An Inf-Sup Stabilized Finite Element Method by Multiscale Functions for the Stokes Equations

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Received 15 November 2008; Accepted (in revised version) 16 January 2009

Available online 17 March 2009

Abstract. In the paper, an inf-sup stabilized finite element method by multiscale functions for the Stokes equations is discussed. The key idea is to use a Petrov-Galerkin approach based on the enrichment of the standard polynomial space for the velocity component with multiscale functions. The inf-sup condition for $P_1-P_0$ triangular element (or $Q_1-P_0$ quadrilateral element) is established. The optimal error estimates of the stabilized finite element method for the Stokes equations are obtained.

AMS subject classifications: 76D05, 65N30, 35K60

Key words: stabilized finite element method; multiscale functions; Petrov-Galerkin approach; inf-sup condition.

1 Introduction

In fluid dynamics, the Stokes equations model the slow flows of incompressible fluids or alternatively isotropic incompressible elastic materials. The Stokes equations have also become an important model for designing and analyzing finite element algorithms because some of the problems encountered for solving the Navier-Stokes equations already appear in the Stokes equations which are of simpler form. In particular, it gives the right setting for studying the stability problems connected with the choice of finite element spaces for the velocity and the pressure. It is well known that the finite element spaces cannot be chosen independently when the discretization...
is based on the Galerkin variational form, because it is very important to ensure the compatibility of the approximations of velocity and pressure (see, e.g., [19]).

It is well known that the simplest conforming low order elements like the $P_1(Q_1) - P_0$ (linear(bilinear) velocity, constant pressure) element is not stable. To overcome the limitation, many kinds of stabilized finite element methods have been proposed for the Stokes or Navier-Stokes equations. Brezzi and Pitkäranta in [3] firstly proposed the stabilized finite element method for $P_1 - P_1$ triangular element. Later, many stabilized methods have been proposed by relaxing the incompressibility constraint (i.e., modifying the second equation of (2.3)), see, e.g., [1–3, 6, 14–16, 22]. Furthermore, a general locally stabilized mixed finite element method was provided by Kechkar and Silvester in [20]. In [12], a new locally stabilized method based on the idea of [20] containing the jump terms across the inter-element boundaries of the macro elements was derived, which is called bubble condensation procedure. A particular kind of bubble functions of the velocity space is obtained by the residual free bubble method (RFBM) (see, e.g., [1, 9]), in which the bubble functions are the solutions of a problem containing the residual of the continuous equation at the element level. At the same time, the stabilized finite element method by multiscale functions was derived by [11], and a priori error analysis can be found in [10]. A main characteristic of the above methods is to use the Petrov-Galerkin approach to split the solution into two parts, i.e., the trial function space is enriched with the bubble functions which are the solutions to a local problem containing the residual of the momentum equation and special boundary conditions so that the local problem can be solved analytically.

In the paper, we use the Petrov-Galerkin approach based on the enrichment of the standard polynomial space for the velocity component with multiscale functions to propose a new stabilized finite element method for the Stokes equations. Although the main idea is derived from [10] and [11], our method is different from the one in [11] because the multiscale functions are new and the jump term which is introduced to the Galerkin variational formulation can be calculated no longer on the element boundary.

The remaining part of this paper is organized as follows. In the next section, we present the general framework and derive a stabilized finite element method for the Stokes equations. We then analyze the inf-sup stable condition for $P_1(Q_1) - P_0$ element and obtain the optimal error estimate.

2 Stabilized FEM by multiscale functions

Let $\Omega$ be an open bounded domain in $\mathbb{R}^d$ ($d=2$ or $3$) with Lipschitz boundary $\partial \Omega$. We consider the following Stokes equations:

\[-\nu \Delta u + \nabla p = f, \quad \text{in } \Omega, \quad (2.1)\]
\[\nabla \cdot u = 0, \quad \text{in } \Omega, \quad (2.2)\]
\[u = 0, \quad \text{on } \partial \Omega, \quad (2.3)\]
where \( u=(u_1(x), \ldots, u_d(x)) \) represents the velocity vector, \( p=p(x) \) the pressure, \( f=f(x) \in L^2(\Omega)^d \) the prescribed body force, and \( \nu \) the viscosity coefficient.

For the mathematical setting of the problem (2.1)-(2.3), we introduce Hilbert spaces:

\[
X = H^1_0(\Omega)^d, \quad M = L^2(\Omega) \triangleq \{ q \in L^2(\Omega) : \int_{\Omega} q dx = 0 \}.
\]

Furthermore, the space \( L^2(\Omega)^d \) are endowed with the \( L^2 \)-scalar product and \( L^2 \)-norm denoted by \( (\cdot, \cdot)_\Omega \) and \( \| \cdot \|_{0, \Omega} \). The space \( X \) are equipped with their usual scalar product and norm

\[
((u, v)) = (\nabla u, \nabla v)_\Omega, \quad |u|_{1, \Omega} = ((u, u))^{1/2}.
\]

Define Laplace operator \( A \) by

\[
Au = -\Delta u, \quad \forall u \in D(A) = H^2(\Omega)^d \cap X.
\]

Let

\[
B_0((u, p); (v, q)) = \nu(\nabla u, \nabla v)_\Omega - (p, \nabla \cdot v)_\Omega + (q, \nabla \cdot u)_\Omega.
\]

It is easy to check that \( B_0 \) satisfies the following important properties (see, e.g., [5, 13, 20]):

\[
\begin{align*}
\nu|u|^2_{1,\Omega} &= B_0((u, p); (u, p)), \quad (2.4a) \\
|B_0((u, p); (v, q))| &\leq \gamma (|u|_{1,\Omega} + \|p\|_{0,\Omega}) (|v|_{1,\Omega} + \|q\|_{0,\Omega}), \quad (2.4b) \\
\alpha_0(|u|_{1,\Omega} + \|p\|_{0,\Omega}) &\leq \sup_{(v, q) \in (X, M)} \frac{B_0((u, p); (v, q))}{|v|_{1,\Omega} + \|q\|_{0,\Omega}}, \quad (2.4c)
\end{align*}
\]

where \( \gamma > 0 \) and \( \alpha_0 > 0 \).

