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A Priori and a Posteriori Error Analysis of the Discontinuous Galerkin Methods for Reissner-Mindlin Plates

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Abstract. In this paper, we apply an a posteriori error control theory that we develop in a very recent paper to three families of the discontinuous Galerkin methods for the Reissner-Mindlin plate problem. We derive robust a posteriori error estimators for them and prove their reliability and efficiency.

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1 Introduction

This paper will consider a posteriori error analysis of finite element methods for the Reissner-Mindlin plate problem: given $g \in L^2(\Omega)$ find

$$(w, \boldsymbol{\phi}) \in W \times \boldsymbol{\Theta} := H_0^1(\Omega) \times H_0^1(\Omega)^2,$$

with

$$a(\boldsymbol{\phi}, \boldsymbol{\psi}) + (\gamma, \nabla v - \boldsymbol{\psi})_{L^2(\Omega)} = (g, v)_{L^2(\Omega)}, \quad \text{for all } (v, \boldsymbol{\psi}) \in W \times \boldsymbol{\Theta}, \tag{1.1}$$

and the shear force

$$\gamma = \lambda t^{-2}(\nabla w - \boldsymbol{\phi}). \tag{1.2}$$

Here and throughout this paper, t denotes the plate thickness, $\lambda = Ek/2(1+\nu)$ the shear modulus, E the Young modulus, ν the Poisson ratio, and κ the shear correction factor. Given $\boldsymbol{\phi} \in \boldsymbol{\Theta}$, the linear Green strain $\varepsilon(\boldsymbol{\phi}) = 1/2[\nabla \boldsymbol{\phi} + \nabla \boldsymbol{\phi}^T]$ is the symmetric

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part of gradient field $\nabla\boldsymbol{\phi}$. For all 2×2 symmetric matrices the linear operator $\mathcal{C}$ is defined by

$$\mathcal{C}\boldsymbol{\tau} := \frac{E}{12(1-\nu^2)}\big[(1-\nu)\boldsymbol{\tau} + \nu\,\mathrm{tr}(\boldsymbol{\tau})I\big].$$

The bilinear form $a(\cdot,\cdot)$ models the linear elastic energy defined as

$$a(\boldsymbol{\phi},\boldsymbol{\psi}) = (\mathcal{C}\varepsilon(\boldsymbol{\phi}),\varepsilon(\boldsymbol{\psi}))_{L^2(\Omega)}, \quad \text{for any } \ \boldsymbol{\phi}, \ \ \boldsymbol{\psi}\in\boldsymbol{\Theta}, \tag{1.3}$$

which gives rise to the norm

$$\|\boldsymbol{\psi}\|_{\mathcal{C}}^2 := a(\boldsymbol{\psi},\boldsymbol{\psi}), \quad \text{for any } \ \boldsymbol{\psi}\in\boldsymbol{\Theta}, \tag{1.4}$$

while $\|\cdot\|_{\mathcal{C}_h}$ denotes the broken version with the piecewise defined operator ε_h taking the place of ε, and $(\cdot,\cdot)_{L^2(\Omega)}$ the L^2 scalar product.

This plate theory has become a popular plate bending model in the engineering community due to its simplicity and effectiveness. However, a direct finite element approximation usually yields poor numerical results, i.e., they are too small compared with the continuous solutions. Such a phenomenon is usually referred to as *shear locking*. To weaken or even overcome the locking, many methods have been proposed, most of them can be regarded as reduction integration methods. Very recently, three class of the discontinuous Galerkin methods are used to discretize the Reissner-Mindlin plate problems [1,2]. The aim of this paper is to provide a robust a prior and a posteriori error analysis for these methods.

