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A Family of Methods of the DG-Morley Type for Polyharmonic Equations

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Abstract. Discontinuous Galerkin methods as a solution technique of second order elliptic problems, have been increasingly exploited by several authors in the past ten years. It is generally claimed the alledged attractive geometrical flexibility of these methods, although they involve considerable increase of computational effort, as compared to continuous methods. This work is aimed at proposing a combination of DGM and non-conforming finite element methods to solve elliptic m-harmonic equations in a bounded domain of $\mathbb{R}^n$, for $n = 2$ or $n = 3$, with $m \geq n+1$, as a valid and reasonable alternative to classical finite elements, or even to boundary element methods.

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1 Introduction

Let Ω be a bounded domain of $\mathbb{R}^n$ for $n = 2$ or $n = 3$, with boundary Γ. For a given $f \in L^2(\Omega)$ we consider the model polyharmonic equation: Find $u \in H_0^m(\Omega)$, such that

$$(-\Delta)^m u = f, \quad \text{for} \quad m \geq 2. \tag{1.1}$$

In the two-dimensional case and for $m = 2$, this equation has several applications in Physics and in Mechanics, while in the three-dimensional case it can be useful in Fluid Mechanics whenever $m = 2$ too (see [7]). As far as the case $m \geq 3$ is concerned, applications of the polyharmonic equation (1.1) were not addressed in the litterature until

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very recently. However in the past few years triharmonic equations have been studied as applied to fluid flow problems [5] or to image processing [11].

If we consider the solution of Eq. (1.1) with conforming finite element methods, functions in the Sobolev spaces $H^m(\Omega)$ for $m \geq 3$ must be approximated by piecewise polynomial functions of the C^{m-1} class. Whenever $n \geq 2$ the construction of such function spaces is a matter of great algebraic complexity. Even in the case where $n = 2$ and $m = 2$ the known constructions are rather elaborated (cf. [2]), let alone the case $m \geq 3$, where the use of such approximation methods becomes unreasonable. This fact naturally leads to external approximations, that is, to the so-called non-conforming methods. In this case the use of polynomials of lower degree is admissible, as long as some conditions are fulfilled in order to ensure the quality of the approximations. More specifically the traces of the polynomials at element interfaces should have suitable continuity properties. Actually for two-dimensional problems a wide spectrum of options of this type has been proposed by several authors since the late sixties, and in this respect we refer to the celebrated Ciarlet's book [2]. For three-dimensional problems only a few non-conforming finite element methods are known for $m = 2$, such as [8]. In the case $n = m = 3$ a classical non-conforming finite element solution method was studied in [9].

Although to date there seems to be little practical use of the m-harmonic equation for high values of m, we address in this work the numerical solution of (1.1), by a method that combines discontinuous Galerkin techniques with classical non-conforming finite elements, for any $m \geq n+1$. One of the main merits of this method is the fact that it reduces to a minimum the intrinsic complexity of solving the m-harmonic equation in arbitrary domains, even for $m = n + 1$.

In the case $n = 2$ and $m = 3$, a first solution method combining both techniques was proposed in [10]. Here we recall this method as a starting point of a family of methods of this type applying to the case $m \geq n+1$. As we should say, for two-dimensional problems, the non conforming part of the methodology is aimed at interpolating derivatives of order r with $m-2 \leq r \leq m-1$ of the numerical solution, whereas its lower order derivatives and the solution itself are represented by completely discontinuous functions. As a matter of fact, the non conforming part of the approximation method is based on the well-known Morley triangle for solving biharmonic problems (cf. [6]). The idea is extended to the three-dimensional case, in which the non-conforming part is used to interpolate derivatives of order r with $m-3 \leq r \leq m-1$, while the lower order derivatives and the function itself are represented by fully discontinuous functions. Here the non-conforming part generalizes the non-conforming tetrahedron introduced in [9] for the case $m = 3$, which in turn are related to the Morley triangle. Indeed it was established in that work that the traces over element interfaces of the cubic functions this finite element is built upon, are nothing but Morley triangles, whenever they happen to be just quadratic. As this property remains valid in our methodology for the natural extension of Morley triangles to the case $m \geq 4$, this explains why we decided to call the new methods a DG-Morley family of methods.

An outline of the paper is as follows. In Section 2 we introduce some notations and recall the definition of the Morley triangle, together with the one of its tetrahedral analogue [8]. Next in Section 3 we consider the general concepts that lie behind our methodology for solving equation (1.1). In Section 4 we address the two-dimensional case: after recalling the member of the family corresponding to the value $m = 3$ (cf. [10]), we treat in detail the one corresponding to $m = 4$. In Section 5 we consider three-dimensional problems, by treating the case $m = 4$ as well. We conclude in Section 6 with some remarks.

2 Notations and basic finite elements

Before starting our study we present some notations and conventions used throughout this paper: We shall represent by ∇ the gradient operator, and by $\nabla^r v$ the set of partial derivatives of r-th order of a function v, arranged in the form of an r-th order tensor over $\mathbb{R}^n$. We further denote by $|\cdot|_m$ and $\|\cdot\|_m$ the standard seminorm and norm of Sobolev space $H^m(\Omega)$, for $m \in \mathbb{N}$. We denote by $\mathcal{A} \cdot \mathcal{B}$ the standard euclidean inner product of two tensors $\mathcal{A}$ and $\mathcal{B}$ of arbitrary order and by $|\mathcal{A}|$ the associated norm $[\mathcal{A} \cdot \mathcal{A}]^{1/2}$. In all the sequel the letters i, j, k and l denote an integer belonging to the set $\{1, \cdots, n+1\}$. Also unless otherwise specified, whenever i, j, k and l appear together in the same expression, notation or definition as either a subscript or a superscript, they represent distinct numbers. In so doing we further introduce the following notations associated with a given non degenerated n-simplex T:

1. G_T denotes the barycenter of T;
2. S_i is one out of the $n+1$ vertices of T;
3. F_i denotes the face (or edge if $n = 2$) of T opposite to S_i;
4. e_{ij} represents the edge of T joining S_i and S_j (for $n = 2$ $e_{ij} \equiv F_k$);
5. M_i denotes the mid-point or the barycenter of F_i;
6. $\vec{n}_i$ is the unit outer normal vector to F_i;
7. λ_i is the barycentric coordinate of T corresponding to vertex S_i;
8. $\partial^r_{n_i}$ denotes the r-th order derivative in the direction of $\vec{n}_i$ at M_i;
9. d_{ij} is the length of e_{ij};
10. $\vec{\tau}^j_i$ is the unit vector along e_{ij} directed from S_i towards S_j;
11. h_i denotes the height of T corresponding to S_i.

As previously stated the family of methods studied in this paper for solving Eq. (1.1) has the Morley triangle as its member for $m = n = 2$. For $m = 2$ and $n = 3$ the tetrahedron introduced by the first author in [8] plays this role. Therefore to begin with we briefly recall the definition of both elements: The non-conforming Morley triangle introduced in [6] is based on polynomials p of degree less than or equal to 2 defined by the set of six degrees of freedom $\{\mathcal{F}_i, \mathcal{F}^i\}$ for $i = 1, 2, 3$, where

$$\mathcal{F}_i(p) := p(S_i), \qquad \text{and} \qquad \mathcal{F}^i(p) := \partial^1_{n_i} p.$$

Its non-conforming tetrahedral counterpart [8] is also based on quadratic functions p, defined by the set of ten degrees of freedom $\{\mathcal{F}_{ij}, \mathcal{F}^i\}$ for $i, j \in \{1,2,3,4\}$ with $i < j$, where

$$\mathcal{F}_{ij}(p) := \frac{1}{d_{ij}} \int_{e_{ij}} p de, \qquad \text{and} \qquad \mathcal{F}^i(p) := \partial^1_{n_i} p.$$

Both metods are known to be first order convergent in the discrete H^2-norm, when employed to solve problem (1.1) for $m = 2$, recast in a suitable equivalent variational form (cf. [2] and [8]).

3 Combining non-conforming and fully discontinuous finite elements

Henceforth we assume that $m > n$. Let

$$\alpha := \{\alpha_1, \cdots, \alpha_n\},$$

be an integer multi-index with $\alpha_j \geq 0$ and setting

$$|\alpha| := \sum_{i=1}^{n} \alpha_i, \quad \text{whenever} \quad |\alpha| = r, \quad \text{for} \quad r \geq 1,$$

∂_α denotes the r-th order partial derivative operator $\partial^r / \partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}$. We extend the definition of ∂_α to the case where $|\alpha| = 0$, by letting $\partial_{\{0,\cdots,0\}}$ be the identity operator. We further introduce for $i \in \{1, 2, \cdots, n+1\}$, a functional $\partial^{\beta_i}_{m-n}$ defined upon functions $q \in C^{m-n}(T)$, in connection with a multi-index $\beta_i \in \mathbb{N}^n$, i.e.,

$$\beta^i := \{\beta^i_1, \cdots \beta^i_n\}, \quad \text{where} \quad 0 \leq \beta^i_r \leq m - n, \quad \text{for} \quad r = 1, \cdots, n,$$

with

$$|\beta^i| := \sum_{r=1}^{n} \beta^i_r = m - n, \qquad \forall i,$$

in the following manner:

$$\begin{aligned}
&\partial^{m-n}_{\beta^1} q := \nabla^{m-n} q(S_1) [\circ \overrightarrow{\tau}^{2}_{1}]^{\beta^1_1} \cdots [\circ \overrightarrow{\tau}^{n+1}_{1}]^{\beta^1_n},\\
&\partial^{m-n}_{\beta^i} q := \nabla^{m-n} q(S_i) [\circ \overrightarrow{\tau}^{1}_{i}]^{\beta^i_1} \cdots [\circ \overrightarrow{\tau}^{i-1}_{i}]^{\beta^i_{i-1}} [\circ \overrightarrow{\tau}^{i+1}_{i}]^{\beta^i_i} \cdots [\circ \overrightarrow{\tau}^{n+1}_{i}]^{\beta^i_n},\\
&\partial^{m-n}_{\beta^{n+1}} q := \nabla^{m-n} q(S_{n+1}) [\circ \overrightarrow{\tau}^{1}_{n+1}]^{\beta^{n+1}_1} \cdots [\circ \overrightarrow{\tau}^{n}_{n+1}]^{\beta^{n+1}_n},
\end{aligned}$$

where $2 \leq i \leq n$, and $\mathcal{A} \circ \overrightarrow{v}$ represents the product of a tensor $\mathcal{A}$ of arbitrary order over $\mathbb{R}^n$ with vector $\overrightarrow{v} \in \mathbb{R}^n$, and $[\circ \overrightarrow{v}]^r$ means that this operation is reiterately performed r times (from left to right) if $r > 0$ or not at all if $r = 0$.

In so doing we define the following functionals that characterize the methodology in use, applied to a generic function $p \in C^m(T)$, corresponding to three types of degrees of freedom (DOF) attached to an element T of functions belonging to the space P_m, where P_r is the space of polynomials of degree less than or equal to r defined in T.

For $n = 2$:

$$\begin{aligned}
&\text{DOF of type A: } \mathcal{F}_\alpha(p) := \partial_\alpha p(G_T), && 0 \leq |\alpha| \leq m-3,\\
&\text{DOF of type B: } \mathcal{F}_{\beta^i}(p) := \partial^{m-2}_{\beta^i} p, && |\beta^i| = m-2, \quad 1 \leq i \leq 3,\\
&\text{DOF of type N: } \mathcal{F}^i(p) := \partial^{m-1}_{n_i} p, && 1 \leq i \leq 3.
\end{aligned}$$

As one can easily check there are $(m-1)(m-2)/2$ DOF of Type A, $3(m-1)$ DOF of Type B, and 3 DOF of Type N. Hence the total number of DOF is $(m+1)(m+2)/2$, that is, the dimension of P_m for $n = 2$.