Under the above notations, the standard Galerkin variational formulation of the problem (2.3) reads as follows: find \((u, p) \in (X, M)\) such that

\[
B_0((u, p); (v, q)) = (f, v)_\Omega, \quad \forall (v, q) \in (X, M). \quad (2.5)
\]

As for the existence and uniqueness of the solution of Stokes equations, we have the classical results as follows (see [13] Chapter IV and [23] Chapter II):

**Theorem 2.1.** Assume that \( \Omega \) is an open bounded domain in \( \mathbb{R}^d \) with Lipschitz boundary \( \partial \Omega \). Then the problem (2.3) admits a unique solution \((u, p)\) such that

\[
|u|_{2,\Omega} + |p|_{1,\Omega} \leq C(v) \|f\|_{0,\Omega}. \quad (2.6)
\]

**Remarks 2.1.** The validity of Theorem 2.1 is known (see [17, 21]) if \( \partial \Omega \) is of \( C^2 \), or if \( \Omega \) is a two-dimensional convex polygon.
Let $\Omega$ be characterized by $\{\mathcal{T}_h\}_{h>0}$ into triangles (quadrilaterals) in the usual sense (see [19], [20]), i.e., for some $\sigma$ and $\lambda$ with $\sigma>1$ and $0<\lambda<1$,
\[
\begin{align*}
     h_K &\leq \sigma \rho_K, & \forall K \in \mathcal{T}_h, \\
    |\cos \theta_{ik}| &\leq \lambda, & i=1,2,3,4, \forall K \in \mathcal{T}_h,
\end{align*}
\]
where $h_K$ is the diameter of element $K$, $\rho_K$ is the diameter of the inscribed circle of element $K$, and $\theta_{ik}$ are the angles of $K$ in the case of a quadrilateral partitioning. The mesh parameter $h$ is given by $h=\max \{h_K : K \in \mathcal{T}_h\}$, and the set of inter-element boundaries is denoted by $\Gamma_h$.

The finite element subspaces in this paper are defined by
\[
\begin{align*}
     X_h &= \{v \in C^0(\Omega)^d \cap X : v|_K \in R_1(K)^d, \forall K \in \mathcal{T}_h\}, \\
     M_h &= \{q \in M : q|_K \in P_0(K), \forall K \in \mathcal{T}_h\},
\end{align*}
\]
where $R_1(K)$ is defined by
\[
R_1(K) = \begin{cases} 
    P_1(K) & \text{if } K \text{ is triangular,} \\
    Q_1(K) & \text{if } K \text{ is quadrilateral.}
\end{cases}
\]
Here $P_n(K)$ and $Q_n(K)$ are the set of all polynomials on $K$ of degree less than $n$ and $n = 1, 2, \cdots$. Let $E_h$ be a finite dimensional space, called multiscale space, such that
\[
E_h \subset H^1(\mathcal{T}_h)^d, \quad E_h \cap X_h = \{0\},
\]
where
\[
H^1(\mathcal{T}_h)^d = \{v \in L^2(\Omega)^d : v|_K \in H^1(K)^d\}.
\]
Under the above notations, we have the Petrov-Galerkin variational formulation of the Stokes equations: find $u_h + u_e \in X_h \oplus E_h$ and $p_h \in M_h$ such that
\[
\nu\left(\nabla(u_h + u_e), \nabla v\right)_\Omega - (p_h, \nabla \cdot v)_\Omega + (q_h, \nabla \cdot (u_h + u_e))_\Omega = (f, v)_\Omega, \quad (2.7)
\]
for all $v \in X_h \oplus E_h^0$ and $q_h \in M_h$, where
\[
E_h^0 = \{v \in H^1(\mathcal{T}_h)^d : v|_K \in H^1_0(K)^d\}.
\]
It follows from (2.7) that
\[
\begin{align*}
    \nu\left(\nabla(u_h + u_e), \nabla v_h\right)_\Omega - (p_h, \nabla \cdot v_h)_\Omega + (q_h, \nabla \cdot (u_h + u_e))_\Omega &= (f, v_h)_\Omega, \quad \forall (v_h, q_h) \in X_h \times M_h, \\
    \nu\left(\nabla(u_h + u_e), \nabla v_K\right)_K - (p_h, \nabla \cdot v_K)_K &= (f, v_K)_K, \quad \forall v_K \in H^1_0(K)^d, \forall K \in \mathcal{T}_h.
\end{align*}
\]
From (2.9), it follows that
\[
(-\nu \Delta u_h, v_K)_K = (f + \nu \Delta u_h, v_K)_K, \quad \forall v_K \in H^1_0(K)^d, \forall K \in \mathcal{T}_h, \quad (2.10)
\]
which implies that $u_e$ is the strong solution of the local problem

$$-\nu \Delta u_e = f + \nu \Delta u_h, \quad \text{in } K,$$  \hspace{1cm} (2.11)

for all $(u_h, p_h) \in X_h \times M_h$.