2 Notation and preliminary results

We use the standard differential operators:

$$\nabla r = \Big(\frac{\partial r}{\partial x},\frac{\partial r}{\partial y}\Big), \qquad \mathrm{curl}\, p = \Big(\frac{\partial p}{\partial y},-\frac{\partial p}{\partial x}\Big).$$

Given any $2D$ vector function

$$\boldsymbol{\psi} = (\psi_1,\psi_2),$$

its divergence reads

$$\mathrm{div}\,\boldsymbol{\psi} = \frac{\partial\psi_1}{\partial x} + \frac{\partial\psi_2}{\partial y}.$$

With the differential operator

$$\mathrm{rot}\,\boldsymbol{\psi} = \frac{\partial\psi_2}{\partial x} - \frac{\partial\psi_1}{\partial y},$$

for a vector function $\boldsymbol{\psi}=(\psi_1,\psi_2)$, the space $H_0(\mathrm{rot},\Omega)$ is defined as

$$H_0(\mathrm{rot},\Omega) := \big\{\boldsymbol{v}\in L^2(\Omega)^2, \ \ \mathrm{rot}\,\boldsymbol{v}\in L^2(\Omega) \text{ and } \boldsymbol{v}\cdot\boldsymbol{\tau} = 0 \text{ on } \partial\Omega\big\},$$

with the unit tangential vector $\boldsymbol{\tau}$ of the boundary $\partial\Omega$ and the endowed norm $\|\boldsymbol{v}\|_{H(\mathrm{rot},\Omega)}$. The dual space for $H_0(\mathrm{rot},\Omega)$ reads

$$H^{-1}(\mathrm{div},\Omega) := \big\{\boldsymbol{v} \in H^{-1}(\Omega)^2,\ \mathrm{div}\,\boldsymbol{v} \in H^{-1}(\Omega)\big\},$$

with the norm $\|\boldsymbol{v}\|_{H^{-1}(\mathrm{div},\Omega)}$.

Suppose that the closure $\overline{\Omega}$ is covered exactly by a regular triangulation $\mathcal{T}_h$ of $\overline{\Omega}$ into (closed) triangles or quadrilaterals in $2D$ or other unions of simplices, that is

$$\overline{\Omega} = \cup\mathcal{T}_h \quad \text{and} \quad |K_1 \cap K_2| = 0, \quad \text{for } K_1, K_2 \in \mathcal{T}_h, \quad \text{with } K_1 \neq K_2, \tag{2.1}$$

where $|\cdot|$ denotes the volume (as well as the length of an edge and the modulus of a vector etc, when there is no real risk of confusion). Let $\mathcal{E}$ denote the set of all edges in $\mathcal{T}_h$ with $\mathcal{E}(\Omega)$ the set of interior edges, and $\mathcal{N}(\Omega)$ the set of interior nodes. $\mathcal{E}(K)$ is the set of edges of the element K, and h_K is the diameter of the element $K\in\mathcal{T}_h$. Also, we denote by ω_K the element patch defined as

$$\omega_K := \big\{T \in \mathcal{T}_h : \bar{T} \cap \bar{K} \neq \emptyset\big\},$$

and by ω_E the union of elements having in common the edge E. Given any edge $E\in\mathcal{E}(\Omega)$ with length $h_E=|E|$ we assign one fixed unit normal

$$\boldsymbol{\nu}_E := (\nu_1,\ \nu_2),$$

and tangential vector

$$\boldsymbol{\tau}_E := (-\nu_2,\ \nu_1).$$

For E on the boundary we choose $\boldsymbol{\nu}_E=\boldsymbol{\nu}$ the unit outward normal to Ω. Once $\boldsymbol{\nu}_E$ and $\boldsymbol{\tau}_E$ have been fixed on E, in relation to $\boldsymbol{\nu}_E$ one defines the elements $K_-\in\mathcal{T}_h$ and $K_+\in\mathcal{T}_h$, with

$$E = K_+ \cap K_- \quad \text{and} \quad \omega_E = K_+ \cup K_-.$$

Given $E\in\mathcal{E}(\Omega)$ and some $\mathbb{R}^d$-valued function v defined in Ω, with d=1, 2, we denote by

$$[v] := (v|_{K_+})|_E - (v|_{K_-})|_E,$$

the jump of v across E.

