	1.58E-01	1.00324
4	1.87E-03	1.99890	4.91E-02	0.99928	2.10E-03	1.94351	7.88E-02	0.99938
5	4.67E-04	1.99948	2.46E-02	0.99982	5.63E-04	1.89702	3.94E-02	0.99890

Table 6: The errors of the Q_1-Q_1 pair for $\gamma_1=1$, $\gamma_2=1$.

	$e_{u,0}$	order	$e_{u,1}$	order	$e_{\sigma,0}$	order	$e_{\sigma,\text{div}}$	order
1	4.21E-02		2.28E-01		1.81E-01		2.99E-01	
2	9.73E-03	2.11259	1.14E-01	1.00323	9.23E-02	0.96983	1.61E-01	0.89393
3	2.32E-03	2.06813	5.69E-02	1.00139	3.30E-02	1.48275	8.62E-02	0.89965
4	5.68E-04	2.03057	2.84E-02	1.00052	1.03E-02	1.68553	4.45E-02	0.95529
5	1.41E-04	2.01203	1.42E-02	1.00016	3.11E-03	1.72534	2.25E-02	0.98412

Table 7: The errors of the Q_1-Q_1 pair for $\gamma_1=1$, $\gamma_2=0.1$.

	$e_{u,0}$	order	$e_{u,1}$	order	$e_{\sigma,0}$	order	$e_{\sigma,\text{div}}$	order
1	1.54E-01		3.05E-01		1.80E-01		2.99E-01	
2	4.60E-02	1.73963	1.29E-01	1.24828	9.20E-02	0.96724	1.61E-01	0.89450
3	1.29E-02	1.83006	5.93E-02	1.11723	3.29E-02	1.48231	8.62E-02	0.89987
4	3.41E-03	1.92173	2.88E-02	1.04294	1.02E-02	1.68469	4.45E-02	0.95540
5	8.72E-04	1.96805	1.43E-02	1.01251	3.10E-03	1.72423	2.25E-02	0.98416

Table 8: The errors of the Q_1-Q_1 pair for $\gamma_1=1$, $\gamma_2=0.01$.

	$e_{u,0}$	order	$e_{u,1}$	order	$e_{\sigma,0}$	order	$e_{\sigma,\text{div}}$	order
1	4.45E-01		5.66E-01		1.79E-01		3.01E-01	
2	1.26E-01	1.81919	1.93E-01	1.55374	9.14E-02	0.96785	1.61E-01	0.90263
3	3.47E-02	1.86463	7.31E-02	1.39898	3.27E-02	1.48239	8.63E-02	0.90222
4	9.06E-03	1.93639	3.11E-02	1.23437	1.02E-02	1.68376	4.45E-02	0.95609
5	2.30E-03	1.97624	1.46E-02	1.09299	3.09E-03	1.72271	2.25E-02	0.98435

Table 9: The errors of the Q_1-Q_1 pair for $\gamma_1=0.1$, $\gamma_2=1$.

	$e_{u,0}$	order	$e_{u,1}$	order	$e_{\sigma,0}$	order	$e_{\sigma,\text{div}}$	order
1	4.09E-02		2.29E-01		6.99E-02		3.35E-01	
2	9.87E-03	2.05164	1.14E-01	1.00510	1.99E-02	1.81240	1.76E-01	0.92819
3	2.44E-03	2.01847	5.69E-02	1.00127	5.66E-03	1.81366	8.96E-02	0.97472
4	6.07E-04	2.00637	2.84E-02	1.00034	1.64E-03	1.78262	4.51E-02	0.99160
5	1.51E-04	2.00221	1.42E-02	1.00009	4.98E-04	1.72293	2.26E-02	0.99716

estimate of L^2-norm for the stress is suboptimal, while that of the L^2-norm and H^1-norm for the displacement, and the $H(div)$-norm for the stress can all achieve optimal.

Table 10: The errors of the Q_1-Q_1 pair for $\gamma_1=0.01$, $\gamma_2=1$.