For $n = 3$:

Here we further represent by h_i^j the height of F_i corresponding to S_j, or yet its length whenever applicable. Moreover we denote by $\overrightarrow{\sigma}_i^j$ the unit vector of the direction of h_i^j, oriented from e_{kl} towards S_j (recall that i,j,k,l are distinct). Additionally for every pair

$$\{i,j\} \in \{1,2,3,4\}^2,$$

we consider another integer multi-index with two components, namely

$$\gamma^{ij} = \{\gamma_1^{ij}, \gamma_2^{ij}\}, \quad \text{satisfying} \quad 0 \leq \gamma_r^{ij} \leq m-2, \quad \text{for} \quad r = 1,2,$$

together with

$$\gamma_1^{ij} + \gamma_2^{ij} = m-2, \quad \text{and} \quad \gamma_1^{ij} = \gamma_2^{ji}, \ \text{(we recall that } i \text{ and } j \text{ are distinct).}$$

We associate with γ^{ij} the scalar differential operator $\partial^{m-2}_{\gamma^{ij}}$ defined by:

$$\begin{aligned}
&\partial^{m-2}_{\gamma^{ij}} q(\mathbf{x}) := \left[\nabla^{m-2} q\right](\mathbf{x}) \left[\circ \overrightarrow{\sigma}_i^j\right]^{\gamma_1^{ij}} \left[\circ \overrightarrow{\sigma}_j^i\right]^{\gamma_2^{ji}}, && \forall q \in C^{m-2}(T), \ \forall \mathbf{x} \in T,\\
&\text{DOF of type A: } \mathcal{F}_\alpha(p) := \partial_\alpha p(G_T), && 0 \leq |\alpha| \leq m-4,\\
&\text{DOF of type B: } \mathcal{F}_{\beta_i}(p) := \partial^{m-3}_{\beta_i} p, && |\beta_i| = m-3, \ 1 \leq i \leq 4,\\
&\text{DOF of type C: } \mathcal{F}_{\gamma^{ij}}(p) := \frac{1}{d_{kl}} \Big(\int_{e_{kl}} \partial^{m-2}_{\gamma^{ij} p} de \Big), && |\gamma^{ij}| = m-2, \ 1 \leq i,j \leq 4,\\
&\text{DOF of type N: } \mathcal{F}^i(p) := \partial^{m-1}_{n_i} p, && 1 \leq i \leq 4.
\end{aligned}$$

Notice that due to the symmetry of partial derivatives and to the fact that

$$\gamma_1^{ij} = \gamma_2^{ji}, \qquad \text{and} \qquad \mathcal{F}_{\gamma^{ij}} \equiv \mathcal{F}_{\gamma^{ji}},$$

Then, similarly to the case $n = 2$, we can easily establish that there are $(m-1)(m-2)(m-3)/6$ DOF of Type A, $2(m-1)(m-2)$ DOF of Type B, $6(m-1)$ DOF of Type C, and 4 DOF of Type N. It follows that there are exactly $(m+1)(m+2)(m+3)/6$ DOF, which is the dimension of P_m for $n = 3$.

Henceforth we assume that Ω is a polygon if $n = 2$ or a polyhedron if $n = 3$. We consider a partition $\mathcal{T}_h$ of Ω into n-simplexes, satisfying the usual compatibility conditions for the finite element method. Moreover, we assume that $\mathcal{T}_h$ belongs to a quasiuniform family of partitions $\mathcal{P}$. h denotes the largest edge of all the simplexes of $\mathcal{T}_h$.

Our partly non-conforming and partly fully discontinuous finite element, generates a subspace V_h of

$$W_h := \{v/v_{/T} \in P_m, \forall T \in \mathcal{T}_h\}.$$

Its definition is as follows:

<u>For $n = 2$:</u>

Definition 3.1. *V_h is the subspace of W_h of those functions v, whose normal derivatives of the $(m-1)$-th order determined on each side of every edge common to two triangles of the partition are the same at this edge's mid-point, and such that the tensors resulting from the application of operator ∇^{m-2} to the restrictions of v to the triangles of the partition containing a given vertex, coincide at this point.*

The actual possibility of constructing such space V_h is a consequence of

Proposition 3.1. *For any non degenerated triangle T the set Σ_m^2 of degrees of freedom, namely*

$$\Sigma_m^2 := \Big\{\mathcal{F}_\alpha, \mathcal{F}_{\beta^i}, \mathcal{F}^i, \;\; i = 1,2,3, \;\; \alpha, \beta^i \in \mathbb{N}^2, \;\; |\alpha| \leq m-3, \;\; |\beta^i| = m-2\Big\},$$

is P_m-unisolvent.

Proof. In order to prove this Proposition it suffices to establish that, if all the $dimP_m$ functionals of Σ_m^2 for a triangle T applied to a function $p \in P_m$ vanish then $p \equiv 0$.

Let us assume that

$$p \in P_m, \qquad \mathcal{F}_\alpha(p) = 0, \qquad \mathcal{F}_{\beta^i}(p) = 0, \qquad \text{and} \qquad \mathcal{F}^i(p) = 0,$$

for $i = 1,2,3$, $\alpha, \beta^i \in \mathbb{N}^2$, $|\alpha| \leq m-3$, $|\beta^i| = m-2$. First we observe that all the components of $\nabla^{m-2}p$ belong to P_2. Since by assumption their values at the vertices of T vanish, their tangential derivatives along each edge of T also vanish at this edge's mid-point by a well-known property of quadratic functions depending on one single variable. Since the $(m-1)-$th order normal derivative of p at this mid-point also vanish, all the components of $\nabla^{m-1}p$ must vanish at this point too. Since each component of the latter tensor is a function of P_1 that vanishes at the three edge mid-points of T, necessarily all the $(m-1)-$th order derivatives of p vanish identically, which implies that $p \in P_{m-2}$. It follows that all the components of $\nabla^{m-2}p$ belong to P_0. But since they vanish at the vertices of T by assumption, all of them must vanish identically. Thus

p is a function of P_{m-3} that vanishes, together with its derivatives of order $r \leq m-3$ at the same point G_T. As a consequence p vanishes identically in T and the result follows. $\square$

For $n = 3$:

Definition 3.2. *V_h is the subspace of W_h of those functions v, whose normal derivatives of the $(m-1)$-th order determined on each side of every face common to two tetrahedra of the partition are the same at this face's barycenter, whose sets of mean values along an edge of all partial derivatives of order $m-2$ with respect to directions of the plane orthogonal to this edge are the same for all the tetrahedra containing this edge, and such that the tensors resulting from the application of ∇^{m-3} to the restrictions of v to the tetrahedra of the partition containing a given vertex, coincide at this point.*

The actual possibility of constructing such space V_h is a consequence of

Proposition 3.2. *For any non degenerated tetrahedron T the set Σ^3_m of degrees of freedom, namely,*

$$\Sigma^3_m := \Big\{ \mathcal{F}_\alpha,\ \mathcal{F}_{\beta^i},\ \mathcal{F}_{\gamma^{ij}},\ \mathcal{F}^i,\ \ i,j \in \{1,2,3,4\},\ \ \alpha, \beta^i \in \mathbb{N}^3,\ \ \textit{with}\ \ |\alpha| \leq m-4,$$
$$|\beta^i| = m-3,\ \ \textit{and}\ \ \gamma^{ij} \in \mathbb{N}^2,\ \ \textit{with}\ \ |\gamma^{ij}| = m-2,\ \ \textit{and}\ \ \gamma^{ij}_1 = \gamma^{ji}_2 \Big\},$$

is P_m-unisolvent.

Proof. Here again this result holds provided any function $p \in P_m$ (defined in T) such that all the $dim P_m$ functionals of Σ^3_m applied to it vanish, necessarily vanishes everywhere in T. The argument is true because a result in [8], states that any polynomial in P_2 whose mean values along the three edges of given a face of tetraheron T vanish, then its tangential derivatives at this face's barycenter also vanish. Indeed in view of this result, using our assumptions on $\mathcal{F}_{\gamma^{ij}}(p)$ and $\mathcal{F}^i(p)$, we conclude that all the components of $\nabla^{m-1}p$ are functions of P_1 that vanish at the barycenters of the faces of T. Hence all of them vanish identically and $p \in P_{m-2}$. Next we note that by assumption for any edge e of T with ends S^e_1, S^e_2 and unit vector $\vec{\tau}_e$ along it,

$$\Big| \int_e \frac{\partial q}{\partial \tau_e} \Big| = \big| q(S^e_1) - q(S^e_2) \big| = 0,$$

for every $(m-3)$-th order derivative q of p; hence by assumption again the mean values along any edge of T of all the components of $\nabla^{m-2}p$ vanish. Since every such component belongs to P_0, it follows that $\nabla^{m-2}p$ vanishes identically in T and that $p \in P_{m-3}$. The fact that every component of $\nabla^{m-3}p$ vanishes at the vertices of T by assumption, implies that it vanishes everywhere in T and hence $p \in P_{m-4}$. Finally recalling that by assumption p vanishes together with its derivatives of order $r \leq m-4$ at the same point G_T, it must vanish identically in T and the proof is complete. $\square$

The fact that both Propositions 3.1 and 3.2 hold does not help much in the practical implementation of the methods. Indeed for this purpose it is necessary to determine complete sets of basis functions associated with the functionals of Σ_m^2 and Σ_m^3. Unfortunately this involves very intrincate calculations for an arbitrary m. That is why we refrain from exhibiting such sets here. Nevertheless, since the basis functions f_α and f^i associated with the degrees of freedom $\mathcal{F}_\alpha$ and $\mathcal{F}^i$ respectively are not so complicated, they are supplied below:

Denoting the cartesian coordinates of G_T by $(g_1, \cdots, g_n)$, we first have:

$$f_\alpha := \prod_{r=1}^{n} \frac{(x_r - g_r)^{\alpha_r}}{\alpha_r!}, \qquad \forall \alpha, \quad \text{such that} \quad |\alpha| < m - n,$$

It is easy to check that $\mathcal{F}_\mu(f_\alpha) = 0$ for every integer multi-index $\mu \in \mathbb{N}^n$ with $0 \leq |\mu| < m - n$, such that $\mu \neq \alpha$, whereas $\mathcal{F}_\alpha(f_\alpha) = 1$. Moreover we trivially have

$$\begin{aligned} &\mathcal{F}_{\beta^i}(f_\alpha) = 0, \quad \mathcal{F}_{\gamma^{ij}}(f_\alpha) = 0, \quad \text{and} \quad \mathcal{F}^i(f_\alpha) = 0, \\ &\forall (i,j) \in \{1, \cdots, n+1\}^2, \quad \beta^i \in \mathbb{N}^n, \quad \text{with} \quad |\beta^i| = m - n, \end{aligned}$$

and for n = 3 only,

$$\gamma^{ij} \in \mathbb{N}^2, \quad \text{with} \quad |\gamma^{ij}| = m - 2, \quad \text{and} \quad \gamma_1^{ij} = \gamma_2^{ji}.$$

As for f^i we first define:

$$\tilde{f}^i := (-1)^m \frac{h_i^{m-1}}{m!} (m\lambda_i^{m-1} - n\lambda_i^m).$$

Since the first order partial derivative of λ_i in the direction of $\overrightarrow{n}_i$ equals $-h_i^{-1}$, and λ_i vanishes identically on F_i, one can readily derive $\mathcal{F}^i(\tilde{f}^i)$=1. Moreover since $\lambda_i(M_j) = 1/n$ (recalling that $i \neq j$), we trivially have $\mathcal{F}^j(\tilde{f}^i)$=$\delta_{ij}$, with i, j not necessarily distinct.