Let $\hat{K}$ be a reference element. In the case of triangles

$$\hat{K} := \big\{(\xi,\eta) \in \mathbb{R}^2 :\ 0 \le \xi \le 1,\ 0 \le \eta \le 1-\xi\big\},$$

and quadrilaterals $\hat{K} :=[-1,1]^2$. The invertible linear (resp. bilinear) transformation $\hat{K}\to K$ is denoted by F_K for any triangle (resp. quadrilateral) $K\in\mathcal{T}_h$ with the Jacobian matrix DF_K and its determinant J_K.

Let $S_0^1(\mathcal{T}_h)$ denote the lowest order conforming finite element space over $\mathcal{T}_h$ which reads

$$S_0^1(\mathcal{T}_h) := \big\{v \in H_0^1(\Omega),\ v|_K \circ F_K \in Q_1(\hat{K}),\ \forall K \in \mathcal{T}_h\big\}.$$

Given a nonnegative integer k, the space $Q_k(\omega)$ consists of polynomials of total degree at most k defined over ω in the case $\omega=K$ is a triangle whereas it denotes polynomials of degree at most k in each variable in the case K is a quadrilateral.

With the first order conforming finite element space $S_0^1(\mathcal{T}_h)$, we consider the Clément-type interpolation operator or any other regularized conforming finite element approximation operator

$$\mathcal{J} : H_0^1(\Omega) \mapsto S_0^1(\mathcal{T}_h),$$

with the properties

$$\|\nabla \mathcal{J}\varphi\|_{L^2(K)} + \|h_K^{-1}(\varphi - \mathcal{J}\varphi)\|_{L^2(K)} \lesssim \|\nabla\varphi\|_{L^2(\omega_K)}, \tag{2.2a}$$

$$\|h_E^{-\frac{1}{2}}(\varphi - \mathcal{J}\varphi)\|_{L^2(E)} \lesssim \|\nabla\varphi\|_{L^2(\omega_K)}, \tag{2.2b}$$

for all $K\in\mathcal{T}_h$, $E\in\mathcal{E}(K)$, and $\varphi\in H_0^1(\Omega)$. The existence of such operators is guaranteed, for instance, in [4,11,17,23].

The piecewise defined gradient operator is denoted by ∇_h, and ε_h is the piecewise counterpart of ε for elements in $\mathbf{\Theta}_h$. The broken H^1 norm $\|\cdot\|_{1,h}$ is defined as

$$\|v_h\|_{1,h} := \Big(\sum_{K\in\mathcal{T}_h} \|v_h\|^2_{H^1(K)}\Big)^{\frac{1}{2}}, \quad \text{for all } \ v_h \in W + W_h.$$

Here and throughout this paper, $\mathbf{\Theta}_h\subset L^2(\Omega)^2$ and $W_h\subset L^2(\Omega)$ denote some finite element spaces over some regular partition $\mathcal{T}_h$ while $\boldsymbol{R}_h$ denotes the reduction integration operator in the context of shear locking with values in the discrete shear force space $\mathbf{\Gamma}_h$.

We need some further notation for discontinuous Galerkin methods under consideration. Define

$$H^k(\mathcal{T}_h) := \big\{v \in L^2(\Omega),\ v|_K \in H^k(K), \text{ for any } K \in \mathcal{T}_h\big\}, \quad k = 1,2, \tag{2.3a}$$

$$\boldsymbol{H}^2(\mathcal{T}_h) := H^2(\mathcal{T}_h) \times H^2(\mathcal{T}_h). \tag{2.3b}$$