	$e_{u,0}$	order	$e_{u,1}$	order	$e_{\sigma,0}$	order	$e_{\sigma,\text{div}}$	order
1	4.24E-02		2.28E-01		4.56E-02		3.54E-01	
2	1.05E-02	2.01318	1.14E-01	1.00370	1.11E-02	2.04061	1.80E-01	0.97901
3	2.62E-03	2.00335	5.69E-02	1.00064	2.77E-03	2.00307	9.03E-02	0.99392
4	6.55E-04	2.00096	2.84E-02	1.00014	6.98E-04	1.98581	4.52E-02	0.99821
5	1.64E-04	2.00029	1.42E-02	1.00004	1.78E-04	1.96795	2.26E-02	0.99944

When reduce γ_2 and keep γ_1 unchanged, the L^2-norm errors of the displacement increase, the H^1-norm errors of the displacement increase slightly, and the L^2-norm and $H(div)$- norm errors of the stress have no obvious change. When reduce γ_1 and keep γ_2 unchanged, the L^2-norm errors of the stress reduce, the $H(div)$-norm errors of the stress increase slightly, and the L^2-norm and H^1-norm errors of the displacement have no obvious change.

Then we give the errors and convergence orders of the P_2-P_2 pair and Q_2-Q_2 pair in Tables 11-12 and Fig. 3. In the computation, we take $\gamma_1=1$, $\gamma_2=1$ for both pairs.

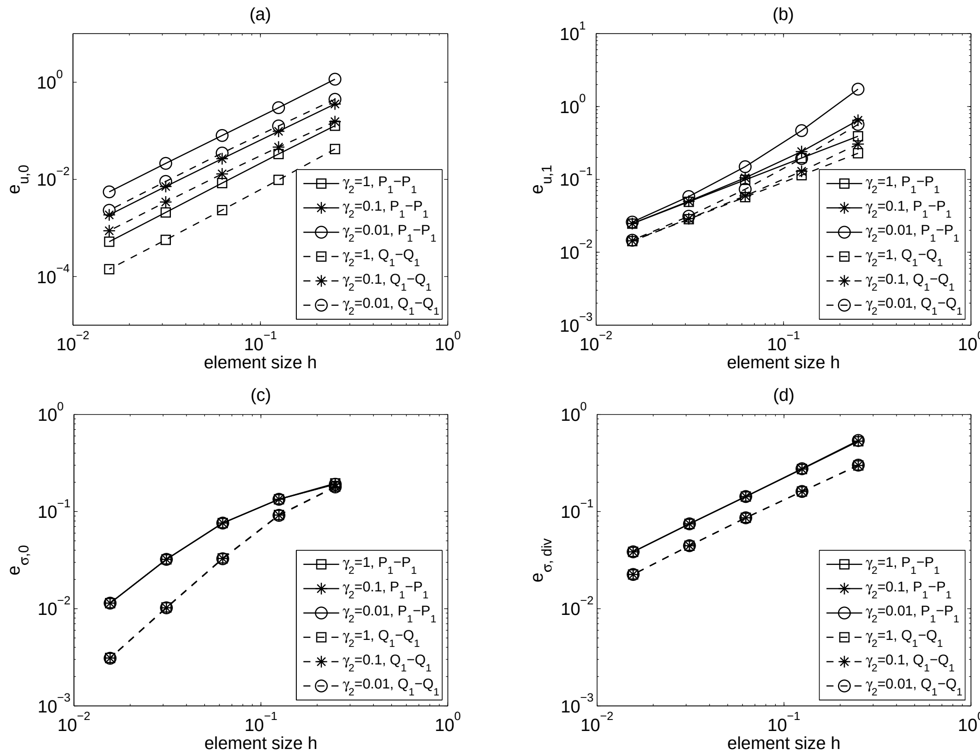

Figure 1: The errors of the P_1-P_1 pair and Q_1-Q_1 pair with $\gamma_1=1$ and $\gamma_2=1,0.1,0.01$. (a) L^2-norm of the displacement; (b) H^1-norm of the displacement; (c) L^2-norm of the stress; (d) $H(div)$-norm of the stress.