On the other hand

$$\partial_{\beta^j}^{m-n} \tilde{f}^i = (-1)^m c_{ij} \left\{ \frac{m!}{(n-1)!} \Big[\lambda_i(S_j)\Big]^{n-1} - \frac{nm!}{n!} \Big[\lambda_i(S_j)\Big]^n \right\},$$

where

$$\begin{aligned} c_{i1} &:= \prod_{r=1}^{n} \Big(\frac{\partial \lambda_i}{\partial \tau_1^{r+1}} \Big)^{\beta_r^j}, \\ c_{ij} &:= \prod_{r=1}^{j-1} \Big(\frac{\partial \lambda_i}{\partial \tau_j^{r}} \Big)^{\beta_r^j} \prod_{r=j}^{n} \Big(\frac{\partial \lambda_i}{\partial \tau_j^{r+1}} \Big)^{\beta_r^j}, \qquad \text{for} \quad 2 \leq j \leq n, \\ c_{i,n+1} &:= \prod_{r=1}^{n} \Big(\frac{\partial \lambda_i}{\partial \tau_{n+1}^{r}} \Big)^{\beta_r^j}. \end{aligned}$$

Then we can see that

$$\mathcal{F}_{\beta^j}(\tilde{f}^i) = 0, \quad \text{for every} \quad j \in \{1, \cdots, n+1\},$$

since $\lambda_i(S_j) = \delta_{ij}$, with i, j not necessarily distinct. Now for n = 3 only we have

$$\partial^{m-2}_{\gamma^{rs}} \tilde{f}^i = (-1)^m c_{irs}\Big[m!\lambda_i - \frac{3m!\lambda_i^2}{2}\Big], \quad \text{with} \quad r,s \in \{1,\cdots,n+1\},$$

r and s being distinct (but not necessarily $r \neq i$ or $s \neq i$), where

$$c_{irs} := \Big(\frac{\partial\lambda_i}{\partial\sigma_r^s}\Big)^{\gamma_1^{rs}} \Big(\frac{\partial\lambda_i}{\partial\sigma_s^r}\Big)^{\gamma_2^{rs}}.$$

It follows that, if either r or s equals i, $\mathcal{F}_{\gamma^{rs}}(\tilde{f}^i) = 0$ since λ_i vanishes identically on any edge of tetrahedron T that does not have S_i as an end. On the other hand if neither $i = r$ nor $i = s$, S_i must be an end of e_{kl} with k, l, r and s distinct. In this case integration of λ_i and λ_i^2 along e_{kl} yields $d_{kl}/2$ and $d_{kl}/3$ respectively, and hence we have $\mathcal{F}_{\gamma^{rs}}(\tilde{f}^i) = 0$ in this case too. Finally noticing that

$$\partial_\mu \tilde{f}^i(G_T) \neq 0, \quad \forall \mu \in \mathbb{N}^n, \quad \text{such that} \quad 0 \leq |\mu| < m-n,$$

we define the basis functions f^i by

$$f^i := \tilde{f}^i - \sum_{0\leq|\mu|<m-n} \mathcal{F}_\mu(\tilde{f}^i) f_\mu.$$

Clearly enough the so defined basis functions satisfy

$$\mathcal{F}^j(f^i) = \delta_{ij},$$

with i, j not necessarily distinct, together with

$$\mathcal{F}_\alpha(f^i) = 0, \quad \forall \alpha \in \mathbb{N}^n, \quad \text{satisfying} \quad 0 \leq |\alpha| < m-n,$$
$$\mathcal{F}_{\beta^i}(f^i) = 0, \quad \forall \beta^i \in \mathbb{N}^n, \quad \text{with} \quad |\beta^i| = m-n, \quad i = 1,\cdots,n+1,$$

and for n = 3 only

$$\mathcal{F}_{\gamma^{ij}}(f^i) = 0, \quad \forall \gamma^{ij} \in \mathbb{N}^2,$$

with

$$|\gamma^{ij}| = m-2, \quad \gamma_1^{ij} = \gamma_2^{ji}, \quad i,j \in \{1,\cdots,n+1\}.$$

Since in practical situations m cannot be very large, we prefer to exhibit the rather complex set of basis functions f_{β^i} and $f_{\gamma^{ij}}$ in Sections 4 and 5, just for small values of m, including the minimum one, i.e., $m = n+1$.

We conclude this Section by presenting our technique, together with expected convegence results, applicable to any value of m.

First we consider some properties of finite element spaces V_h.

Lemma 3.1. *Let n = 2 and v be an arbitrary function of V_h. For every edge e common to two elements of $\mathcal{T}_h$ the tensors resulting from the application of operator ∇^{m-1} to the restrictions of v to both triangles, coincide at the mid-point of e.*

Proof. Let T_1 and T_2 be the triangles of $\mathcal{T}_h$ having e as a common edge, and M_e be the mid-point of e. Denoting by $\vec{n}_e$ the unit vector normal to e directed in a given sense, let $\vec{\tau}_e$ be a unit vector parallel to e, such that $(\vec{n}_e, \vec{\tau}_e)$ form a direct orthonormal basis of $\Re^2$. In the expressions that follow the notation employed for the partial derivatives in connection with these two directions are self-explanatory.

By construction we have

$$\frac{\partial^{m-1}(v_{/T_1})}{\partial n_e^{m-1}(M_e)} = \frac{\partial^{m-1}(v_{/T_2})}{\partial n_e^{m-1}(M_e)}.$$

Moreover the operator ∇^{m-2} applied to v restricted to every triangle of the partition is a quadratic tensor continuous at the ends of e by assumption. Thus from well-known properties of quadratic functions in a single variable we have

$$\frac{\partial\left[\nabla^{m-1}(v_{/T_1})\right]}{\partial \tau_e(M_e)} = \frac{\partial\left[\nabla^{m-1}(v_{/T_2})\right]}{\partial \tau_e(M_e)}.$$

This means that any partial derivative of v of order $m-1$ other than the purely normal one is also continuous at point M_e, and the result follows. □

Lemma 3.2. *Let $n = 3$ and v be an arbitrary function of V_h. For every face F common to two elements of $\mathcal{T}_h$ the tensors resulting from the application of operator ∇^{m-1} to the restrictions of v to both tetrahedra coincide at the barycenter of F.*

Proof. Let T_1 and T_2 be the tetrahedra of $\mathcal{T}_h$ having F as a common face, and M_F be the barycenter of F. Denoting by $\overrightarrow{n}_F$ the unit vector normal to F directed in a given sense, let $\overrightarrow{\tau}_F$ and $\overrightarrow{\sigma}_F$ be two unit vectors parallel to F, such that $(\overrightarrow{n}_F, \overrightarrow{\tau}_F, \overrightarrow{\sigma}_F)$ form a direct orthonormal basis of $\mathbb{R}^3$. In the expressions that follow the notation employed for the partial derivatives in connection with these three directions are self-explanatory.

By construction we have

$$\frac{\partial^{m-1}(v_{/T_1})}{\partial n_F^{m-1}(M_F)} = \frac{\partial^{m-1}(v_{/T_2})}{\partial n_F^{m-1}(M_F)}.$$

Moreover the tensor operator ∇^{m-2} applied to v restricted to every tetrahedron of the partition is a quadratic tensor whose mean values along the edges of F coincide for both T_1 and T_2. Indeed, this is true by construction for the components (partial derivatives) involving only directions of the plane orthogonal to each edge. As for the other components, they involve at least one differentiation along a given edge. Since the integral along this edge of such components is necessarily the difference between partial derivatives of order $m-3$ at the edge's ends, and the former are all continuous at such points by assumption as vertices of T, the mean value along every edge of F of the whole tensor of partial derivatives of order $m-2$ are the same for both T_1 and T_2. Then recalling a result of [8] for the non-conforming tetrahedral element whose quadratic functions are constructed in the same way as the $(m-3)$-th order partial derivatives of functions in our space V_h, the first order partial derivatives in the direction of $\overrightarrow{\tau}_F$ or $\overrightarrow{\sigma}_F$ of all the $(m-2)$-th order partial derivatives of v coincide at M_F too. This means that any partial derivative of v of order $m-1$ other than the purely normal one is also continuous at point M_F, and the result follows. $\square$

Next, let

$$E_h := \Big\{T \in \mathcal{T}_h, \quad \partial T \cap \Gamma \neq \emptyset\Big\},$$

and S denote the measure of the boundary of a generic n-simplex. We introduce the functional $[\cdot]_{m,n,h}$ on V_h given by

$$\begin{aligned}[v]^2_{m,n,h} := |v|^2_{m,h} + \frac{1}{h^4}\sum_{i=0}^{m-n-1}\Big[&\sum_{T\in E_h}\int_{\partial T\cap\Gamma}\big|\nabla^i(v_{/T})\big|^2 dS \\ &+\frac{1}{2}\sum_{T\in\mathcal{T}_h}\sum_{T'\in\mathcal{T}_h}\int_{\partial T\cap\partial T'}\big|\nabla^i(v_{/T}-v_{/T'})\big|^2 dS\Big], \qquad (3.1)\end{aligned}$$

where ∂T represents the boundary of an n-simplex T, and $|v|_{m,h}$ denotes the standard discrete H^m-seminorm, i.e.,

$$|v|_{m,h} := \Big[\sum_{T\in\mathcal{T}_h}\big|\nabla^m(v_{/T})\big|^2 dx\Big]^{\frac{1}{2}}.$$

Functional $[\cdot]_{m,n,h}$ is clearly a seminorm for any space of piecewise polynomials. Moreover it trivially extends to $H^m(\Omega)$, for which the interelement boundary jump terms necessarily vanish. Actually $[\cdot]_{m,n,h}$ is a non standard norm of $H^m(\Omega)$ (at least as far as Ω is a polygonal or a polyhedral domain).

Definition 3.3. *V_h^0 is defined as the subspace of V_h consisting of functions v, such that $\forall T \in E_h$,*

$$\begin{aligned}&\mathcal{F}_\alpha(v_{/T}) = 0, \qquad \forall\alpha\in\mathbb{N}^n, \qquad \textit{with}\ \ 0\le|\alpha|<m-n,\\ &\mathcal{F}_{\beta^i}(v_{/T}) = 0, \qquad \forall\beta^i\in\mathbb{N}^n, \qquad \textit{with}\ \ 0\le|\beta^i|=m-n,\end{aligned}$$

for $i=1,\cdots,n+1$, whenever the vertex S_i of T belongs to Γ, $\mathcal{F}^i(v_{/T})=0$ whenever the face F_i of T is contained in Γ, and for $n=3$ only, such that

$$\mathcal{F}_{\gamma^{ij}}(v_{/T}) = 0, \quad \forall\gamma^{ij}\in\mathbb{N}^2, \quad \textit{with} \quad 0\le|\gamma^{ij}|=m-2,$$

and

$$\gamma_1^{ij}=\gamma_2^{ji}, \quad \textit{for} \quad i,j\in\{1,\cdots,n+1\},$$

whenever the edge e_{kl} of T is contained in Γ.

Proposition 3.3. *Seminorm $[v]_{m,n,h}$ is a norm over V_h^0.*

Proof. Let $v\in V_h^0$ and $[v]_{m,n,h}=0$. The latter condition implies that $\nabla^{m-1}(v_{/T})$ is constant in every element T of $\mathcal{T}_h$. From Lemmata 3.1 and 3.2 $\nabla^{m-1}v$ is continuous at the barycenters of the faces of the n-simplexes belonging to $\mathcal{T}_h$. Moreover $\nabla^{m-1}(v_{/T})$

vanishes at the barycenter of any n-simplex face contained in Γ, as one can easily check. Hence $\nabla^{m-1}v = \mathcal{O}$ everywhere in Ω. This implies in turn that $\nabla^{m-2}(v_{/T})$ is constant in every element T of $\mathcal{T}_h$.

Let us momentarily consider the case $n = 2$: The continuity of $\nabla^{m-2}v$ at the vertices of the triangulation $\mathcal{T}_h$ implies that this quantity is constant everywhere in Ω. Since $\nabla^{m-2}v = \mathcal{O}$ at every vertex of $\mathcal{T}_h$ belonging to Γ, it must vanish everywhere in Ω too. Then $\nabla^{m-3}v$ must be constant in every element of the triagulation.

Next, we turn our attention to the case $n = 3$. Recalling an argument in the proof of Lemma 3.2, we know that different mean values of $\nabla^{m-2}(v/T)$ along an edge common to several tetrahedra T of the partition $\mathcal{T}_h$ coincide and moreover, as one can easily check, these mean values vanish if such an edge happens to be contained in Γ. This certainly implies not only that $\nabla^{m-2}v$ is constant everywhere in Ω, but also that this quantity vanishes everywhere in Ω. Then $\nabla^{m-3}(v/T)$ must be constant in every element of $\mathcal{T}_h$, and since these quantities coincide at vertices common to several tetrahedra $T \in \mathcal{T}_h$, $\nabla^{m-3}v$ must be constant everywhere in Ω. Since $\nabla^{m-3}v$ vanishes at every vertex of $\mathcal{T}_h$ belonging to Γ, it must vanish everywhere in Ω too. It follows that $\nabla^{m-4}v$ is constant in every tetrahedron of the partition.