If $\psi\in H^1(\mathcal{T}_h)$ (or possibly the vector- or tensor-valued analogue), we define the average $\{\psi\}$ on $E\in\mathcal{E}(\Omega)$ as usual:

$$\{\psi\} = \frac{\psi^+ + \psi^-}{2}, \tag{2.4}$$

with

$$\psi^+ = \psi|_{K^+} \quad \text{and} \quad \psi^- = \psi|_{K^-}.$$

On the boundary $E\subset\partial\Omega$, the average $\{\psi\}$ is defined simply as the trace of ψ. The symmetric jump $[\boldsymbol{\psi}]_S$ is defined as

$$[\boldsymbol{\psi}]_S = (\boldsymbol{\psi}^+ \otimes \boldsymbol{\nu}_E^+)_S + (\boldsymbol{\psi}^- \otimes \boldsymbol{\nu}_E^-)_S, \tag{2.5}$$

where $(\boldsymbol{\psi}\otimes\boldsymbol{\nu}_E)_S$ denotes the symmetric part of the tensor, and $\boldsymbol{\nu}_E^+$ (resp. $\boldsymbol{\nu}_E^-$) is the outward unit normal to $E\subset\partial K^+$ (resp. $E\subset\partial K^-$). Given $\boldsymbol{\phi}_h\in\boldsymbol{H}^2(\mathcal{T}_h)$ and $\boldsymbol{\psi}_h\in\boldsymbol{H}^2(\mathcal{T}_h)$, we define the following bilinear form:

$$\begin{aligned}\mathfrak{A}_h(\boldsymbol{\phi}_h,\boldsymbol{\psi}_h) :=&\big(\mathcal{C}\varepsilon_h(\boldsymbol{\phi}_h),\varepsilon_h(\boldsymbol{\psi}_h)\big)_{L^2(\Omega)}-\sum_{E\in\mathcal{E}}\big(\{\mathcal{C}\varepsilon_h(\boldsymbol{\phi}_h)\},[\boldsymbol{\psi}_h]\big)_{L^2(E)}\\ &-\sum_{E\in\mathcal{E}}\big(\{\mathcal{C}\varepsilon_h(\boldsymbol{\psi}_h)\},[\boldsymbol{\phi}_h]\big)_{L^2(E)}+\sum_{E\in\mathcal{E}}\frac{\gamma_E}{h_E}\big([\boldsymbol{\phi}_h]_S,[\boldsymbol{\psi}_h]_S\big)_{L^2(E)},\end{aligned}\tag{2.6}$$

where γ_E are some positive constants to ensure the stability of the discrete problem. Let β_E be some positive constant, we define the following penalty term

$$P_W(u,v)=\sum_{E\in\mathcal{E}}\frac{\beta_E}{h_E}\int_E[u][v]ds,\quad \text{for any } u\in W_h \text{ and } v\in W_h. \tag{2.7}$$

For all methods considered herein, their discrete problems read: find $(w_h,\boldsymbol{\phi}_h,\gamma_h)\in W_h\times\boldsymbol{\Theta}_h\times\boldsymbol{\Gamma}_h$ with

$$\begin{aligned}&\mathfrak{A}_h(\boldsymbol{\phi}_h,\boldsymbol{\psi}_h)+P_W(w_h,v_h)+\big(\gamma_h,\boldsymbol{R}_h(\nabla_h v_h-\boldsymbol{\psi}_h)\big)_{L^2(\Omega)}\\ &\quad-\sum_{E\in\mathcal{E}}\int_E\{\gamma_h\}\cdot[v_h]\boldsymbol{\nu}_E ds=(g,v_h)_{L^2(\Omega)},\quad \text{for any } (v_h,\boldsymbol{\psi}_h)\in W_h\times\boldsymbol{\Theta}_h,\end{aligned}\tag{2.8a}$$

$$\begin{aligned}&\big(\boldsymbol{R}_h(\nabla_h w_h-\boldsymbol{\phi}_h),\boldsymbol{\sigma}\big)_{L^2(\Omega)}-\sum_{E\in\mathcal{T}_h}\int_E[w_h]\boldsymbol{\nu}_E\cdot\{\boldsymbol{\sigma}\}ds\\ &\quad-\lambda^{-1}t^2(\gamma_h,\boldsymbol{\sigma})_{L^2(\Omega)}=0,\qquad \text{for any } \boldsymbol{\sigma}\in\boldsymbol{\Gamma}_h.\end{aligned}\tag{2.8b}$$