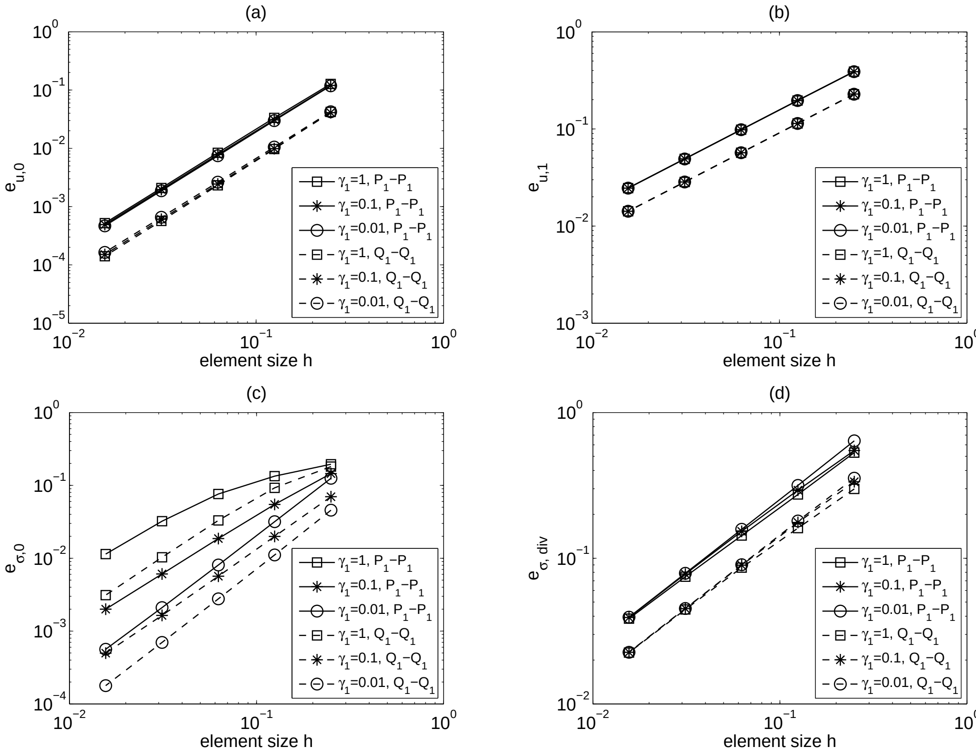

Figure 2: The errors of the P_1-P_1 pair and Q_1-Q_1 pair with $\gamma_1=1,0.1,0.01$ and $\gamma_2=1$. (a) L^2-norm of the displacement; (b) H^1-norm of the displacement; (c) L^2-norm of the stress; (d) $H(div)$-norm of the stress.

Table 11: The errors of P_2-P_2 element for $\gamma_1=1$, $\gamma_2=1$.

	$e_{u,0}$	order	$e_{u,1}$	order	$e_{\sigma,0}$	order	$e_{\sigma,\text{div}}$	order
1	7.36E-03		8.52E-02		3.06E-02		1.29E-01	
2	5.59E-04	3.71772	2.04E-02	2.06077	7.12E-03	2.10405	3.33E-02	1.95726
3	4.27E-05	3.71032	5.02E-03	2.02468	1.73E-03	2.03814	8.41E-03	1.98622
4	3.84E-06	3.47779	1.25E-03	2.00729	4.30E-04	2.01024	2.11E-03	1.99514
5	4.16E-07	3.20584	3.12E-04	2.00194	1.07E-04	2.00259	5.28E-04	1.99809

Table 12: The errors of Q_2-Q_2 element for $\gamma_1=1$, $\gamma_2=1$.