Now treating again both cases $n = 2$ and $n = 3$ alltogether, from our assumptions all the interface jump terms in (3.1) must vanish. Hence $\forall\, i \in \{n+1, \cdots, m\}$, $\nabla^{m-i}v$ must not only be continuous everywhere in Ω, but also vanish in the elements belonging to E_h. Since from the first part of the proof $\nabla^{m-n-1}v$ is constant in Ω, it follows that $\nabla^{m-n-1}v = \mathcal{O}$ everywhere in Ω. Then it suffices to apply recursion on i to conclude that v vanishes everywhere in Ω, and the result follows. □

Now we apply the finite dimensional space V_h^0 to approximate the polyharmonic problem (1.1) by searching for $u_h \in V_h^0$ such that

$$a_{hm}^n(u_h, v) = L_f(v), \qquad \forall v \in V_h^0, \tag{3.2}$$

where $\forall u, v \in V_h + H^m(\Omega)$, a_{hm}^n is given by

$$a_{hm}^n(u,v) := \sum_{T\in\mathcal{T}_h} \int_T \nabla^m(u_{/T}) \cdot \nabla^m(v_{/T})dx + J_{h,m-n-1}(u,v), \tag{3.3}$$

and for every non-negative integer r and a given $f \in L^2(\Omega)$, we define respectively:

$$\begin{aligned} J_{hr}(u,v) := \sum_{i=0}^{r} \frac{1}{h^4}\Big[&\sum_{T\in E_h} \int_{\partial T\cap\Gamma} \nabla^i(u_{/T}) \cdot \nabla^i(v_{/T})dS \\ &+ \frac{1}{2} \sum_{T\in\mathcal{T}_h} \sum_{T'\in\mathcal{T}_h} \int_{\partial T\cap\partial T'} \nabla^i(u_{/T} - u_{/T'}) \cdot \nabla^i(v_{/T} - v_{/T'})dS\Big], \end{aligned} \tag{3.4}$$

$$L_f(v) := \int_\Omega f v dx, \qquad \forall v \in L^2(\Omega). \tag{3.5}$$

Proposition 3.4. *Problem* (3.2)-(3.3)-(3.5) *has a unique solution* $u_h \in V_h^0$. *Moreover, the following error bound applies*

$$[u-u_h]_{m,n,h} \leq \inf_{v\in V_h^0}[u-v]_{m,n,h} + \sup_{v\in V_h^0,\, v\neq 0} \frac{a_{hm}^n(u,v)-L_f(v)}{[v]_{m,n,h}}. \tag{3.6}$$

Proof. The continuity of both a_{hm}^n and L_f over the finite dimensional space V_h^0 equipped with any norm is obvious. Taking into account Proposition 3.3, if $[\cdot]_{m,n,h}$ is chosen as a norm, the continuity constant M for a_{hm}^n can be taken equal to 1. Indeed $a_{hm}^n(v,v)$ is nothing but the square of $[v]_{m,n,h}$, $\forall v \in V_h$, which also implies that a_{hm}^n is a coercive bilinear form over $V_h^0 \times V_h^0$ for the norm $[\cdot]_{m,n,h}$ with constant $\alpha = 1$. It follows that the approximate problem (3.2)-(3.3)-(3.5) has a unique solution $u_h \in V_h^0$. Moreover from the celebrated second Strang's inequality (cf. [2]), the following error bound applies, for a constant $\mathcal{C}$ depending only on α and M:

$$[u-u_h]_{m,n,h} \leq \mathcal{C}\left[\inf_{v\in V_h^0}[u-v]_{m,n,h} + \sup_{v\in V_h^0,\, v\neq 0} \frac{a_{hm}^n(u,v)-L_f(v)}{[v]_{m,n,h}}\right].$$

Since here $M = \alpha = 1$, we may take $\mathcal{C} = 1$, using a fine evaluation of $\mathcal{C}$ due to Dupire [3] recalled in the Appendix. This completes the proof. □

Now exploiting and extending to the case of an arbitrary value of m, $m > n$, the arguments developped in [10], we are able to establish the following convergence result for problem (3.2):

Theorem 3.1. *There exists a constant C independent of h such that the following estimate holds, provided $u \in H^{m+2}(\Omega)$, and $\Delta^i u \in H^{m-i+1}(\Omega)$, for $i = 2,\cdots,m-1$,*

$$[u-u_h]_{m,n,h} \leq Ch\Big[\|u\|_{m+2} + \sum_{i=2}^{m-1} \|\Delta^i u\|_{m-i+1}\Big]. \tag{3.7}$$

Remark 3.1. An estimate sharper than (3.7) can be derived, in which powers of h greater than one appear before the Sobolev norm of the laplacian of u to the power i. However we discarded these details here since in any case the method remains first order convergent. This is because we are limited by the interpolation error with polynomials of P_m measured in the H^m-norm. Nevertheless in the next two sections we treat in detail the particular case where $m = 4$, for $n = 2$ and $n = 3$ respectively, thereby illustrating how to derive such finer results, though still qualitatively equivalent to (3.7).

4 Two-dimensional problems

In order to widen the scope of the study that follows, we will treat simultaneously the cases $m = 3$ and $m = 4$ for $n = 2$. We recall that the case $m = 3$ was the object of [10] for $n = 2$.

First, we recall that for $m = 3$, three types of degrees of freedom applied to a generic function $p \in C^2(T)$, in the total amount of $dimP_3 = 10$, characterize the local approximation functions in a triangle $T \in \mathcal{T}_h$, namely,

$$\mathcal{F}_0(p) := p(G_T), \tag{4.1a}$$

$$\mathcal{F}_i^j(p) := \frac{\partial p}{\partial \tau_i^j}(S_i), \quad 1 \leq i, j \leq 3 \ \ (i \neq j), \tag{4.1b}$$

$$\mathcal{F}^i(p) := \partial_{n_i}^2 p, \qquad 1 \leq i \leq 3. \tag{4.1c}$$

As for the case $m = 4$, four types of degrees of freedom applied to a generic function $p \in C^3(T)$, in the total amount of $dimP_4 = 15$, play this role, namely

$$\mathcal{F}_0(p) := p(G_T), \tag{4.2a}$$

$$\mathcal{F}_r(p) := \frac{\partial p}{\partial x_r}(G_T), \qquad r = 1, 2, \tag{4.2b}$$

$$\mathcal{F}_i^{jk}(p) := \frac{\partial^2 p}{\partial \tau_i^j \partial \tau_i^k}(S_i), \qquad 1 \leq j \leq k \leq 3, \ \ 1 \leq i \leq 3, \ \ (j, k \neq i), \tag{4.2c}$$

$$\mathcal{F}^i(p) := \partial_{n_i}^3 p, \qquad 1 \leq i \leq 3. \tag{4.2d}$$

Next, we consider the particular versions of subspace V_h of W_h, corresponding to the two-dimensional case for the values $m = 3$ and $m = 4$, namely, V_{2h3} and V_{2h4} defined as follows:

Definition 4.1. *V_{2h3} is the subspace of W_h of those functions v, whose second order normal derivatives determined on each side of every edge common to two triangles of $\mathcal{T}_h$ have the same value at the edge's mid-point, and such that the gradient of their restrictions to all the triangles of the partition containing a given vertex coincide at this point.*

Definition 4.2. *V_{2h4} is the subspace of W_h of those functions v, whose third order normal derivatives determined on each side of every edge common to two triangles of $\mathcal{T}_h$ have the same value at the edge's mid-point, and such that the hessian of their restrictions to all the triangles of the partition containing a given vertex coincide at this point.*

The construction of the above defined space V_{2h3} is possible, according to a result whose proof can be found in [10], namely:

Proposition 4.1. *For any non degenerated triangle T, the set of ten functions given by*

$$f_0 := 1,$$

$$f_i^j := \tilde{f}_i^j - \Big[\mathcal{F}_0(\tilde{f}_i^j) f_0 + \sum_{r=1}^{3} \mathcal{F}^r(\tilde{f}_i^j) f^r\Big], \qquad 1 \leq i, j \leq 3, \ \ (j \neq i),$$

$$f^i := \frac{h_i^2}{6}\Big[3\lambda_i^2 - 2\lambda_i^3 - \frac{7}{27}\Big], \qquad 1 \leq i \leq 3.$$

where

$$\tilde{f}_i^j := d_{ij}\lambda_i^2\lambda_j, \quad \textit{for} \quad 1 \le i,j \le 3, \ (j \ne i),$$

are canonical basis functions respectively associated with the elements of the set Σ_3 *of ten degrees of freedom, namely*

$$\Sigma_3 := \Big\{\mathcal{F}_0;\ \mathcal{F}_i^j,\ 1 \le i,j \le 3\ \ (j \ne i);\ \ \mathcal{F}^i,\ 1 \le i \le 3\Big\}.$$

Remark 4.1. Quoting [10], we have

$$\begin{aligned}
&\mathcal{F}_0(\tilde{f}_i^j) = \frac{d_{ij}}{27}, && \mathcal{F}^i(\tilde{f}_i^j) = d_{ij}h_i^{-2},\\
&\mathcal{F}^j(\tilde{f}_i^j) = 2d_{ij}h_j^{-1}\nu_j^i, && \mathcal{F}^k(\tilde{f}_i^j) = d_{ij}\left[h_k^{-2} - (\nu_k^i)^2\right],
\end{aligned}$$

where ν_i^j denotes the (first order) derivative of λ_j in the direction of $\overrightarrow{n}_i$.

The actual possibility of constructing space V_{2h4} in turn is a consequence of the following:

Proposition 4.2. *For any non degenerated triangle* T*, the set of fifteen functions given by*

$$\begin{aligned}
&f_0 := 1,\\
&f_r := x_r - g_r, \qquad r = 1,2,\\
&f_i^{jk} := \tilde{f}_i^{jk} - \sum_{r=1}^{3}\Big[\mathcal{F}_{r-1}(\tilde{f}_i^{jk})f_{r-1} + \mathcal{F}^r(\tilde{f}_i^{jk})f^r\Big],\\
&\qquad\qquad 1 \le j \le k \le 3,\ 1 \le i \le 3,\ (j,k \ne i),\\
&f^i := \frac{-h_i^3}{12}\Big(2\lambda_i^3 - \lambda_i^4 - \frac{5}{81}\Big), \qquad 1 \le i \le 3,
\end{aligned}$$

where:

$$\begin{aligned}
&\tilde{f}_i^{jj} := -\frac{d_{ij}^2}{6}\lambda_i^3\lambda_j, && 1 \le i,j \le 3,\ (j \ne i),\\
&\tilde{f}_i^{jk} := d_{ij}d_{ik}\lambda_i^2\lambda_j\lambda_k, && 1 \le j < k \le 3, \quad 1 \le i \le 3,\ (j,k \ne i),
\end{aligned}$$

are the canonical basis functions respectively associated with the elements of the set Σ_4 *of fifteen degrees of freedom, namely*

$$\Sigma_4 := \Big\{\mathcal{F}_0;\ \mathcal{F}_r,\ r = 1,2;\ \mathcal{F}_i^{jk},\ 1 \le j \le k \le 3\ (j,k \ne i);\ \mathcal{F}^i,\ 1 \le i \le 3\Big\}.$$

Here it is useful to state again Lemma 3.1 restricted to the particular case of spaces V_{2h3} and V_{2h4}:

Lemma 4.1. *Let* v *be an arbitrary function of* V_{2h3}*. For every edge* e *common to two elements of* $\mathcal{T}_h$ *the hessian of the restrictions of* v *to both triangles coincide at the mid-point of* e*.*