We define norms

$$|||\boldsymbol{\psi}|||_{\Theta}^2:=\|\boldsymbol{\psi}\|_{1,h}^2+\sum_{E\in\mathcal{E}}\frac{1}{h_E}\|[\boldsymbol{\psi}]_S\|_{L^2(E)}^2,\qquad \boldsymbol{\psi}\in\boldsymbol{H}^2(\mathcal{T}_h),\tag{2.9a}$$

$$|||v|||_W^2:=\|v\|_{1,h}^2+\sum_{E\in\mathcal{E}}\frac{1}{h_E}\|[v]\|_{L^2(E)}^2,\qquad v\in H^1(\mathcal{T}_h),\tag{2.9b}$$

$$|||\boldsymbol{\sigma}|||_{\Gamma}^2:=\|\boldsymbol{\sigma}\|_{L^2(\Omega)}^2+\sum_{E\in\mathcal{E}}h_E\|[\boldsymbol{\sigma}]_S\|_{L^2(E)}^2,\qquad \boldsymbol{\sigma}\in\boldsymbol{H}^1(\mathcal{T}_h).\tag{2.9c}$$

In a very recent paper [18], we develop an a posteriori error control theory of finite element methods for Reissner-Mindlin plates. Given some positive function

$$\alpha\in L^\infty(\Omega),\quad \text{with } \|\alpha\|_{L^\infty(\Omega)}^2<\frac{\lambda}{t^2},$$

we define a positive function $\beta\in L^\infty(\Omega)$ satisfying

$$\frac{1}{\beta^2}=\frac{\lambda}{t^2}-\alpha^2.$$

We define the following norm on the space $W \times \mathbf{\Theta}$:

$$|||\boldsymbol{\phi}, w|||^2 := \|\boldsymbol{\phi}\|_{\mathcal{C}}^2 + \|\alpha(\nabla w - \boldsymbol{\phi})\|_{L^2(\Omega)}^2, \quad \text{for any } (w, \boldsymbol{\phi}) \in W \times \mathbf{\Theta}. \tag{2.10}$$

Define the following mesh dependent norm for the space $\boldsymbol{Q}$:

$$\|\delta\|_Q := \sup_{0 \neq (\boldsymbol{\psi}, v) \in H_0^1(\Omega)^3} \frac{(\nabla v - \boldsymbol{\psi}, \delta)_{L^2(\Omega)}}{|||\boldsymbol{\psi}, v|||} + \|\beta\delta\|_{L^2(\Omega)}, \quad \text{for any } \delta \in \boldsymbol{Q}. \tag{2.11}$$

Let $a_h(\cdot, \cdot)$ be the discrete bilinear form which will be specified for various finite element methods with a suitable linear operator L (the so called linking operator). The discrete problem reads: find $(w_h, \boldsymbol{\phi}_h) \in W_h \times \mathbf{\Theta}_h$ such that

$$a_h(\boldsymbol{\phi}_h, \boldsymbol{\psi}_h) + \big(\gamma_h, \boldsymbol{R}_h(\nabla_h(v_h + L\boldsymbol{\psi}_h) - \boldsymbol{\psi}_h)\big)_{L^2(\Omega)} = (g, v_h + L\boldsymbol{\psi}_h)_{L^2(\Omega)}, \tag{2.12}$$

for any $(v_h, \boldsymbol{\psi}_h) \in W_h \times \mathbf{\Theta}_h$. The discrete shear force γ_h is defined as

$$\gamma_h = \lambda t^{-2} \boldsymbol{R}_h\big(\nabla_h(w_h + L\boldsymbol{\phi}_h) - \boldsymbol{\phi}_h\big). \tag{2.13}$$