	$e_{u,0}$	order	$e_{u,1}$	order	$e_{\sigma,0}$	order	$e_{\sigma,\text{div}}$	order
1	4.02E-03		2.33E-02		1.40E-02		3.13E-02	
2	4.97E-04	3.01773	5.78E-03	2.01059	2.38E-03	2.55726	8.27E-03	1.91985
3	6.18E-05	3.00734	1.44E-03	2.00426	4.71E-04	2.33833	2.11E-03	1.97387
4	7.71E-06	3.00215	3.60E-04	2.00125	1.02E-04	2.20213	5.30E-04	1.98995
5	9.64E-07	3.00058	8.99E-05	2.00033	2.38E-05	2.10525	1.33E-04	1.99592

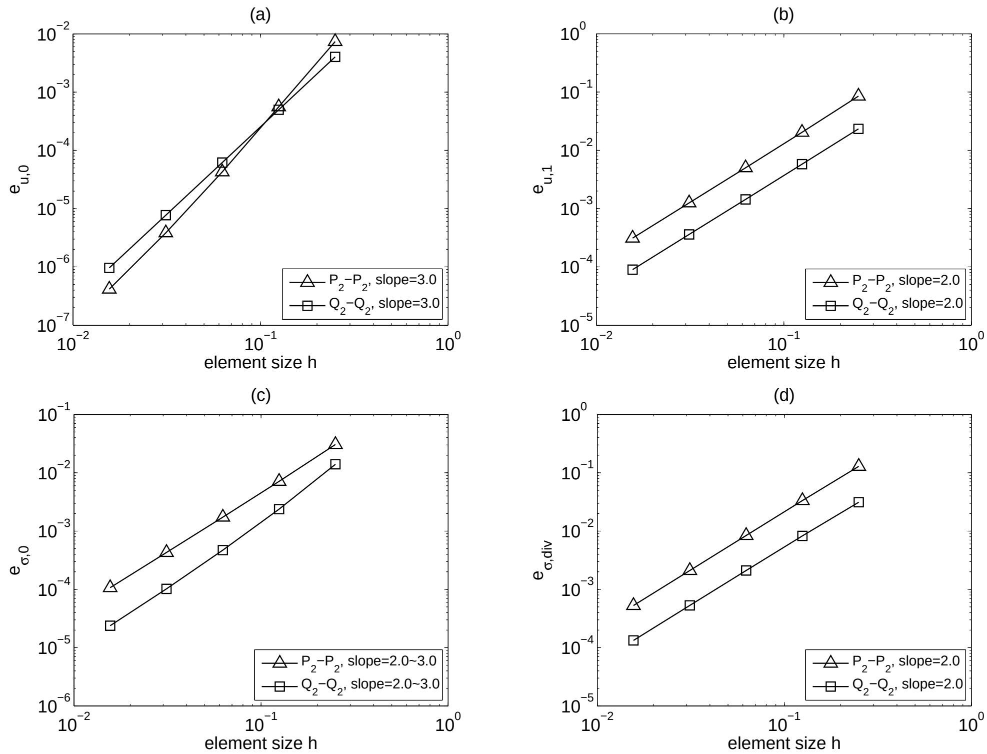

Figure 3: The errors of the P_2-P_2 pair and Q_2-Q_2 pair. (a) L^2-norm of the displacement; (b) H^1-norm of the displacement; (c) L^2-norm of the stress; (d) $H(div)$-norm of the stress.

From Tables 11-12 and Fig. 3, we can see that the convergence orders of the L^2-norm for the displacement and the stress are 3.0 and 2.0-3.0, respectively, which are higher than our theoretical analysis. The H^1-norm error of the displacement and $H(\text{div})$-norm error of the stress can achieve two order convergence, which accord with our theoretical analysis.

Example 4.2. We compute the following three dimensional example with $\Omega=[0,1]^3$

$$\boldsymbol{u}=\left(\begin{array}{c} 2x_1(1-x_1)x_2(1-x_2)x_3(1-x_3) \\ 4x_1(1-x_1)x_2(1-x_2)x_3(1-x_3) \\ 6x_1(1-x_1)x_2(1-x_2)x_3(1-x_3) \end{array}\right). \tag{4.3}$$

Let $E=1$, $\nu=0.3$. The mesh size h is taken as $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, $\frac{1}{32}$. We give the errors and convergence orders of P_1-P_1 and Q_1-Q_1 pairs in Tables 13-14 and Fig. 4. In the computation, we take $\gamma_1=0.1$, $\gamma_2=1$ for P_1-P_1 pair and $\gamma_1=1$, $\gamma_2=1$ for Q_1-Q_1.