Lemma 4.2. *Let v be an arbitrary function of V_{2h4}. For every edge e common to two elements of $\mathcal{T}_h$ the gradient of the hessian of the restrictions of v to both triangles coincide at the mid-point of e.*

Next, we denote the space V_h^0 corresponding to V_{2hm} by V_{2hm}^0, for $m = 3$ and $m = 4$ respectively. For convenience in this Section we also represent the semi-norm $[\cdot]_{m,2,h}$ of V_{2hm} (which is known to be a norm of V_{2hm}^0), by $\{\cdot\}_{m,h}$, for $m = 3$ and $m = 4$, i.e.,

$$
\begin{aligned}
\{v\}_{3,h}^2 &:= |v|_{3,h}^2 + \frac{1}{h^4}\Big[\sum_{T\in E_h}\int_{\partial T\cap\Gamma} v_{/T}^2 dS \\
&\qquad + \frac{1}{2}\sum_{T\in\mathcal{T}_h}\sum_{T'\in\mathcal{T}_h}\int_{\partial T\cap\partial T'}(v_{/T}-v_{/T'})^2 dS\Big], \qquad (4.3)\\
\{v\}_{4,h}^2 &= |v|_{4,h}^2 + \frac{1}{h^4}\Bigg\{\sum_{T\in E_h}\int_{\partial T\cap\Gamma}\Big[v_{/T}^2 + \big|\nabla(v_{/T})\big|^2\Big]dS \\
&\qquad + \frac{1}{2}\sum_{T\in\mathcal{T}_h}\sum_{T'\in\mathcal{T}_h}\int_{\partial T\cap\partial T'}\Big[(v_{/T}-v_{/T'})^2 + \big|\nabla(v_{/T}-v_{/T'})\big|^2\Big]dS\Bigg\}. \qquad (4.4)
\end{aligned}
$$

Now we briefly recall the application of the method described in the previous Section, to solve the polyharmonic equation (1.1) for $m = 3$ (cf. [10]), by slightly adapting the notation.

Here we search for $u_h \in V_{2h3}^0$ satisfying (4.5) below,

$$a_{h3}^2(u_h, v) = L_f(v), \qquad \forall v \in V_{2h3}^0, \tag{4.5}$$

where L_f is given by (3.5) and $\forall u, v \in \big[V_{2h3} + H^3(\Omega)\big]$, a_{h3}^2 is given by

$$
\begin{aligned}
a_{h3}^2(u,v) &:= \sum_{T\in\mathcal{T}_h}\int_T \nabla^3(u_{/T})\cdot\nabla^3(v_{/T})dx + J_{h0}(u,v), \qquad (4.6a)\\
J_{h0}(u,v) &:= \frac{1}{h^4}\Big[\sum_{T\in E_h}\int_{\partial T\cap\Gamma} u_{/T}v_{/T}dS \\
&\qquad + \frac{1}{2}\sum_{T\in\mathcal{T}_h}\sum_{T'\in\mathcal{T}_h}\int_{\partial T\cap\partial T'}(u_{/T}-u_{/T'})(v_{/T}-v_{/T'})dS\Big]. \qquad (4.6b)
\end{aligned}
$$

As pointed out in Section 3, the approximate problem (4.5)-(4.6)-(3.5) has a unique solution $u_h \in V_{2h3}^0$, and the following error bound holds,

$$\{u-u_h\}_{3,h} \le \inf_{v\in V_{2h3}^0}\{u-v\}_{3,h} + \sup_{v\in V_{2h3}^0,\, v\neq 0}\frac{a_{h3}^2(u,v)-L_f(v)}{\{v\}_{3,h}}. \tag{4.7}$$

In all the sequel the letter C, combined or not with other symbols, represents different constants independent of h.

The two terms on the right hand side of (4.7) can be estimated according to the following propositions:

Proposition 4.3. $\exists C_3^1$, *such that*

$$\inf_{v \in V_{2h3}^0} \{u - v\}_{3,h} \leq C_3^1 h ||u||_5,$$

holds for $u \in H^5(\Omega)$.

Proof. First we recall that $\{u - v\}_{3,h}^2$ is the sum of two terms, namely, $|u - v|_{3,h}^2$ and the summation term, which is nothing but $J_{h0}(u - v, u - v)$ according to (4.6). It is not difficult to check that for the first term it holds (cf. the proof of Proposition 4.5 hereafter):

$$\inf_{v \in V_{2h3}^0} |u - v|_{3,h}^2 \leq 2\Big[||u - \Pi_{h3} u||_{3,h}^2 + \sum_{T \in E_h} \int_T |u(G_T)|^2 dx\Big],$$

where $\Pi_{h3} w$ denotes the V_{2h3}-interpolate of a function $w \in H^4(\Omega)$ and $|| \cdot ||_{3,h}$ the stan -dard discrete H^3-norm. Since

$$card(E_h) \leq \bar{C} h^{-1},$$

and from the Sobolev Embedding Theorem and standard results (cf. [2]) whenever

$$u \in H^5(\Omega) \cap H_0^3(\Omega), \qquad |u(G_T)| \leq \tilde{C}_3 h^3 \sup_{\mathbf{x} \in \Omega} |\nabla^3 u(\mathbf{x})|, \quad for \;\; T \in E_h,$$

we derive:

$$\inf_{v \in V_{2h3}^0} |u - v|_{3,h}^2 \leq C_3^{'} \Big[h^2 |u|_4^2 + h^7 ||u||_5^2\Big].$$

As for the jump terms, by straightforward calculations we obtain:

$$\inf_{v \in V_{2h3}^0} J_{h0}(u - v, u - v) \leq \frac{C_3^0}{h^4} \Big[\sum_{T \in \mathcal{T}_h} \int_{\partial T} |(u - \Pi_{h3} u)_{/T}|^2 dS + \sum_{T \in E_h} \int_{\partial T} |u(G_T)|^2 dS\Big].$$

Hence, using the Trace Theorem, we derive an estimate in all similar to the above one, that is,

$$\inf_{v \in V_{2h3}^0} J_{h0}(u - v, u - v) \leq C_3^* \Big[h^3 |u|_4^2 + h^2 ||u||_5^2\Big].$$

This completes the proof. □

Proposition 4.4. $\exists C_3^2$ *such that the following estimate holds, provided* $u \in H^5(\Omega)$ *and* $\Delta^2 u \in H^2(\Omega)$,

$$\sup_{v \in V_{2h3}^0,\, v \neq 0} \frac{a_{h3}^2(u, v) - L_f(v)}{\{v\}_{3,h}} \leq C_3^2 \Big(h \|u\|_5 + h^{\frac{3}{2}} \|\Delta^2 u\|_2\Big). \tag{4.8}$$

Proof. First we note that the term $J_{h0}(u,v)$ in the expression of a^2_{h3} vanishes if $u \in H^3_0(\Omega)$. Bearing this in mind we next expand the remaining term in the numerator of the fraction on the left hand side of (4.8) using reiterately integration by parts. For a given $T \in \mathcal{T}_h$, let us denote by $\overrightarrow{n}_T$ the unit outer normal vector to ∂T, and by $\partial_{n_T}(\cdot)$ the first order partial derivative in the direction of $\overrightarrow{n}_T$. In so doing under our assumptions on u and taking into account (4.6), we obtain:

$$a^2_{h3}(u,v) - L_f(v) = b_{h3}(u,v) + c_{h3}(u,v), \tag{4.9a}$$

$$b_{h3}(u,v) := \sum_{T\in\mathcal{T}_h} \int_{\partial T} \Big[\partial_{n_T}\nabla^2 u \cdot \nabla^2(v_{/T}) - \partial_{n_T}\nabla\Delta u \cdot \nabla(v_{/T})\Big] dS, \tag{4.9b}$$

$$c_{h3}(u,v) := \sum_{T\in\mathcal{T}_h} \int_{\partial T} \partial_{n_T}\Delta^2 u \cdot v_{/T} dS. \tag{4.9c}$$

Thanks to Lemma 4.1, together with the continuity of the gradient of $v \in V^0_{2h3}$ at the vertices of the partition, we may employ the same arguments as in [9]. It follows that,

$$\big|b_{h3}(u,v)\big| \le C^3_3 h\Big[\parallel u \parallel_4 + \parallel \Delta u \parallel_3\Big]\{v\}_{3,h}. \tag{4.10}$$

Now in order to estimate $c_{h3}(u,v)$, we first observe that the assumed regularity of $\Delta^2 u$ implies that,

$$\begin{aligned} c_{h3}(u,v) = &\sum_{T\in E_h} \int_{\partial T\cap\Gamma} \partial_{n_T}\Delta^2 u \cdot v_{/T} dS \\ &+ \frac{1}{2}\sum_{T\in\mathcal{T}_h}\sum_{T'\in\mathcal{T}_h} \int_{\partial T\cap\partial T'} \partial_{n_T}\Delta^2 u(\cdot v_{/T} - v_{/T'})dS, \quad \forall v \in V^0_{2h3}. \end{aligned}$$

Hence, using the Cauchy-Schwartz inequality and our assumptions on $\mathcal{T}_h$, we obtain:

$$|c_{h3}(u,v)| \le \underline{C}_3 h^2\Big[\sum_{T\in\mathcal{T}_h}\int_{\partial T}\big|\partial_{n_T}\Delta^2 u\big|^2 dS\Big]^{\frac{1}{2}}\big[J_{h0}(v,v)\big]^{\frac{1}{2}}.$$

Then, going to the unit reference triangle $\hat{T}$ and using the Trace Theorem for $H^1(\hat{T})$, together with standard estimates for quasi-uniform families of triangulations (cf. [2]), we derive:

$$\int_{\partial T}\big|\partial_{n_T}\Delta^2 u\big|^2 dS \le \hat{C}_3 h^{-1}\Big[\int_T \big|\nabla\Delta^2 u\big|^2 + \big|\nabla^2\Delta^2 u\big|^2 dx\Big], \qquad \forall T \in \mathcal{T}_h.$$

Recalling (4.3), this readily yields,

$$\big|c_{h3}(u,v)\big| \le C^4_3 h^{\frac{3}{2}}\|\Delta^2 u\|_2\{v\}_{3,h}. \tag{4.11}$$

Finally (4.8) results from the combination of (4.9), (4.10) and (4.11). □

As a direct consequence of Propositions 4.3 and 4.4, we have the following convergence result,

Theorem 4.1. *$\exists C_3$ such that the following error estimate holds, provided $u \in H^5(\Omega)$ and $\Delta^2 u \in H^2(\Omega)$,*

$$\{u-u_h\}_{3,h} \le C_3\Big[h\|u\|_5 + h^{\frac{3}{2}}\|\Delta^2 u\|_2\Big]. \tag{4.12}$$

Next we consider the case $m = 4$. In this case, we search for $u_h \in V^0_{2h4}$ satisfying

$$a^2_{h4}(u_h, v) = L_f(v), \qquad \forall v \in V^0_{2h4}, \tag{4.13}$$

where $\forall u, v \in [V_{2h4} + H^4(\Omega)]$, a^2_{h4} is given by

$$a^2_{h4}(u,v) := \sum_{T\in\mathcal{T}_h}\Big[\int_T \nabla^4(u_{/T})\cdot\nabla^4(v_{/T})dx + J_{h1}(u,v), \tag{4.14a}$$

$$\begin{aligned} J_{h1}(u,v) := \sum_{r=0}^{1}\frac{1}{h^4}\Big[&\sum_{T\in E_h}\int_{\partial T\cap\Gamma}\nabla^r(u_{/T})\cdot\nabla^r(v_{/T})dS \\ &+\frac{1}{2}\sum_{T\in\mathcal{T}_h}\sum_{T'\in\mathcal{T}_h}\int_{\partial T\cap\partial T'}\nabla^r(u_{/T}-u_{/T'})\cdot\nabla^r(v_{/T}-v_{/T'})dS\Big]. \end{aligned} \tag{4.14b}$$

Clearly enough here too approximate problem (4.13)-(4.14)-(3.5) has a unique solution $u_h \in V^0_{2h4}$, for which, it holds,

$$\{u-u_h\}_{4,h} \le \inf_{v\in V^0_{2h4}}\{u-v\}_{4,h} + \sup_{v\in V^0_{2h4},\, v\neq 0}\frac{a^2_{h4}(u,v)-L_f(v)}{\{v\}_{4,h}}. \tag{4.15}$$

In order to estimate both terms on the right hand side of (4.15) we prove the following propositions:

Proposition 4.5. *$\exists C^1_4$ such that*

$$\inf_{v\in V^0_{2h4}}\{u-v\}_{4,h} \le C^1_4 h||u||_6,$$

holds for $u \in H^6(\Omega)$.