Herein and throughout this paper, $\boldsymbol{R}_h$ is the reduction integration operator from $L^2(\Omega)^2$ to the discrete shear force space $\mathbf{\Gamma}_h$. The role of $\boldsymbol{R}_h$ is to reduce the effect of the shear force and then weaken or overcome the locking. Let $(\tilde{w}_h, \tilde{\boldsymbol{\phi}}_h, \tilde{\gamma}_h') \in W \times \mathbf{\Theta} \times \boldsymbol{Q}$ be some approximation to $(w, \boldsymbol{\phi}, \gamma')$ over some regular partition $\mathcal{T}_h$, which are *undetermined* and *not* necessarily discrete functions. With the residual

$$\tilde{\boldsymbol{r}}_h = \beta^2 \tilde{\gamma}_h' - (\nabla \tilde{w}_h - \tilde{\boldsymbol{\phi}}_h),$$

and the usual Clément interpolation operator $\mathcal{J}$ [4, 11, 17], we define the following abstract estimators:

$$
\begin{aligned}
\eta_K &:= \alpha_K^{-1} h_K \|\operatorname{div} \gamma_h + g\|_{L^2(K)} + h_K \|\operatorname{div} \mathcal{C}\varepsilon_h(\boldsymbol{\phi}_h) + \gamma_h\|_{L^2(K)}, \\
\eta_E &:= \alpha_E^{-1} h_E^{\frac{1}{2}} \|[\gamma_h] \cdot \boldsymbol{\nu}_E\|_{L^2(E)} + h_E^{\frac{1}{2}} \|[\mathcal{C}\varepsilon_h(\boldsymbol{\phi}_h)] \cdot \boldsymbol{\nu}_E\|_{L^2(E)}, \\
\mu_h(\gamma_h) &:= \sup_{0 \neq (v, \boldsymbol{\psi}) \in W \times \mathbf{\Theta}} \frac{\big(\gamma_h, (\boldsymbol{R}_h - I)(\nabla(\mathcal{J}v + (I - \mathcal{J})L\mathcal{J}\boldsymbol{\psi}) - \mathcal{J}\boldsymbol{\psi})\big)_{L^2(\Omega)}}{|||\boldsymbol{\psi}, v|||}, \\
\tilde{\eta}(\tilde{\boldsymbol{r}}_h) &:= \sup_{0 \neq p \in \hat{H}^1(\Omega)} \frac{(\tilde{\boldsymbol{r}}_h, \operatorname{curl} \mathcal{J}p)_{L^2(\Omega)}}{\|p\|_{L^2(\Omega)} + \|t\nabla p\|_{L^2(\Omega)}} + \Big(\sum_{E \in \mathcal{E}} \min\Big(\frac{1}{t}, \frac{h_E}{t^2}\Big) \|[\tilde{\boldsymbol{r}}_h] \cdot \boldsymbol{\tau}_E\|_{L^2(E)}^2\Big)^{\frac{1}{2}} \\
&\quad + \Big(\sum_{K \in \mathcal{T}_h} \min\Big(1, \frac{h_K}{t}\Big)^2 \|\operatorname{rot} \tilde{\boldsymbol{r}}_h\|_{L^2(K)}^2\Big)^{\frac{1}{2}} + \|\alpha \tilde{\boldsymbol{r}}_h\|_{L^2(\Omega)}, \\
\tilde{\eta}_R &:= \Big\|\frac{1}{\alpha}(\gamma_h - \alpha^2(\nabla \tilde{w}_h - \tilde{\boldsymbol{\phi}}_h) - \tilde{\gamma}_h')\Big\|_{L^2(\Omega)},
\end{aligned}
$$

where $\alpha_E=\alpha|_E$ and $\alpha_K=\alpha|_K$. Then we can define the abstract estimator as

$$\begin{aligned}\tilde{\eta}_h^2 :=& \sum_{K\in\mathcal{T}_h}\eta_K^2 + \sum_{E\in\mathcal{E}(\Omega)}\eta_E^2 + \mu_h(\gamma_h)^2 + \sum_{E\in\mathcal{E}} h_E^{-1}\|[\boldsymbol{\phi}_h]\|^2_{L^2(E)} \\ &+ \|\varepsilon_h(\boldsymbol{\phi}_h - \tilde{\boldsymbol{\phi}}_h)\|^2_{L^2(\Omega)} + \tilde{\eta}(\tilde{\boldsymbol{r}}_h)^2 + \tilde{\eta}_R^2, \end{aligned} \tag{2.14}$$

for any $\tilde{w}_h\in W$, $\tilde{\boldsymbol{\phi}}_h\in\boldsymbol{\Theta}$, and $\tilde{\gamma}'_h\in\boldsymbol{Q}$. As mentioned above, L is a suitable linear operator which will be specified for various methods. Under conditions (H1)-(H3) of [18], the theory of [18, Theorem 3.3] states that