From Tables 13-14 and Fig. 4, we can see that our method can keep the accuracy and the convergence order in 3D as that in 2D.

Table 13: The errors of P_1-P_1 element with $\gamma_1=0.1$, $\gamma_2=1$.

	$e_{u,0}$	order	$e_{u,1}$	order	$e_{\sigma,0}$	order	$e_{\sigma,\text{div}}$	order
1	1.41E-01		4.34E-01		1.75E-01		2.54E-01	
2	3.27E-02	2.10783	2.22E-01	0.96851	7.41E-02	1.24200	1.39E-01	0.87278
3	7.84E-03	2.05988	1.12E-01	0.98986	3.14E-02	1.24099	7.43E-02	0.89963
4	1.91E-03	2.03296	5.61E-02	0.99529	1.29E-02	1.27710	3.90E-02	0.93017

Table 14: The errors of Q_1-Q_1 element with $\gamma_1=1$, $\gamma_2=1$.

	$e_{u,0}$	order	$e_{u,1}$	order	$e_{\sigma,0}$	order	$e_{\sigma,\text{div}}$	order
1	7.71E-02		2.55E-01		1.52E-01		8.04E-02	
2	1.90E-02	2.02371	1.26E-01	1.02014	5.16E-02	1.56034	4.28E-02	0.90808
3	4.74E-03	2.00155	6.26E-02	1.00516	1.88E-02	1.45699	2.23E-02	0.94460
4	1.19E-03	1.98990	3.13E-02	1.00129	7.50E-03	1.32630	1.14E-02	0.96668

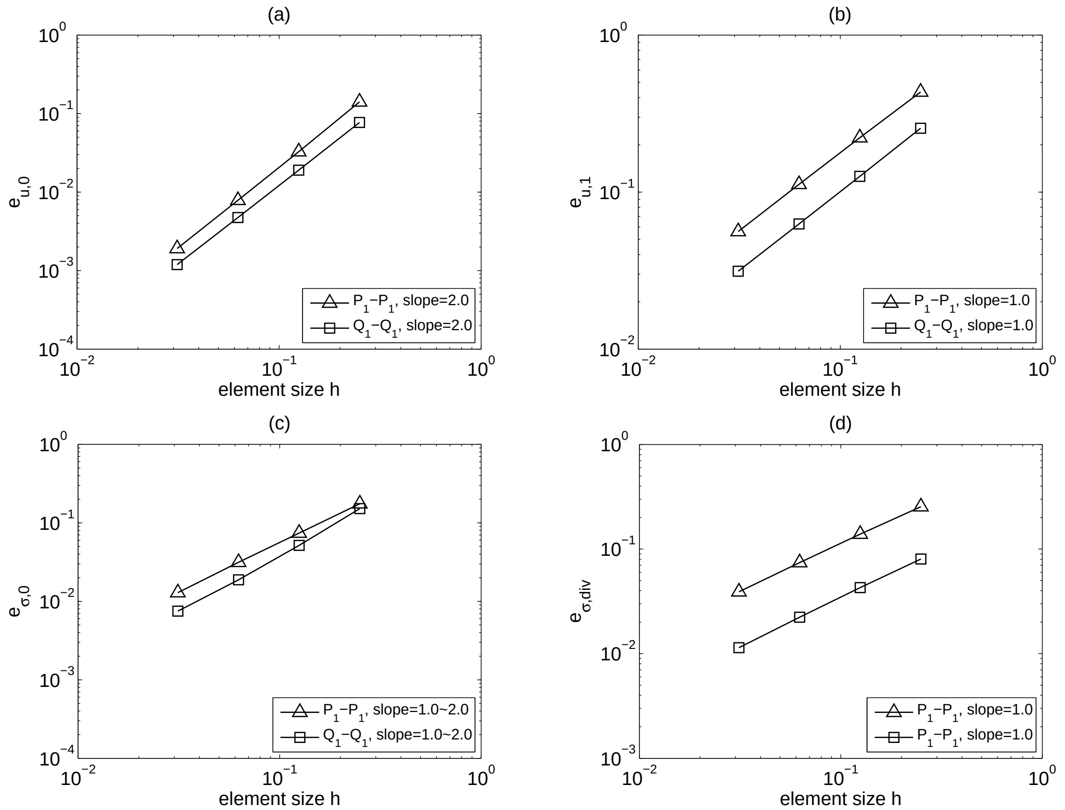

Figure 4: The errors of the P_1-P_1 pair and Q_1-Q_1 pair in 3D. (a) L^2-norm of the displacement; (b) H^1-norm of the displacement; (c) L^2-norm of the stress; (d) $H(div)$-norm of the stress.