Proof. First we estimate the term $|u-v|_{4,h}$ by

$$\inf_{v\in V^0_{2h4}}|u-v|_{4,h} \le ||u-\Pi^0_{h4}u||_{4,h},$$

as usual, where $\Pi^0_{h4}w$ is the V^0_{2h4}-interpolate of a function $w \in H^5(\Omega)$ and $||\cdot||_{4,h}$ denotes the standard discrete H^4-norm. Further denoting by $\Pi_{h4}w$ the V_{2h4}-interpolate of a function $w \in H^5(\Omega)$ and recalling the basis functions f_r for $r = 0,1,2$, we may rewrite $\forall \mathbf{x} \in \Omega$,

$$\big(u-\Pi^0_{h4}u\big)(\mathbf{x}) = \big(u-\Pi_{h4}u\big)(\mathbf{x}) - \sum_{T\in E_h}\chi_T(\mathbf{x})\Big[u(G_T)+\nabla u(G_T)\cdot(\mathbf{x}-G_T)\Big], \tag{4.16}$$

where χ_T is the characterisitic function of a triangle $T \in \mathcal{T}_h$. It easily follows the existence of a constant $\bar{C}_4$ such that

$$\inf_{v \in V^0_{2h4}} |u-v|_{4,h} \leq ||u - \Pi_{h4} u||_{4,h} + \bar{C}_4 \sum_{T \in E_h} \int_T \Big[\big|u(G_T)\big|^2 + \big(\big|\nabla u(G_T)\big|^2 dx \Big].$$

However, $\forall u \in H^6(\Omega) \cap H^4_0(\Omega)$, we have

$$\begin{aligned} &\Big[\big|u(G_T)\big|^2 + \big|(\mathbf{x}-G_T)\cdot \nabla u(G_T)\big|^2\Big]^{\frac{1}{2}} + h\big|\nabla u(G_T)\big| \\ \leq &\tilde{C}_4 h^4 \sup_{\mathbf{y}\in\Omega}\big|\nabla^4 u(\mathbf{y})\big|, \qquad \forall T \in E_h, \qquad \forall \mathbf{x} \in T. \end{aligned} \tag{4.17}$$

Then from standard estimates (cf. [2]), and recalling that $card(E_h) \leq \bar{C}_2 h^{-1}$, for n = 2, we readily derive,

$$\inf_{v \in V^0_{2h4}} \{u-v\}^2_{4,h} \leq C_4^{'} \Big[h^2 |u|^2_5 + h^7 ||u||^2_6 \Big].$$

As for the jump terms which are nothing but $J_{h1}(u-v, u-v)^{1/2}$, taking into account (4.16), by straightforward calculations we obtain,

$$\begin{aligned} \inf_{v \in V^0_{2h4}} J_{h1}(u-v,u-v) \leq \frac{C^0_4}{h^4} \Bigg[&\sum_{T\in\mathcal{T}_h} \int_{\partial T} \big|(u - \Pi_{h4}u)_{/T}\big|^2 dS \\ &+ \sum_{T \in E_h} \int_{\partial T} \Big[\big|u(G_T)\big|^2 + \big|\nabla u(G_T)\cdot(\mathbf{x}-G_T)\big|^2\Big] dS \Bigg]. \end{aligned}$$

Hence thanks to (4.17), similarly to Proposition 4.3, we derive the estimate

$$\inf_{v \in V^0_{2h4}} J_{h1}(u-v,u-v) \leq C_4^* \Big[h^3 |u|^2_5 + h^2 ||u||^2_6 \Big],$$

which completes the proof.

Proposition 4.6. *$\exists C_4^2$ such that the following estimate holds, provided $u \in H^6(\Omega)$, $\Delta^2 u \in H^3(\Omega)$, and $\Delta^3 u \in H^2(\Omega)$,*

$$\sup_{v \in V^0_{2h4},\ v\neq 0} \frac{a^2_{h4}(u,v) - L_f(v)}{\{v\}_{4,h}} \leq C^2_4 \Big[h\|u\|_6 + h^{\frac{3}{2}} \Big(\|\Delta^2 u\|_3 + \|\Delta^3 u\|_2 \Big) \Big]. \tag{4.18}$$

Proof. The jump term $J_{h1}(u,v)$ in the expression of a^2_{h4} vanishes if $u \in H^4_0(\Omega)$. Hen -ce from our assumptions on u we may integrate by parts four times on the left hand side of (4.18). Thus, using the same notation as in Proposition 4.4, we obtain,

$$a^2_{h4}(u,v) - L_f(v) = b_{h4}(u,v) + c_{h4}(u,v), \tag{4.19a}$$

$$b_{h4}(u,v) := \sum_{T\in\mathcal{T}_h} \int_{\partial T} \Big[\partial_{n_T} \nabla^3 u \cdot \nabla^3(v_{/T}) - \partial_{n_T} \nabla^2 \Delta u \cdot \nabla^2 (v_{/T}) \Big] dS, \tag{4.19b}$$

$$c_{h4}(u,v) := \sum_{T\in\mathcal{T}_h} \int_{\partial T} \Big[\partial_{n_T} \nabla \Delta^2 u \cdot \nabla(v_{/T}) - \partial_{n_T} \Delta^3 u \cdot v_{/T} \Big] dS. \tag{4.19c}$$

Thanks to Lemma 3.1, together with the continuity of the hessian of $v \in V^0_{2h4}$ at the vertices of the triangulation, we can treat the term $b_{h4}(u,v)$ in the same manner as in Proposition 4.4, thereby deriving,

$$\big|b_{h4}(u,v)\big| \leq C_4^3 h\Big[\|u\|_5 + \|\Delta u\|_4\Big]\{v\}_{4,h}. \tag{4.20}$$

Now in order to estimate $c_{h4}(u,v)$, we first observe that the assumed regularity of $\Delta^2 u$ and $\Delta^3 u$ implies that $\forall v \in V^0_{2h4}$,

$$\begin{aligned} c_{h4}(u,v) =& \sum_{T\in E_h}\int_{\partial T\cap\Gamma}\Big[\partial_{n_T}\Delta^3 u\cdot v_{/T} + \partial_{n_T}\nabla\Delta^2 u\cdot\nabla(v_{/T})\Big]dS \\ &+\frac{1}{2}\sum_{T\in\mathcal{T}_h}\sum_{T'\in\mathcal{T}_h}\int_{\partial T\cap\partial T'}\Big[\partial_{n_T}\Delta^3 u\cdot(v_{/T}-v_{/T'}) + \partial_{n_T}\nabla\Delta^2 u\cdot\nabla(v_{/T}-v_{/T'})\Big]dS. \end{aligned}$$

Hence, using the Cauchy-Schwartz inequality and our assumptions on $\mathcal{T}_h$, we obtain

$$\big|c_{h4}(u,v)\big| \leq \underline{C}_4 h^2\Big[\sum_{T\in\mathcal{T}_h}\int_{\partial T}\Big[\big|\partial_{n_T}\Delta^3 u\big|^2 + |\partial_{n_T}\nabla\Delta^2 u|^2 dS\Big]^{\frac{1}{2}}\big[J_{h1}(v,v)\big]^{\frac{1}{2}}.$$

Then, going to the unit reference triangle $\hat{T}$, analogously to Proposition 4.4, we obtain, $\forall T \in \mathcal{T}_h$,

$$\begin{aligned} &\int_{\partial T}\Big[\big|\partial_{n_T}\Delta^3 u\big|^2 + \big|\partial_{n_T}\nabla\Delta^2 u\big|^2\Big]dS \\ \leq& \hat{C}_4 h^{-1}\int_T\Big\{\Big[\big|\nabla^2\Delta^2 u\big|^2 + \big|\nabla^3\Delta^2 u\big|^2\Big] + \Big[\big|\nabla\Delta^3 u\big|^2 + \big|\nabla^2\Delta^3 u\big|^2\Big]\Big\}dx. \end{aligned}$$

Recalling (4.4), this readily yields,

$$\big|c_{h4}(u,v)\big| \leq C_4^4 h^{\frac{3}{2}}\Big[\|\Delta^2 u\|_3 + \|\Delta^3 u\|_2\Big]\{v\}_{4,h}. \tag{4.21}$$

Finally (4.18) results from the combination of (4.19), (4.20) and (4.21). □

From Propositions 4.5 and 4.6 it follows that:

Theorem 4.2. *$\exists C_4$ such that the following error estimate holds ,if $u \in H^6(\Omega)$, $\Delta^2 u \in H^3(\Omega)$ and $\Delta^3 u \in H^2(\Omega)$,*

$$\{u-u_h\}_{4,h} \leq C_4\Big[h\|u\|_6 + h^{\frac{3}{2}}\Big(\|\Delta^2 u\|_3 + \|\Delta^3 u\|_2\Big)\Big]. \tag{4.22}$$

5 Three-dimensional problems

In this Section we consider the application of our numerical approach to three-dimensional m-harmonic equations, by taking $m = 4$. Incidentally for $m = 3$ a related method involving just standard non-conformity was introduced in [9]. In view of this, it turns out that DG techniques are not mandatory for three-dimensional triharmonic equations. As we should also explain, the convergence analysis for three-dimensional problems follows the main lines of the one carried out in detail in the previous section for two-dimensional problems. That is why we stress here some points that are really different or new.

First we note that for $m = 4$ the definition of the four types of degrees of freedom applied to a generic function $p \in C^3(T)$ for $T \in \mathcal{T}_h$, in the total amount of $dimP_4 = 35$, can be more conveniently recast in the form of five types of functionals, namely

$$\mathcal{F}_0(p) := p(G_T), \tag{5.1a}$$

$$\mathcal{F}_i^j(p) := \frac{\partial p}{\partial \tau_i^j}(S_i), \qquad 1 \le i,j \le 4, \ \ (i \neq j), \tag{5.1b}$$

$$\mathcal{F}_{ij}^{ii}(p) := \frac{\displaystyle\int_{e_{kl}} \frac{\partial^2 p}{\partial(\sigma_j^i)^2} de}{d_{kl}}, \qquad 1 \le i,j \le 4, \ \ (k,l \neq i, \ k,l \neq j, \ i \neq j), \tag{5.1c}$$

$$\mathcal{F}_{ij}^{ij}(p) := \frac{\displaystyle\int_{e_{kl}} \frac{\partial^2 p}{\partial(\sigma_j^i)\partial(\sigma_i^j)} de}{d_{kl}}, \qquad 1 \le i < j \le 4, \ \ (k,l \neq i, \ k,l \neq j), \tag{5.1d}$$

$$\mathcal{F}^i(p) := \partial_{n_i}^3 p, \qquad 1 \le i \le 4. \tag{5.1e}$$

Next we denote by V_{3h4} the particular version of subspace V_h of W_h, corresponding to the three-dimensional case and to the value $m = 4$. This space is defined as follows:

Definition 5.1. *V_{3h4} is the subspace of W_h of those functions v, whose third order normal derivatives determined on each side of every face common to two tetrahedra of $\mathcal{T}_h$ have the same value at the face's barycenter, whose sets of mean values along an edge of all second order partial derivatives with respect to directions of the plane orthogonal to this edge are the same for all the tetrahedra containing this edge, and such that the gradient of their restrictions to all the tetrahedra of the partition containing a given vertex coincide at this point.*

The construction of the so-defined space V_{3h4} is possible indeed, according to