Denote the mean value operator by $\langle \cdot \rangle$, i.e., if $v$ is a vector field experiencing a discontinuity across an element boundary, then

$$\langle v \cdot n \rangle_{\partial K} = \frac{n^+ v^+ - n^- v^-}{2} = \frac{n^+ v^+ + v^- n^+}{2} = n \cdot \left( \frac{v^+ + v^-}{2} \right),$$

where $n = n^+ - n^-$, $v^+ = v|_{\partial K^+}$ and $v^- = v|_{\partial K^-}$.

In order to give an expression for $u_e$ in terms of $u_h, p_h$ and $f$ on each element $K$, we need to impose some special boundary condition and obtain the following local problem

$$-\nu \Delta u_e^K = f + \nu \Delta u_h, \quad \text{in } K,$$  \hspace{1cm} (2.12a)

$$-\nu \Delta u_e^{\partial K} = 0, \quad \text{in } K,$$  \hspace{1cm} (2.12b)

$$u_e^K|_{\partial K} = 0, \quad u_e^{\partial K} = g_e, \quad \text{on } \partial K,$$  \hspace{1cm} (2.12c)

$$-\nu \partial s g_e = \frac{1}{h_e} \langle \nu \partial_n u_h + p_h I \cdot n \rangle, \quad \forall s \in T,$$  \hspace{1cm} (2.12d)

$$g_e = 0, \quad \text{at the nodes},$$  \hspace{1cm} (2.12e)

where $u_e^K = u_e^K + u_e^{\partial K}, h_e \triangleq \text{mes } T$ denotes the length of the edge $T$, $n$ is the normal outward vector on $\partial K$, $\partial_s, \partial_n$ are the tangential and normal derivative operators, respectively, and $I$ is the $\mathbb{R}^{d \times d}$ identity matrix.

It is easy to check that problem (2.12) is well posed, i.e., $u_e$ can be solved in terms of $u_h, p_h$ and $f$ on each element $K$.

For convenience, we define two operators

$$\mathcal{H}_K : L^2(K)^d \rightarrow H^1_0(K)^d, \quad \mathcal{J}_K : L^2(\partial K)^d \rightarrow H^1_0(\partial K)^d,$$

by

$$u_e^K = \frac{1}{\nu} \mathcal{H}_K (f + \nu \Delta u_h), \quad \forall K \in T_h,$$  \hspace{1cm} (2.13a)

$$u_e^{\partial K} = \frac{1}{h_e \nu} \mathcal{J}_K \left( \langle \nu \partial_n u_h + p_h I \cdot n \rangle \right), \quad \forall K \in T_h.$$  \hspace{1cm} (2.13b)

Integrating by parts on $K \in T_h$, we have

$$\nu \langle \nabla u_e, \nabla v_h \rangle_K = -\nu (u_e, \Delta v_h)_K + (u_e, \nu \partial_n v_h)_{\partial K}, \quad \forall v_h \in X_h,$$  \hspace{1cm} (2.14a)

$$\langle q_h, \nabla \cdot u_e \rangle_K = -(u_e, \nabla q_h)_K + (u_e, q_h I \cdot n)_{\partial K}, \quad \forall q_h \in M_h.$$  \hspace{1cm} (2.14b)
Substituting (2.14) into (2.8) yields
\[
\sum_{K \in T_h} \left[ \nu (\nabla u_h, \nabla v_h)_K - (p_h, \nabla \cdot v_h)_K + (q_h, \nabla \cdot u_h)_K - (u_e, \nu \Delta v_h)_K
\right.
\]
\[
+ (u_e, \nu \partial_n v_h)_{\partial K} + \sum_{K \in T_h} \left[ - (u_e, \nabla q_h)_K + (u_e, q_h I \cdot n)_{\partial K} \right]
\]
\[
= \sum_{K \in T_h} (f, v_h)_K,
\]  
(2.15)
which implies that
\[
v (\nabla u_h, \nabla v_h)_K - (p_h, \nabla \cdot v_h)_K + (q_h, \nabla \cdot u_h)_K - (u_e, \nu \Delta v_h + \nabla q_h)_K
\]
\[
+ (u_e, \nu \partial_n v_h + q_h I \cdot n)_{\partial K} = (f, v_h)_K,
\]  
(2.16)
for \((u_h, p_h), (v_h, q_h) \in X_h \times M_h, u_e \in E_h\) and for each \(K \in T_h\). Combining (2.13) with (2.16) yields
\[
v (\nabla u_h, \nabla v_h)_K - (p_h, \nabla \cdot v_h)_K + (q_h, \nabla \cdot u_h)_K
\]
\[
+ \frac{1}{v} (\mathcal{H}_K (-v \Delta u_h) - \frac{1}{h_e} \mathcal{J}_K (v \partial_n u_h + p_h I \cdot n), v \Delta v_h + \nabla q_h)_K
\]
\[
+ \frac{1}{h_e v} (\mathcal{J}_K ((v \partial_n u_h + p_h I \cdot n), v \partial_n v_h + q_h I \cdot n))_{\partial K}
\]
\[
= (f, v_h)_K + \frac{1}{v} (\mathcal{H}_K (f), v \Delta v_h + \nabla q_h)_K.
\]  
(2.17)
From (2.17), we propose the following approach: find \((u_h, p_h) \in X_h \times M_h\) such that
\[
B((u_h, p_h); (v_h, q_h)) = (f, v_h)_{\Omega_h}, \quad \forall (v_h, q_h) \in X_h \times M_h,
\]  
(2.18)
where
\[
B((u_h, p_h); (v_h, q_h)) = B_0 ((u_h, p_h); (v_h, q_h))
\]
\[
+ \sum_{T \in T_h} \beta_T \left[ (v \partial_n u_h + p_h I \cdot n), (v \partial_n v_h + q_h I \cdot n) \right]_T.
\]  
(2.19)
Next, we calculate the parameter \(\beta_T\). Firstly, we define the matrix function \(A_K\) by
\[
A_K = \left( \mathcal{J}_K (\phi_1) | \mathcal{J}_K (\phi_2) | \cdots | \mathcal{J}_K (\phi_d) \right),
\]
where \(\phi_i (i = 1, \cdots, d)\) is a group of basis of \(\mathbb{R}^d\). From the definition, we have \(A_K = a_K I\), where \(a_K\) is the solution of
\[
-\Delta a_K = 0, \quad \text{in } K, \quad a_K = g(s), \quad \text{on each } T \subset \partial K,
\]  
(2.20)
where \(g = 0\) if \(T \subset \partial \Omega\), and in the internal edges \(g\) satisfies
\[
-\partial_n g(s) = \frac{1}{h_e}, \quad \text{on } T, \quad g = 0, \quad \text{at the nodes}.
\]  
(2.21)
Since \((u_h, p_h), (v_h, q_h) \in X_h \times M_h\), (2.17) reduces to