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References

[1] C. JOHNSON AND B. MERCIER, *Some equilibrium finite element methods for two-dimensional elasticity problems,* Numer. Math., 30 (1978), pp. 103–116.

[2] D. N. ARNOLD, F. BREZZI AND J. DOUGLAS, *PEERS: a new mixed finite element for plane elasticity,* Japan J. Appl. Math., 1(2) (1984), pp. 347–367.

[3] R. STENBERG, *On the construction of optimal mixed finite element methods for the linear elasticity problem,* Numer. Math., 48(4) (1986), pp. 447–462.

[4] D. N. ARNOLD, R. S. FALK AND R. WINTHER, *Mixed finite element methods for linear elasticity with weakly imposed symmetry,* Math. Comput., 76(260) (2007), pp. 1699–1723.

[5] D. BOFFI, F. BREZZI AND M. FORTIN, *Reduced symmetry elements in linear elasticity,* Commun. Pure Appl. Anal., 8(1) (2009), pp. 95–121.

[6] D. N. ARNOLD, G. AWANOU AND W. QIU, *Mixed finite elements for elasticity on quadrilateral meshes,* Adv. Comput. Math., 41(3) (2015), pp. 553–572.

[7] D. N. ARNOLD AND R. WINTHER, *Mixed finite element for elasticity,* Numer. Math., 92(3) (2002), pp. 401–419.

[8] D. N. ARNOLD AND G. AWANOU, *Rectangular mixed finite elements for elasticity,* Math. Models Methods Appl. Sci., 15(9) (2005), pp. 1417–1429.

[9] D. N. ARNOLD AND R. WINTHER, *Nonconforming mixed elements for elasticity,* Math. Models Methods Appl. Sci., 13(3) (2003), pp. 295–307.

[10] J. HU AND Z. C. SHI, *Lower order rectangular nonconforming mixed elements for plane elasticity,* SIAM J. Numer. Anal., 46(1) (2007), pp. 88–102.

[11] D. N. ARNOLD, G. AWANOU AND R. WINTHER, *Finite elements for symmetric tensors in three dimensions,* Math. Comput., 77(263) (2008), pp. 1229–1251.

[12] S. C. CHEN AND Y. N. WANG, *Conforming rectangular mixed finite elements for elasticity,* J. Sci. Comput., 47(1) (2011), pp. 93–108.

[13] H. MAN, J. HU AND Z. SHI, *Lower order rectangular nonconforming mixed finite element for the three dimensional elasticity problem,* Math. Models Methods Appl. Sci., 19(1) (2009), pp. 51–65.

[14] G. AWANOU, *Two remarks on rectangular mixed finite elements for elastic1ity,* J. Sci. Comput., 50(1) (2012), pp. 91–102.

[15] J. GOPALAKRISHNAN AND J. GUZMÁN, *Symmetric nonconforming mixed finite elementsfor linear elasticity,* SIAM J. Numer. Anal., 49(4) (2011), pp. 1504–1520.

[16] S. Y. YI, *Nonconforming mixed finite element methods for linear elasticity using rectangular elements in two and three dimensions,* Calcolo, 42(2) (2005), pp. 115–133.

[17] D. SHI AND M. H. LI, *Superconvergence analysis of a stable conforming rectangular mixed finite elements for the linear elasticity,* J. Comput. Math., 32(2) (2014), pp. 205–214.