Proposition 5.1. *For any non degenerated tetrahedron T, the set of thirty-five functions given by*

$$f_0 := 1,$$

$$f_i^j := \tilde{f}_i^j - \sum_{r=1}^{4} \Big[\mathcal{F}^r(\tilde{f}_i^j) f^r + \sum_{s=1,\ s\neq r}^{4} \mathcal{F}_{rs}^{rr}(\tilde{f}_i^j) f_{rs}^{rr} + \sum_{s=r+1}^{4} \mathcal{F}_{rs}^{rs}(\tilde{f}_i^j) f_{rs}^{rr} + \frac{1}{256} \Big], \qquad 1 \le i,j \le 4,$$

$$f_{ij}^{ii} := h_i^j h_j^i \Big(\lambda_i \lambda_j + \frac{3\lambda_i^2 \lambda_j^2}{2} - \lambda_i^2 \lambda_j - \lambda_i \lambda_j^2 - \frac{19}{512} \Big), \qquad 1 \le i,j \le 4, \ \ (j \neq i),$$

$$f_{ij}^{ij} := \frac{1}{2} (h_j^i)^2 \Big(\lambda_i^2 - \frac{2\lambda_i^3}{3} + 2\lambda_i^3 \lambda_j - 2\lambda_i^2 \lambda_j - \frac{11}{384} \Big), \qquad 1 \le i < j \le 4,$$

where:

$$\tilde{f}_i^j := -\frac{d_{ij}}{3}\lambda_i^3\lambda_j, \qquad 1 \leq i,j \leq 4, \ \ (j \neq i),$$

are the canonical basis functions respectively associated with the elements of the set Σ_5 *of thirty-five degrees of freedom, namely*

$$\Sigma_5 := \Big\{\mathcal{F}_0;\ \big[\mathcal{F}_i^j;\ \mathcal{F}_{ij}^{ii},\ 1 \leq i,j \leq 4\ (j \neq i)\big];\ \mathcal{F}_{ij}^{ij},\ 1 \leq i < j \leq 4;\ \mathcal{F}^i,\ 1 \leq i \leq 4\Big\}.$$

Here we recall Lemma 3.2 restricted to the particular case of space V_{3h4}:

Lemma 5.1. *Let* v *be an arbitrary function of* V_{3h4}. *For every face* F *common to two elements of* $\mathcal{T}_h$ *the tensors resulting from the application of the operator* ∇^3 *to the restrictions of* v *to both tetrahedra, coincide at the barycenter of* F.

We also need the following Lemma, whose proof results from the arguments developped in the proof of Lemma 3.2 restricted to the case $m = 4$.

Lemma 5.2. *Let* v *be an arbitrary function of* V_{3h4}. *The mean values along every edge* e *of* $\mathcal{T}_h$, *of the hessian of the restrictions of* v *to the tetrahedra of the partition containing* e *are all the same.*

Next, we denote the space V_h^0 corresponding to $m = 4$ and $n = 3$ by V_{3h4}^0. Notice that the functional $[\cdot]_{4,3,h}$ is a norm of V_{3h4}^0, rewritten here as $[\cdot]_{4,h}$, i.e.,

$$[v]_{4,h}^2 := |v|_{4,h}^2 + \frac{1}{h^4}\Big[\sum_{T \in E_h}\int_{\partial T \cap \Gamma} v_{/T}^2 dS + \frac{1}{2}\sum_{T \in \mathcal{T}_h}\sum_{T' \in \mathcal{T}_h}\int_{\partial T \cap \partial T'}(v_{/T} - v_{/T'})^2 dS\Big]. \tag{5.2}$$

Now we study the application of the method described in Section 3, to solve the polyharmonic equation (1.1) for $m = 4$ and $n = 3$.

We wish to find $u_h \in V_{3h4}^0$ satisfying (5.3) below,

$$a_{h4}^3(u_h, v) = L_f(v), \qquad \forall v \in V_{3h4}^0, \tag{5.3}$$

where L_f is given by (3.5) and a_{h4}^3 by (3.3) (with $m = 4$ and $n = 3$).

As pointed out in Section 3, the approximate problem (5.3)-(3.3)-(3.5) has a unique solution $u_h \in V_{3h4}^0$, and the following error bound holds

$$[u - u_h]_{4,h} \leq \inf_{v \in V_{3h4}^0}[u - v]_{4,h} + \sup_{v \in V_{3h4}^0,\ v \neq 0}\frac{a_{h4}^3(u,v) - L_f(v)}{[v]_{4,h}}. \tag{5.4}$$

The two terms on the right hand side of (5.4) can be estimated according to the following propositions:

Proposition 5.2. $\exists C_5^1$ *such that*

$$\inf_{v \in V_{3h4}^0}[u - v]_{4,h} \leq C_5^1 h||u||_5,$$

for $u \in H^5(\Omega)$.

Proof. First we recall that $[u-v]^2_{4,h}$ is the sum of two terms, namely, $|u-v|^2_{4,h}$ and the summation term, which is nothing but $J_{h0}(u-v,u-v)$ according to (4.6). For the first term, it holds

$$\inf_{v\in V^0_{3h4}} |u-v|^2_{4,h} \leq 2\Big[||u-\Pi^0_{3h4}u||^2_{4,h} + \sum_{T\in E_h}\int_T |u(G_T)|^2 dx\Big],$$

where $\Pi^0_{3h4}w$ denotes the V^0_{3h4}-interpolate of a function $w \in H^5(\Omega)$, and $||\cdot||_{4,h}$ the standard discrete H^4-norm. Since $card(E_h) \leq \bar{C}_3 h^{-2}$ for n = 3, and for $T \in E_h$,

$$|u(G_T)| \leq \tilde{C}_5 h^3 \sup_{\mathbf{x}\in\Omega}\big|\nabla^3(u)(\mathbf{x})\big|,$$

whenever $u \in H^5(\Omega)\cap H^4_0(\Omega)$, we have

$$\inf_{v\in V^0_{3h4}} |u-v|^2_{4,h} \leq C_5'\Big[h^2|u|^2_5 + h^7||u||^2_5\Big].$$

As for the jump terms, by straightforward calculations we obtain

$$\inf_{v\in V^0_{3h4}} J_{h0}(u-v,u-v) \leq \frac{C^0_5}{h^4}\Big[\sum_{T\in\mathcal{T}_h}\int_{\partial T}\big|(u-\Pi_h u)_{/T}\big|^2 dS + \sum_{T\in E_h}\int_{\partial T}\big|u(G_T)\big|^2 dS\Big].$$

Hence, like in Proposition 4.3 we derive the estimate

$$\inf_{v\in V^0_{3h4}} J_{h0}(u-v,u-v) \leq C^*_5\Big[h^5|u|^2_5 + h^2||u||^2_5\Big],$$

and the proof is complete. □

Proposition 5.3. *$\exists C^2_5$ such that the following estimate holds , provided $u \in H^5(\Omega)$, $\Delta u \in H^4(\Omega)$, $\Delta^2 u \in H^3(\Omega)$ and $\Delta^3 u \in H^2(\Omega)$,*

$$\begin{aligned}&\sup_{v\in V^0_{3h4},\ v\neq 0}\frac{a^3_{h4}(u,v)-L_f(v)}{[v]_{4,h}}\\ \leq& C^2_5\Big[h\Big(\|u\|_5+\|\Delta u\|_4\Big)+h^2\|\Delta^2 u\|_3 + h^{\frac{3}{2}}\|\Delta^3 u\|_2\Big].\end{aligned} \tag{5.5}$$

Proof. First we note that the term $J_{h0}(u,v)$ in the expression of a^3_{h4} vanishes if $u \in H^4_0(\Omega)$. Then expanding the remaining term in the numerator of the fraction on the left hand side of (5.5) using integration by parts, under our assumptions on u and taking into account (4.6), we obtain

$$a^3_{h4}(u,v) - L_f(v) = b_{1h}(u,v) + b_{2h}(u,v) + c_h(u,v), \tag{5.6a}$$

$$b_{1h}(u,v) := \sum_{T\in\mathcal{T}_h}\int_{\partial T}\Big[\partial_{n_T}\nabla^3 u\cdot\nabla^3(v_{/T}) - \partial_{n_T}\nabla^2\Delta u\cdot\nabla^2(v_{/T})\Big]dS, \tag{5.6b}$$

$$b_{2h}(u,v) := \sum_{T\in\mathcal{T}_h}\int_{\partial T}\partial_{n_T}\nabla\Delta^2 u\cdot\nabla(v_{/T})dS, \tag{5.6c}$$

$$c_h(u,v) := -\sum_{T\in\mathcal{T}_h}\int_{\partial T}\partial_{n_T}\Delta^3 u\cdot v_{/T}dS. \tag{5.6d}$$

First we deal with bilinear form b_{1h}, and for this purpose we associate with every face F of tetrahedron T the operators Π_0^F and Π_1^F defined respectively on $L^2(F)$ and $P_2(F)$ as follows: $\Pi_0^F(w)$ is the mean value of function w over F and $\Pi_1^F(w)$ is the unique function in $P_1(F)$ whose values at the mid-point of each edge of T coincides with the mean value of function w along this edge, the notation $P_r(F)$ indicating the space of polynomials of degree less than or equal to r defined on F. Notice that $\Pi_0^F(w)$ coincides with $w(M_F)$ if $w \in P_1(F)$.

Thanks to Lemmata 5.1 and 5.2 and to the continuity properties of the traces of u and its derivatives resulting from our assumptions, we may write (cf. [8]):

$$b_{1h}(u,v) = \sum_{T\in\mathcal{T}_h}\sum_{F\subset\partial T}\int_F \Big\{ \Big[\partial_{n_T}\nabla^3 u - \Pi_0^F(\partial_{n_T}\nabla^3 u)\Big]\cdot\Big[\nabla^3 v_{/T} - \Pi_0^F(\nabla^3 v_{/T})\Big] - \partial_{n_T}\nabla^2\Delta u\cdot\Big[\nabla^2 v_{/T} - \Pi_1^F(\nabla^2 v_{/T})\Big]\Big\} dS.$$

Hence by standard arguments for non-conforming methods, we obtain in exactly the same way as in [8]:

$$|b_{h1}(u,v)| \leq C_5^3 h\Big[\|u\|_5 + \|\Delta u\|_4\Big][v]_{4,h}. \tag{5.7}$$

The term $b_{h2}(u,v)$ can be handled by extending to the case $m = 4$ the argument employed in [9] for the case $m = 3$. We recall it below, since it is not so classical: Given a tetrahedron $T \in \mathcal{T}_h$, for every face F of ∂T, we introduce a quadratic interpolant of the trace of $\nabla(v_{/T})$ over F denoted by $\Pi_F^T[\nabla(v_{/T})]$ defined as follows:

$$\begin{aligned} &\Pi_F^T\Big[\nabla(v_{/T})\Big](S) = \nabla v(S), && \forall \text{ vertex } S\in F,\\ &\frac{\partial\Pi_F^T\Big[\nabla(v_{/T})\Big]}{\partial\rho_F^e}(M_e) = \frac{1}{meas(e)}\int_e \frac{\partial\nabla(v_{/T})}{\partial\rho_F^e} de, && \forall \text{ edge } \ e\subset F, \end{aligned}$$

where M_e is the mid-point of e, and the first order partial derivative means the one in the direction of the unit vector $\overrightarrow{\rho}_F^e$ parallel to the median of F corresponding to edge e, oriented from e outwards F. Notice that the interpolating field $\Pi_F^T[\nabla(v_{/T})]$ is uniquely defined $\forall v \in C^2(T)$, and $\forall F \subset \partial T$ for any non degenerated tetrahedron T, since the six interpolation degrees of freedom on which it is based upon, are nothing but those of the affine-equivalent element derived from the Morley element associated with the triangle F, by replacing the (first order) normal derivative at M_e with the derivative at this point in the direction of $\overrightarrow{\rho}_F^e$, for every edge e of F (cf. [2] and [4]). In so doing we recall that, according to Lemma 5.2, the mean values along any edge e of face F of the second order derivatives of v, are the same for all the tetrahedra containing e, in particular those with respect to directions of the plane of this face. Hence due to the continuity of ∇v at the vertices of F, for every pair of tetrahedra T and T' of $\mathcal{T}_h$ having F as a common face, we have

$$\Pi_F^T\Big[\nabla(v_{/T})\Big] = \Pi_F^{T'}\Big[\nabla(v_{/T'})\Big].$$

Furthermore by a similar argument $\Pi_F^T[\nabla(v_{/T})]$ vanishes identically for every F contained in Γ from the construction of V_{3h4}^0.