\[
\sum_{K \in T_h} \left[ \nu \left( \nabla u_h, \nabla v_h \right)_K - \left( p_h, \nabla \cdot v_h \right)_K + \left( q_h, \nabla \cdot u_h \right)_K \right] + \sum_{T \in \Gamma_h} \frac{1}{h_T} \nu \left( J_K \left( \langle \nu \partial_n u_h + p_h I \cdot n \rangle, \nu \partial_n v_h + q_h I \cdot n \right) \right)_T = \sum_{K \in T_h} (f, v_h)_K. \tag{2.22}
\]

Secondly, we claim that \(\langle \nu \partial_n u_h + p_h I \cdot n \rangle |_{T \subset \partial K}\) is a constant vector function on every \(T\). By using (2.20) and (2.21), we have

\[
\frac{(a_K, 1)_T}{h_T} = \frac{h_T}{6}.
\]

It is easy to check that

\[
\frac{1}{h_T} (J_K \left( \langle \nu \partial_n u_h + p_h I \cdot n \rangle, \langle \nu \partial_n v_h + q_h I \cdot n \rangle \right)_T \right)
= \left( \int_T A_K \langle \nu \partial_n u_h + p_h I \cdot n \rangle \right) \langle \nu \partial_n v_h + q_h I \cdot n \rangle |_T
= \frac{h_T}{6} \left( \langle \nu \partial_n u_h + p_h I \cdot n \rangle, \langle \nu \partial_n v_h + q_h I \cdot n \rangle \right). \tag{2.23}
\]

Comparing (2.23) with (2.19) gives \(\beta_T = h_T / (6 \nu)\).

### 3 Error estimates

To derive the error estimates of the approximate solution \((u_h, p_h)\), we use the canonical interpolation operator \(I_h\): \(X \rightarrow X_h\) defined by

\[
\int_T \left( I_h u - u \right) ds = 0, \quad \forall T \in \Gamma_h,
\]

and the \(L^2\)-projection operator \(P_h\): \(L^2(\Omega) \rightarrow M_h\). Then the following approximation estimates hold (see, e.g., [5,7,13]):

\[
|v - I_h v|_{m,K} \leq C_1 h_K^{2-m} |v|_{2,K}, \quad \forall v \in H^2(K), \tag{3.1}
\]

\[
|v - I_h v|_{m,T} \leq C_2 h_T^{2-m-1/2} |v|_{2,\partial K}, \quad \forall v \in H^2(\partial K), \tag{3.2}
\]

\[
\|v\|_{0,\partial K} \leq C_3 (h_K^{-1} \|v\|_{0,K} + h_K |v|_{1,K}), \quad \forall v \in H^1(K), \tag{3.3}
\]

\[
\|q - P_h q\|_{0,\Omega} \leq C_4 h |q|_{1,\Omega}, \quad \forall q \in H^1(\Omega), \tag{3.4}
\]

where \(m=0,1\) and \(C_i \ (i=1, \cdots, 4)\) are positive constants independent of \(h\).
Remarks 3.1. The inequality (3.3) is very important which is called by the local trace theorem.

Remarks 3.2. From (3.3) and (3.4), we deduce that
\[
\begin{align*}
  h_c \| \langle q - P_h q \rangle \|_{0,T}^2 \\
  \leq C_3 \left( \| q - P_h q \|_{0,K}^2 + h_k^2 \| \langle q - P_h q \rangle \|_{1,K}^2 \right) \leq Ch^2 |q|_{1,K}^2,
\end{align*}
\]
which leads to
\[
\left( \sum_{T \in T_h} h_c \| \langle q - P_h q \rangle \|_{0,T}^2 \right)^{1/2} \leq C_3 h |q|_{1,\Omega}, \quad \forall q \in H^1(\Omega). \tag{3.5}
\]

Define the mesh-dependent norm
\[
\|(u, p)\|_h = \left( v|u|^2_{1,\Omega} + \sum_{T \in T_h} \beta_T \| (v \partial_n u + p I \cdot n) \|_{0,T}^2 \right)^{1/2}, \tag{3.6}
\]
Next, we give the inf-sup stable result for $P_1 - P_0$ element as follows:

Lemma 3.1. There exists $\beta_1 > 0$ such that
\[
\sup_{0 \neq (v_h, q_h) \in X_h \times M_h} \frac{B((u_h, p_h); (v_h, q_h))}{\| (v_h |v_h|^2_{1,\Omega} + \| q_h \|_{0,\Omega}^2) \|_{1/2}} \geq \beta_1 \left( |u_h|_{1,\Omega} + \| p_h \|_{0,\Omega} \right)^{1/2}, \tag{3.7}
\]
for all $(u_h, p_h) \in X_h \times M_h$.