[18] C. CARSTENSEN, D. GÜNTHER, J. REININGHAUS AND J. THIELE, *The Arnold-Winther mixed FEM in linear elasticity: Part I: Implementation and numerical verification,* Comput. Methods Appl. Mech. Eng., 197(33) (2008), pp. 3014–3023.

[19] J. HU, H. MAN, J. WANG AND S. ZHANG, *The simplest nonconforming mixed finite element method for linear elasticity in the symmetric formulation on n-rectanglar grids,* Comput. Math. Appl., 71(7) (2016), pp. 1317–1336.

[20] J. HU, H. MAN AND S. ZHANG, *A simple conforming mixed finite element for linear elasticity on rectangular grids in any space dimension,* J. Sci. Comput., 58(2) (2014), pp. 367–379.

[21] J. HU, *A new family of efficient rectangular conforming mixed finite elements for linear elasticity in the symmetric formulation,* SIAM J. Numer. Anal., 53(3) (2015), pp. 1438–1464.

[22] S. C. CHEN, Y. P. SUN AND J. K. ZHAO, *The simplest conforming anisotropic rectangular and cubic mixed finite elements for elasticity,* Appl. Math. Comput., 265 (2015), pp. 292–303.

[23] J. HU, *Finite element approximations of symmetric tensors on simplicial grids in $\mathbb{R}^n$: the high order case,* J. Comput. Math., 33(3) (2015), pp. 283–296.

[24] J. HU AND S. Y. ZHANG, *A family of symmetric mixed finite elements for linear elasticity on tetrahedral grids,* Sci. China Math., 58(2) (2015), pp. 297–307.

[25] J. HU AND S. ZHANG, *Finite element approximations of symmetric tensors on simplicial grids in $\mathbb{R}^n$: the lower order case,* Math. Model. Meth. Appl. Sci., 26(9) (2016), pp. 1649–1669.

[26] L. P. FRANCE AND R. STENBERG, *Error analysis of some Galerkin least squares methods for the elasticity equations,* SIAM J. Numer. Anal., 28(6) (1991), pp. 1680–1697.

[27] M. H. LI, D. Y. SHI AND Y. DAI, *The Brezzi-Pitkäranta stabilization scheme for the elasticity problem,* J. Comput. Appl. Math., 286(1) (2015), pp. 7–16.

[28] M. CERVERA, M. CHIUMENTI AND R. CODINA, *Mixed stabilized finite element methods in nonlinear solid mechanics: Part I: Formulation,* Comput. Methods Appl. Mech. Eng., 199(37) (2010), pp. 2559–2570.

[29] Z. LI, S. CHEN AND S. QU ET AL., *Stabilization of low-order mixed finite elements for the plane elasticity equations,* Comput. Math. Appl., 73(3) (2017), pp. 363–373.

[30] D. Y. SHI, M. H. LI AND C. XU, *A new stabilization method for the elasticity problem,* J. Sci. Comput., 65(3) (2015), pp. 1025–1038.

[31] L. CHEN, J. HU AND X. HUANG, *Stabilized mixed finite element methods for linear elasticity on simplicial grids in $\mathbb{R}^n$,* Comput. Meth. Appl. Math., 17(1) (2017), pp. 17–31.

[32] Z. CAI AND X. YE, *A mixed nonconforming finite element for linear elasticity,* Numer. Methods Partial Differential Equations, 21 (2005), pp. 1043–1051.

[33] Z. Q. CAI AND G. STARKE, *First-order system least squares for the stress-displacement formulation: linear elasticity,* SIAM J. Sci. Comput., 41(2) (2003), pp. 715–730.

[34] Z. Q. CAI AND G. STARKE, *Least-squares methods for linear elasticity,* SIAM J. Numer. Anal., 42(2) (2004), pp. 826–842.

[35] Z. Q. CAI, J. KORSAWE AND G. STARKE, *An adaptive least squares mixed finite element method for the stress-displacement formulation of linear elasticity,* Numer. Methods Partial Differential Equations, 21 (2005), pp. 32–148.

[36] H.Y. DUAN AND Q. LIN, *Mixed finite elements of least-squares type for elasticity,* Comput. Methods Appl. Mech. Eng., 194(9) (2005), pp. 1093–1112.