On the other hand, the assumption $\Delta^2 u \in H^3(\Omega)$ implies that

$$\partial_{n_T}(\nabla\Delta^2 u) + \partial_{n_{T'}}(\nabla\Delta^2 u) = 0,$$

on every face F of the partition common to two tetrahedra T and T'. Taking into account both arguments above, we may rewrite $b_{h2}(u,v)$ in the following manner:

$$b_{h2}(u,v) = \sum_{T\in\mathcal{T}_h}\sum_{F\subset\partial T}\int_F \partial_{n_T}(\nabla\Delta^2 u)\cdot\Big\{\nabla(v_{/T}) - \Pi_F^T\Big[\nabla(v_{/T})\Big]\Big\}dS. \tag{5.8}$$

Now let $\hat{T}$ be the reference tetrahedron and $\mathcal{F}_T$ be the invertible affine mapping from T onto $\hat{T}$; we define the continuous interpolation operator $\hat{\Pi}_{\hat{F}}$ from $[H^3(\hat{T})]^3$ onto the space of quadratic vector fields on $\hat{F} := \mathcal{F}_T(F)$ through interpolation of fields of $[H^3(\hat{T})]^3$ by using the same set of six degrees of freedom for $\hat{F}$ as those defining Π_F^T for F. Then setting

$$\overrightarrow{\hat{w}} := \overrightarrow{w}\circ\mathcal{F}_T^{-1}, \qquad \forall\overrightarrow{w}\in[H^3(T)]^3,$$

from the affine properties of operator Π_F^T we certainly have

$$\widehat{\Pi_F^T(\overrightarrow{w})} = \hat{\Pi}_{\hat{F}}(\overrightarrow{\hat{w}}), \qquad \forall w\in[H^3(T)]^3.$$

Moreover we have

$$\hat{\Pi}_{\hat{F}}[\overrightarrow{\hat{w}}] = \overrightarrow{\hat{w}},$$

whenever $\overrightarrow{\hat{w}}$ is quadratic, and thus we may apply again standard estimates for non-conforming finite elements (cf. [9]) to $b_{h2}(u,v)$. More specifically taking $\overrightarrow{w} = \nabla(v_{/T})$ and using (5.8), these estimates directly leads to the existence of a constant C_5^4 such that

$$\big|b_{h2}(u,v)\big| \le C_5^4 h^2\|\Delta^2 u\|_3[v]_{4,h}, \tag{5.9}$$

$c_h(u,v)$ in turn can be estimated as follows: First we note that $\forall v\in V_{3h4}^0$,

$$c_h(u,v) = -\sum_{T\in E_h}\Big[\int_{\partial T\cap\Gamma}\partial_{n_T}\Delta^3 u\cdot v_{/T}dS + \frac{1}{2}\sum_{T'\in\mathcal{T}_h}\int_{\partial T\cap\partial T'}\partial_{n_T}\Delta^3 u\cdot(v_{/T}-v_{/T'})\Big]dS.$$

Hence owing to our assumptions, we have:

$$\big|c_h(u,v)\big| \le \underline{C}_5 h^2\Big[\sum_{T\in\mathcal{T}_h}\int_{\partial T}|\partial_{n_T}\Delta^3 u|^2 dS\Big]^{\frac{1}{2}}\Big[J_{h0}(v,v)\Big]^{\frac{1}{2}}.$$

Since

$$\int_{\partial T}\big|\partial_{n_T}\Delta^3 u\big|^2 dS \le \hat{C}_5 h^{-1}\int_T\Big[\big|\nabla\Delta^3 u\big|^2 + \big|\nabla^2\Delta^3 u\big|^2\Big]dx, \qquad \forall T\in\mathcal{T}_h,$$

recalling (5.2), this readily yields

$$\left|c_h(u,v)\right| \leq C_5^5 h^{\frac{3}{2}} \|\Delta^3 u\|_2 [v]_{4,h}, \tag{5.10}$$

Then combining (5.6), (5.7), (5.9) and (5.10) we establish that (5.5) holds. □

From Propositions 5.2 and 5.3 immediately follows,

Theorem 5.1. *$\exists C_5$ such that the following error estimate holds, if $u \in H^5(\Omega)$, $\Delta u \in H^4(\Omega)$, $\Delta^2 u \in H^3(\Omega)$, and $\Delta^3 u \in H^2(\Omega)$,*

$$[u-u_h]_{4,h} \leq C_5\Big[h\Big(\|u\|_5 + \|\Delta u\|_4\Big) + h^2\|\Delta^2 u\|_3 + \ h^{\frac{3}{2}}\|\Delta^3 u\|_2\Big]. \tag{5.11}$$

6 Miscellaneous remarks

1. Though not expressed in exactly the same form, the error estimates given in Sections 4 and 5 are qualitatively compatible with the error estimate in the norm $[\cdot]_{m,n,h}$ for arbitrary m given by (3.7).

2. The authors conjecture that $\mathcal{O}(h^2)$ error estimates in the L^2-norm can be derived from the estimate (3.7), using classical duality arguments. However formal proofs in this sense lack of a rigourous basis, since to the best of their knowledge no regularity results are known for the solution of (1.1) in case Ω is either a polygon or a polyhedron.

3. Boundary element methods could be used to solve polyharmonic equations in two-dimension space, using the boundary integral technique proposed in [5]. In the three-dimensional case such possibility is not known. In view of next remark too, our method if then a good option for solving triharmonic problems in both cases.

4. At first glance the implementation of our method may seem prohibitive to solve even triharmonic problems in $\mathbb{R}^2$, as compared to boundary integral methods or to series expansion techniques (cf. [11]). However at least in this case if one uses efficient iterative methods, such as pre-conditionned conjugate gradients, to solve the linear system of equations resulting from our discretization procedure, the numerical solution should not be so expensive. Indeed for a given mesh there are roughly 70% more degrees of freedom involved with our method than with the Morley element. Moreover a given half matrix row has no more than ten non zero entries, whereas for the Morley triangle there are about six. This means that solving the polyharmonic equation with our method costs barely less than three times as much, as solving a biharmonic equation with the Morley element.

5. In the same way as the Morley triangle and its three-dimensional counterpart [8] are the simplest possible finite elements to solve Eq. (1.1) for $m = 2$ using a single field formulation, as far as we can see this is also the case of the new family for $m > 2$. Furthermore, to the best of the authors' knowledge, except for the case $n = 2$ and $m = 3$ (in which our method is certainly competitive anyway according to the previous remark), no other method has yet been proposed to solve this equation in arbitrary

domains. Notice that for cartesian domains only, suitable finite difference methods could be good solution alternatives too.

Appendix

A refined error bound for stable external approximations

The aim of this Appendix is to supply theoretical support to the error bound (3.4) given in Proposition 3.9.

Suppose that one wishes to determine an approximation u_h of a given element u in a certain Hilbert space V with norm $||\cdot||$, in another Hilbert space V_h that is not necessarily a subspace of V, but has the same null element $\mathbf{0}$ as V. Both V_h and V are assumed to be equipped with a norm denoted by $||\cdot||_h$. Although this is not necessary to derive the error bound below, one may assume that $||v||_h = ||v||$, $\forall v \in V$, which incidentally happens in the application in view.

As a general framework, we consider that the approximation u_h of u is determined by solving a variational problem of the form

$$a_h(u_h, v) = L_h(v), \qquad \forall v \in V_h,$$

where

$$a_h : (V_h + V) \times (V_h + V) \longrightarrow \mathbb{R}, \quad \text{and} \quad L_h : V_h \longrightarrow \mathbb{R},$$

are respectively a continuous bilinear form and a continuous linear form with respect to the norm $||\cdot||_h$. This means that there exist two constants M and N such that:

$$\begin{aligned} &a_h(u,v) \le M||u||_h||v||_h, &&\forall (u,v) \in (V_h+V)\times(V_h+V),\\ &L_h(v) \le N||v||_h, &&\forall v \in V_h. \end{aligned}$$

According to Dupire [3] such variational problem has a unique solution $u_h \in V_h$ if and only if bilinear form a_h is weakly-coercive on $V_h \times V_h$, which means that it satisfies simultaneously the following conditions:

$$\begin{aligned} &\exists \alpha > 0, \quad \text{such that} \quad \forall u \in V_h, \quad \sup_{v \in V_h - \{\mathbf{0}\}} \frac{a_h(u,v)}{||v||_h} \ge \alpha ||u||_h,\\ &\forall v \in V_h - \{\mathbf{0}\}, \ \exists u \in V_h, \quad \text{such that} \quad a_h(u,v) \ne 0. \end{aligned}$$

Clearly enough the second condition above is a consequence of the first one if form a_h is symmetric. Moreover if a_h is coercive, that is, if it satisfies,

$$\exists \alpha^* > 0, \quad \text{such that} \quad a_h(v,v) \ge \alpha^* ||v||_h^2, \ \forall v \in V_h,$$

then both conditions above hold with $\alpha \ge \alpha^*$.

In the classical litterature the error bound known as the second Strang's inequality is used, in order to derive error estimates for non conforming finite element approximations with coercive bilinear forms (cf. [2]). This inequality is recalled below:

$$||u-u_h||_h \leq \frac{1}{\alpha^*}\left[(M+\alpha^*)\inf_{v\in V_h}||u-v||_h + \sup_{v\in V_h-\{\mathbf{0}\}}\frac{a_h(u,v)-L_h(v)}{||v||_h}\right].$$

However on the basis of the work of Dupire [3] this result can be refined. Essentially this refinement allows the coerciveness constant α^* to be dropped in the above brackets. Moreover it extends this improvement to the more general case of weakly coercive bilinear forms, according to

Proposition A.1. Equipping space $V+V_h$ with the norm $||\cdot||_h$, and assuming that bilinear form a_h is continuous on $(V+V_h)\times(V+V_h)$ with constant M, and weakly-coercive on $V_h\times V_h$ with constant α, and that L_h is a continuous linear form on V_h with constant N, the following error bound holds:

$$||u-u_h||_h \leq \mathcal{C}\left[\inf_{v\in V_h}||u-v||_h + \sup_{v\in V_h-\{\mathbf{0}\}}\frac{a_h(u,v)-L_h(v)}{||v||_h}\right].$$

where $\mathcal{C}=1/\alpha\max\{1,M\}$.

Proof. Trivially enough the linear form $L_u: V_h \longrightarrow \mathbb{R}$ defined by

$$L_u(v) := a_h(u,v) - L_h(v), \qquad \forall v\in V_h,$$

is continuous with a continuity constant that may be taken equal to $M||u||_h+N$. Let then $w_h\in V_h$ be the unique solution of the well-posed linear variational problem

$$a_h(w_h,v)=L_u(v), \qquad \forall v\in V_h.$$

By the definition of u_h, we have

$$a_h(u_h,v) = -a_h(w_h,v)+a_h(u,v), \qquad \forall v\in V_h,$$

which means that

$$v_h := u_h + w_h \in V_h,$$

satisfies

$$a_h(v_h,v)=a_h(u,v), \qquad \forall v\in V_h.$$

Applying a result due to Dupire [3] this relation implies that

$$||u-v_h||_h \leq \frac{M}{\alpha}\inf_{v\in V_h}||u-v||_h,$$

(note that if a_h is coercive this bound is simply Céa's Lemma [2]). It follows that

$$||u-u_h||_h \leq \frac{M}{\alpha}\inf_{v\in V_h}||u-v||_h + ||w_h||_h.$$

On the other hand we know that w_h satisfies,

$$\alpha||w_h||_h \leq \sup_{v \in V_h - \{\mathbf{0}\}} \frac{a_h(w_h, v)}{||v||_h}.$$

Replacing in the above fraction $a_h(w_h, v)$ with $a_h(u, v) - L_h(v)$, the result follows. $\square$

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