Theorem 2.1. *There holds*

$$|||\boldsymbol{\phi}-\tilde{\boldsymbol{\phi}}_h, w-\tilde{w}_h||| + \|\gamma'-\tilde{\gamma}'_h\|_Q \lesssim \tilde{\eta}_h. \tag{2.15}$$

3 The Arnold-Brezzi-Falk-Marini element I

Based on the regular triangular partition of Ω, this class of methods read as [1]

$$W_h := \big\{v\in H_0^1(\Omega),\ v|_K\in P_k(K),\ \text{for any } K\in\mathcal{T}_h\big\}, \tag{3.1}$$

for any integer $k\geq 2$. The shear force and the rotation spaces are the Brezzi-Douglas-Marini space of degree $k-1$ which reads

$$\boldsymbol{\Theta}_h = \boldsymbol{\Gamma}_h = \mathrm{BDM}_{k-1} := \big\{\boldsymbol{\sigma}\in H_0(\mathrm{rot},\Omega),\ \boldsymbol{\sigma}|_K\in(P_{k-1}(K))^2,\ K\in\mathcal{T}_h\big\}. \tag{3.2}$$

Since $\boldsymbol{\Theta}_h=\boldsymbol{\Gamma}_h$, no reduction integration is used in this family of methods, namely, $\boldsymbol{R}_h=I$.

3.1 The refined a priori error estimate

Let the energy norm $\mathfrak{E}(\boldsymbol{\phi}-\boldsymbol{\phi}_h, w-w_h, \gamma-\gamma_h)$ be defined as

$$\begin{aligned}\mathfrak{E}(\boldsymbol{\phi}-\boldsymbol{\phi}_h, w-w_h,\gamma-\gamma_h)^2 =& |||\boldsymbol{\phi}-\boldsymbol{\phi}_h|||^2_{\boldsymbol{\Theta}} + \|\nabla w-\nabla w_h\|^2_{L^2(\Omega)} + t^2\|\gamma-\gamma_h\|^2_{L^2(\Omega)} \\ &+ t^4\|\mathrm{rot}(\gamma-\gamma_h)\|^2_{L^2(\Omega)} + \|\gamma-\gamma_h\|^2_{H^{-1}(\Omega)}. \end{aligned}\tag{3.3}$$

One can use the same arguments of MITC methods, for instance, [9,20], to prove the following Helmholtz decomposition

Lemma 3.1. *For any $\boldsymbol{q}\in\boldsymbol{\Gamma}_h$, there exist unique $r\in W_h$, $p\in\mathcal{Q}_{k-2}\cap L_0^2(\Omega)$, and $\boldsymbol{\alpha}\in\boldsymbol{\Gamma}_h$ such that*

$$\boldsymbol{q} = \nabla r + \boldsymbol{\alpha}, \tag{3.4a}$$

$$(\boldsymbol{\alpha},\boldsymbol{\sigma}) = (\mathrm{rot}\,\boldsymbol{\sigma}, p), \quad \forall\boldsymbol{\sigma}\in\boldsymbol{\Gamma}_h, \tag{3.4b}$$

here and in the sequel, the pressure space $\mathcal{Q}_{k-2}$ is defined as

$$\mathcal{Q}_{k-2} := \big\{q\in L_0^2(\Omega),\ \ q|_K\in P_{k-2}(K),\ \ \textit{for any } K\in\mathcal{T}_h\big\}. \tag{3.5}$$