Proof. From the continuous version of inf-sup condition (see [13]), we know that for each $p_h \in M_h \subseteq M$ there exist a function $w \in X$ and a finite element approximation $w_h \in X_h$ of $w$ such that $|w|_{1,\Omega} = \| p_h \|_{0,\Omega}$ and
\[
\begin{align*}
  (\nabla \cdot w, p_h)_{\Omega} &\geq C_6 \| p_h \|_{0,\Omega} |w|_{1,\Omega}, \tag{3.8} \\
  \left( \sum_{K \in T_h} h_k^{-2} \| w - w_h \|_{0,K}^2 \right)^{1/2} &\leq C |w|_{1,\Omega}, \quad |w_h|_{1,\Omega} \leq C |w|_{1,\Omega}. \tag{3.9}
\end{align*}
\]
Using the Cauchy-Schwartz inequality and (3.8), we have
\[
\begin{align*}
  B((u_h, p_h); (-w, 0)) &= -v(\nabla u_h, \nabla w) - \sum_{K \in T_h} \beta_T (\langle v \partial_n u_h \rangle, \langle v \partial_n w \rangle)_{\partial K} \\
  &\quad + (\nabla \cdot w, p_h)_{\Omega} - \sum_{K \in T_h} \beta_T (\langle p_h I \cdot n \rangle, \langle v \partial_n w \rangle)_{\partial K} \\
  &\geq -v|u_h|_{1,\Omega} |w|_{1,\Omega} + C_6 \| p_h \|_{0,\Omega} |w|_{1,\Omega} \\
  &\quad - \sum_{K \in T_h} \beta_T \| (v \partial_n u_h + p_h I \cdot n) \|_{0,T} \| (v \partial_n w) \|_{0,T} \\
  &\geq -\left( v|u_h|^2_{1,\Omega} + \sum_{T \in T_h} \beta_T \| (v \partial_n u_h + p_h I \cdot n) \|_{0,T}^2 \right)^{1/2} |w|_{1,\Omega} \\
  &\quad + \sum_{T \in T_h} \beta_T \| (v \partial_n w) \|_{0,T}^2 \right)^{1/2} + C_6 \| p_h \|_{0,\Omega} |w|_{1,\Omega}. \tag{3.10}
\end{align*}
\]
Using (3.3) and the inverse inequality, we obtain
\[
\beta_T \| \langle \nu \partial_n w \rangle \|_{0,T}^2 \leq \frac{h_T}{6v} (h_K - 1) |v \nabla w \cdot n|^2_{0,K} + h_K |v \nabla w \cdot n|^2_{1,K}
\]
\[
\leq \frac{vh_T}{6h_K} |w|^2_{1,K} + \frac{vh_T}{6} C_K h_K^{-1} |w|^2_{1,K}
\]
\[
\leq \frac{v(1 + C_K)}{6} |w|^2_{1,K}.
\]
(3.11)
Combining (3.10) with (3.11) yields
\[
B((u_h, p_h); (-w, 0)) \geq -\sqrt{C_v} \|w\|_{1, \Omega} \left( |v|_{u_h}^2_{1, \Omega} + \sum_{k \in h} \beta_T \| \langle \partial_n u_h + p_k I \cdot n \rangle \|_{0,T}^2 \right)^{1/2} + C_6 \| p_h \|_{0, \Omega} \| w \|_{1, \Omega}
\]
\[
= -\sqrt{C_v} |w|_{1, \Omega} |(u_h, p_h)|_h + C_6 \| p_h \|_{0, \Omega} \| w \|_{1, \Omega}
\]
\[
= -\sqrt{C_v} \| p_h \|_{0, \Omega} |(u_h, p_h)|_h + C_6 \| p_h \|_{0, \Omega}^2
\]
\[
\geq -C_v \gamma_1^{-1} |(u_h, p_h)|_h^2 + [C_6 - \gamma_1] \| p_h \|_{0, \Omega}^2
\]
(3.12)
where $C = (7 + C_0)/6$ with $C_0 = \max_{k \in h} C_K$, and $\gamma_1$ is chosen small enough. Let
\[
(v_h, q_h) = (u_h - \delta w, p_h), \quad \delta > 0.
\]
Using (3.12) we have
\[
B((u_h, p_h); (v_h, q_h))
\]
\[
= B((u_h, p_h); (u_h, p_h)) + \delta B((u_h, p_h); (-w, 0))
\]
\[
\geq |(u_h, p_h)|_h^2 + \delta \left( -C_v \gamma_1^{-1} |(u_h, p_h)|_h^2 + [C_6 - \gamma_1] \| p_h \|_{0, \Omega}^2 \right)
\]
\[
\geq (1 - \delta C_v \gamma_1^{-1}) |(u_h, p_h)|_h^2 + \delta (C_6 - \gamma_1) \| p_h \|_{0, \Omega}^2
\]
\[
\geq (1 - \delta C_v \gamma_1^{-1}) |u_h|_{1, \Omega}^2 + \delta (C_6 - \gamma_1) \| p_h \|_{0, \Omega}^2
\]
(3.13)
provided that $0 < \delta < \gamma_1/(C_v)$ and $0 < \gamma_1 < C_6$. Denote
\[
C(\nu) \triangleq \min \{ (1 - \delta C_v \gamma_1^{-1}), \delta (C_6 - \gamma_1) \}, \quad C(\delta) \triangleq \max \{ 2, 1 + 2\delta^2 \}.
\]
Then we have
\[
|v_h|_{1, \Omega}^2 + \| q_h \|_{0, \Omega}^2 = |u_h - \delta w|_{1, \Omega}^2 + \| p_h \|_{0, \Omega}^2
\]
\[
\leq 2 |u_h|_{1, \Omega}^2 + (2\delta^2 + 1) \| p_h \|_{0, \Omega}^2 \leq C(\delta) \left( |u_h|_{1, \Omega}^2 + \| p_h \|_{0, \Omega}^2 \right).
\]
(3.14)
Taking $\beta_1 = C(\nu)/C(\delta)$ ends the proof. □
Lemma 3.2. Let \((u, p) \in [H^2(\Omega)^d \cap X] \times [H^1(\Omega) \cap M]\) be the solution of (2.5) and assume that \((u_h, p_h) \in X_h \times M_h\) satisfy (2.18). Then there exists a positive constant \(C\) independent of \(v\) and \(h\) such that

\[
|B((u - u_h, p - p_h); (v_h, q_h))| \leq C h \left( \sqrt{v}|u|_{2,\Omega} + \frac{1}{\sqrt{v}}|p|_{1,\Omega} \right) |(v_h, q_h)|_h. \tag{3.15}
\]

Proof. Since \((u, p) \in [H^2(\Omega)^d \cap X] \times [H^1(\Omega) \cap M]\) is the solution of (2.5) and \((u_h, p_h)\) satisfies (2.18), we have

\[
B((u - u_h, p - p_h); (v_h, q_h)) = B_0((u - u_h, p - p_h); (v_h, q_h)) + \sum_{T \in \Gamma_h} \beta_T \left( \langle v \partial_n (u - u_h) + (p - p_h) I \cdot n, \langle v \partial_n v_h + q_h I \cdot n \rangle \right)_T
= \sum_{T \in \Gamma_h} \beta_T \left( \langle v \partial_n u + p I \cdot n, \langle v \partial_n v_h + q_h I \cdot n \rangle \right)_T
= \sum_{T \in \Gamma_h} \beta_T \left( \langle v \partial_n u + p I \cdot n, \langle v \partial_n v_h + q_h I \cdot n \rangle \right)_T. \tag{3.16}
\]

Using (3.16), (3.3), the Cauchy-Schwartz inequality and Lemma 3.1, we obtain

\[
B((u - u_h, p - p_h); (v_h, q_h)) \leq \sum_{T \in \Gamma_h} \beta_T^{1/2} \| v \partial_n u + p I \cdot n \|_{0,T} \beta_T^{1/2} \| v \partial_n v_h + q_h I \cdot n \|_{0,T}.
\]

\[
\leq \sum_{T \in \Gamma_h} \beta_T^{1/2} \left( \| v \partial_n u \|_{0,T} + \| p I \cdot n \|_{0,T} \right) |(v_h, q_h)|_h
\leq \sum_{T \in \Gamma_h} \beta_T^{1/2} \left( \| v \partial_n u \|_{0,T} + \| p I \cdot n \|_{0,T} \right) |(v_h, q_h)|_h
\leq C h \left( \sqrt{v}|u|_{2,\Omega} + \frac{1}{\sqrt{v}}|p|_{1,\Omega} \right) |(v_h, q_h)|_h, \tag{3.17}
\]

where \(C\) is a positive constant independent of \(v\) and \(h\). □

Lemma 3.3. Let \((u, p) \in [H^2(\Omega)^d \cap X] \times [H^1(\Omega) \cap M]\) be the solution of (2.5). Then there exists a positive constant \(C\) independent of \(v\) and \(h\) such that

\[
|B((u - I_h u, p - P_h p); (v_h, q_h))| \leq C h \left( \sqrt{v}|u|_{2,\Omega} + \frac{1}{\sqrt{v}}|p|_{1,\Omega} \right) |(v_h, q_h)|_h, \tag{3.18}
\]

for all \((v_h, q_h) \in X_h \times M_h\).

Proof. By using the orthogonality of the \(L^2\)-projection \(P_h\), we have

\[
(p - P_h p, \nabla \cdot v_h) = 0, \quad \forall v_h \in X_h. \tag{3.19}
\]
From the definition of the canonical interpolation operator $I_h$ and the fact that $q_h$ is piecewise constant, it follows that
\[
(q_h, \nabla \cdot (u - I_h u))_K = (q_h, (u - I_h u) \cdot n)_{\partial K} = 0.
\] (3.20)

Using Cauchy-Schwarz inequality and (3.1), we have
\[
v(\nabla (u - I_h u), \nabla v_h)_K \\
\leq \sqrt{v} \| \nabla (u - I_h u) \|_{0,K} \sqrt{v} \| \nabla v_h \|_{0,K} \\
\leq C \sqrt{v} h_k \| u \|_{2,K} (v_h, q_h)_h.
\] (3.21)

Again, applying Cauchy-Schwarz inequality and using (3.3) and (3.4), we get
\[
\sum_{T \in \mathcal{G}_h} \beta_T \left( \langle v \partial_n (u - I_h u) + (p - P_h p) I \cdot n \rangle, \langle v \partial_n v_h + q_h I \cdot n \rangle \right)_T \\
\leq \sum_{T \in \mathcal{G}_h} \beta_T^{1/2} \| \langle v \partial_n (u - I_h u) + (p - P_h p) I \cdot n \rangle \|_{0,T} \| (v_h, q_h) \|_h \\
\leq \sum_{K \in \mathcal{T}_h} C h \left( \sqrt{v} \| u \|_{2,\Omega} + \frac{1}{\sqrt{v}} \| p \|_{1,\Omega} \right) (v_h, q_h)_h.
\] (3.22)

From (3.19)-(3.22), it follows that
\[
B((u - I_h u, p - P_h p); (v_h, q_h)) \\
= v(\nabla (u - I_h u), \nabla v_h)_{\Omega} + \sum_{T \in \mathcal{G}_h} \beta_T \left( \langle v \partial_n (u - I_h u) + (p - P_h p) I \cdot n \rangle, \langle v \partial_n v_h + q_h I \cdot n \rangle \right)_T \\
\leq C h \left( \sqrt{v} \| u \|_{2,\Omega} + \frac{1}{\sqrt{v}} \| p \|_{1,\Omega} \right) (v_h, q_h)_h.
\]

\]

**Lemma 3.4.** There exists a positive constant $\beta_2$ independent of $h$ and $v$ such that
\[
\sup_{0 \neq (v_h, q_h) \in X_h \times M_h} \frac{B((u_h, p_h); (v_h, q_h))}{(v_h, q_h)_h} \geq \beta_2 \| (u_h, p_h) \|_h,
\] (3.23)

for all $(u_h, p_h) \in X_h \times M_h$.

**Proof.** Using (3.10), we have
\[
B((u_h, p_h); (-w, 0)) \geq -\| (u_h, p_h) \|_h \| (v_h, q_h) \|_h + C_6 \| p_h \|_{0,\Omega}^2,
\] (3.24)

which by using the Young’s inequality implies that
\[
B((u_h, p_h); (-w, 0)) \geq -\frac{1}{2\gamma_2} \| (u_h, p_h) \|_h^2 - \frac{\gamma_2}{2} \| (v_h, q_h) \|_h^2 + C_6 \| p_h \|_{0,\Omega}^2.
\] (3.25)
provided that $\gamma_2$ is chosen sufficiently small. Denote

$$(v_h, q_h) = (u_h - \delta w, p_h), \quad \delta > 0.$$  

Using (3.25) gives

$$B((u_h, p_h); (v_h, q_h)) = B((u_h, p_h); (u_h, p_h)) + \delta B((u_h, p_h); (-w, 0))$$

$$\geq |(u_h, p_h)|^2_h + \delta \left( - \frac{1}{2\gamma_2} |(u_h, p_h)|^2_h - \frac{\gamma_2}{2} |(v_h, q_h)|^2_h + C_6 \|p_h\|^2_{0, \Omega} \right)$$

$$\geq (1 - \frac{\delta}{2\gamma_2}) |(u_h, p_h)|^2_h + \delta \left( C_6 \|p_h\|^2_{0, \Omega} - \frac{\gamma_2}{2} |(v_h, q_h)|^2_h \right), \quad (3.26)$$

where $\delta$ and $\gamma_2$ are chosen to satisfy $0 < \delta < 2\gamma_2$ and $\gamma_2$ is small enough. Note that

$$|(v_h, q_h)|^2_h = |(u_h - \delta w, p_h)|^2_h \leq |(u_h, p_h)|^2_h. \quad (3.27)$$

Taking $\beta_2 = 1 - \delta / (2\gamma_2)$ ends the proof. \hfill \Box

**Theorem 3.1.** Let $(u, p) \in [H^2(\Omega)^d \cap X] \times [H^1(\Omega) \cap M]$ be the solution of (2.5) and $(u_h, p_h) \in X_h \times M_h$ be the solution of (2.18). Then the following error estimate holds:

$$|(u - u_h, p - p_h)|_h \leq C \left( \sqrt{v} |u|_{2, \Omega} + \frac{1}{\sqrt{v}} |p|_{1, \Omega} \right). \quad (3.28)$$

**Proof.** Starting with Lemma 3.4 we have

$$|(u_h - \mathcal{I}_hu, p_h - \mathcal{P}_hp)|_h$$

$$\leq \frac{1}{\beta_2} \sup_{(v_h, q_h) \in X_h \times M_h} \frac{B((u_h - \mathcal{I}_hu, p_h - \mathcal{P}_hp); (v_h, q_h))}{|(v_h, q_h)|_h}$$

$$\leq \frac{1}{\beta_2} \sup_{(v_h, q_h) \in X_h \times M_h} \frac{B((u - u_h, p - p); (v_h, q_h))}{|(v_h, q_h)|_h} + \frac{1}{\beta_2} \sup_{(v_h, q_h) \in X_h \times M_h} \frac{B((u - \mathcal{I}_hu, p - \mathcal{P}_hp); (v_h, q_h))}{|(v_h, q_h)|_h}. \quad (3.29)$$

Combining (3.15) with (3.18) and (3.29) yields

$$|(u_h - \mathcal{I}_hu, p_h - \mathcal{P}_hp)|_h \leq C \left( \sqrt{v} |u|_{2, \Omega} + \frac{1}{\sqrt{v}} |p|_{1, \Omega} \right), \quad (3.30)$$

which implies that

$$|(u - u_h, p - p_h)|_h \leq |(u - \mathcal{I}_hu, p - \mathcal{P}_hp)|_h + |(u_h - \mathcal{I}_hu, p_h - \mathcal{P}_hp)|_h$$

$$\leq C \left( \sqrt{v} |u|_{2, \Omega} + \frac{1}{\sqrt{v}} |p|_{1, \Omega} \right).$$

This completes the proof of the theorem. \hfill \Box
Theorem 3.2. Let \((u, p) \in [H^2(\Omega)^d \cap X] \times [H^1(\Omega) \cap M]\) be the solution of (2.5) and \((u_h, p_h) \in X_h \times M_h\) be the solution of (2.18). Then the following error estimate holds
\[
\| p - p_h \|_{0,\Omega} \leq Ch. \quad (3.31)
\]

Proof. By using Lemma 3.1, we have
\[
\| p_h - \mathcal{P}_h(p) \|_{0,\Omega} \leq \frac{1}{\beta_1} \sup_{(v_h, q_h) \in X_h \times M_h} \frac{B\left( (u_h - \mathcal{I}_h u, p_h - \mathcal{P}_h p); (v_h, q_h) \right)}{|v_h|_{1,\Omega} + \| q_h \|_{0,\Omega}}.
\]
\[
\leq \frac{1}{\beta_1} \sup_{(v_h, q_h) \in X_h \times M_h} \frac{B\left( (u - u_h, p - p_h); (v_h, q_h) \right)}{|v_h|_{1,\Omega} + \| q_h \|_{0,\Omega}} + \frac{1}{\beta_1} \sup_{(v_h, q_h) \in X_h \times M_h} \frac{B\left( (u - \mathcal{I}_h u, p - \mathcal{P}_h p); (v_h, q_h) \right)}{|v_h|_{1,\Omega} + \| q_h \|_{0,\Omega}}. \quad (3.32)
\]
Since \((v_h, q_h) \in X_h \times M_h\), we conclude that there exists \(C > 0\) such that
\[
|v_h|^2_{1,\Omega} + \| q_h \|^2_{0,\Omega} \leq C, \quad \|(v_h, q_h)\|_h \leq C. \quad (3.33)
\]
Combining (3.15) with (3.18) and (3.32)-(3.33) yields
\[
\| p_h - \mathcal{P}_h p \|_{0,\Omega} \leq Ch. \quad (3.34)
\]
Therefore, by (3.4) and the triangle inequality we have
\[
\| p - p_h \|_{0,\Omega} \leq \| p - \mathcal{P}_h p \|_{0,\Omega} + \| p_h - \mathcal{P}_h p \|_{0,\Omega} \leq Ch. \quad \square
\]

Theorem 3.3. Let \((u, p) \in [H^2(\Omega)^d \cap X] \times [H^1(\Omega) \cap M]\) be the solution of (2.5) and \((u_h, p_h) \in X_h \times M_h\) satisfy (2.18). Then the following error estimate holds
\[
\| u - u_h \|_{0,\Omega} \leq Ch^2 \left( \sqrt{\nu} |u|_{2,\Omega} + \frac{1}{\sqrt{\nu}} |p|_{1,\Omega} \right). \quad (3.35)
\]

Proof. Firstly, we consider the following duality Stokes problem:
\[
\begin{cases}
-\nu \Delta v - \nabla q = u_h - u, & \nabla \cdot v = 0, \quad \text{in } \Omega, \\
v = 0, & \text{on } \partial \Omega.
\end{cases} \quad (3.36)
\]
By Theorem 2.1, we have
\[
\nu \| v \|_{2,\Omega} + \| q \|_{1,\Omega} \leq C \| u - u_h \|_{0,\Omega}. \quad (3.37)
\]
Multiplying (3.36) by \((u_h - u)\) gives
\[
\|u - u_h\|^2_{0,\Omega} = v(\nabla v, \nabla (u_h - u))_\Omega + (q, \nabla \cdot (u_h - u))_\Omega - (p_h - p, \nabla \cdot v)_\Omega \\
\leq B((u_h - u, p_h - p); (v - \mathcal{I}_h v, q - \mathcal{P}_h q)) + |B((u_h - u, p_h - p); (\mathcal{I}_h v, \mathcal{P}_h q))|
\leq C\left|(u - u_h, p - p_h)|_h(v - \mathcal{I}_h v, q - \mathcal{P}_h q)|_h + (p - p_h, \nabla \cdot (v - \mathcal{I}_h v))\right|.
\] (3.38)

Using (3.38) and the Cauchy-Schwartz inequality, we have
\[
\|u - u_h\|^2_{0,\Omega} \\
\leq C\left|(u - u_h, p - p_h)|_h(v - \mathcal{I}_h v, q - \mathcal{P}_h q)|_h + \|p - p_h\|_{0,\Omega} \|\nabla \cdot (v - \mathcal{I}_h v)\|_{0,\Omega}\right) \leq C\left|(u - u_h, p - p_h)|_h(v - \mathcal{I}_h v, q - \mathcal{P}_h q)|_h + \|p - p_h\|^2_{0,\Omega} + \|\nabla \cdot (v - \mathcal{I}_h v)\|^2_{0,\Omega}\right)^{1/2}.
\] (3.39)

Using Theorems 3.1 and 3.2, (3.30), (3.4) and (3.37), we obtain
\[
\|u - u_h\|^2_{0,\Omega} \leq C h^2 \left(\sqrt{\nu} |u|_{2,\Omega}^2 + \frac{1}{\nu} |p|_{1,\Omega}^2\right) \leq C h^2 \left(\sqrt{\nu} |u|_{2,\Omega}^2 + \frac{1}{\nu} |q|_{1,\Omega}^2\right) \leq C h^2 \left(\sqrt{\nu} |u|_{2,\Omega} + \frac{1}{\sqrt{\nu}} |p|_{1,\Omega}\right) \|u - u_h\|_{0,\Omega}.
\] (3.40)

This completes the proof. □

Acknowledgments

The authors gratefully acknowledge the support of the Natural Science Foundation of China (No. 10671154) and the National Basic Research Program (No. 2005CB321703).

References