Lemma 3.2. *It holds that*

$$\|q\|_{L^2(\Omega)} \lesssim \sup_{0\neq\boldsymbol{\theta}\in\boldsymbol{\Theta}_h} \frac{(\operatorname{rot}\boldsymbol{\theta}, q)_{L^2(\Omega)}}{\||\boldsymbol{\theta}|\|_{\Theta}}, \quad \textit{for any } q \in \mathcal{Q}_{k-2}. \tag{3.6}$$

Proof. We define the interpolation operator $\boldsymbol{\Pi}_{\boldsymbol{\Theta}} : \boldsymbol{\Theta} \to \boldsymbol{\Theta}_h$ by

$$\begin{aligned} &\int_E (\boldsymbol{\theta} - \boldsymbol{\Pi}_{\boldsymbol{\Theta}}\boldsymbol{\theta}) \cdot \boldsymbol{t} q ds = 0, && \text{for any } q \in P_{k-1}(E), \\ &\int_K (\boldsymbol{\theta} - \boldsymbol{\Pi}_{\boldsymbol{\Theta}}\boldsymbol{\theta}) \cdot \boldsymbol{q} dx = 0, && \text{for any } \boldsymbol{q} \in \mathbf{RT}_{k-3}(K), \end{aligned}$$

where $\mathbf{RT}_{k-3}(K)$ is the usual Raviart-Thomas space of the index $k-3$. Given any $q\in P_{k-2}(K)$, it follows from the definition of the operator $\boldsymbol{\Pi}_{\boldsymbol{\Theta}}$ that

$$\begin{aligned} \big(\operatorname{rot}(\boldsymbol{\theta} - \boldsymbol{\Pi}_{\boldsymbol{\Theta}}\boldsymbol{\theta}), q\big)_{L^2(K)} &= \sum_{E\subset\partial K} \int_E (\boldsymbol{\theta} - \boldsymbol{\Pi}_{\boldsymbol{\Theta}}\boldsymbol{\theta}) \cdot \boldsymbol{t} q ds + \int_K (\boldsymbol{\theta} - \boldsymbol{\Pi}_{\boldsymbol{\Theta}}\boldsymbol{\theta}) \cdot \mathbf{curl} q dx \\ &= 0. \end{aligned} \tag{3.7}$$

Moreover, we have

$$\|\boldsymbol{\theta} - \boldsymbol{\Pi}_{\boldsymbol{\Theta}}\boldsymbol{\theta}\|_{L^2(K)} \lesssim h_K \|\nabla\boldsymbol{\theta}\|_{L^2(K)}, \qquad \|\boldsymbol{\Pi}_{\boldsymbol{\Theta}}\boldsymbol{\theta}\|_{H^1(K)} \lesssim \|\boldsymbol{\theta}\|_{H^1(K)}, \tag{3.8a}$$

$$\begin{aligned} \|[\boldsymbol{\Pi}_{\boldsymbol{\Theta}}\boldsymbol{\theta}]_S\|_{L^2(E)} &= \|[\boldsymbol{\Pi}_{\boldsymbol{\Theta}}\boldsymbol{\theta} - \boldsymbol{\theta}]_S\|_{L^2(E)} \lesssim h_E^{-\frac{1}{2}} \|\boldsymbol{\Pi}_{\boldsymbol{\Theta}}\boldsymbol{\theta} - \boldsymbol{\theta}\|_{L^2(\omega_E)} + h_E^{\frac{1}{2}} \|\nabla_h(\boldsymbol{\Pi}_{\boldsymbol{\Theta}}\boldsymbol{\theta} - \boldsymbol{\theta})\|_{L^2(\omega_E)} \\ &\lesssim h_E^{\frac{1}{2}} \|\nabla\boldsymbol{\theta}\|_{L^2(\omega_E)}. \end{aligned} \tag{3.8b}$$

A summary of (3.7) and (3.8) completes the proof. □

Applying the discrete Helmholtz decomposition presented in Lemma 3.1 and the discrete inf-sup condition in (3.6), one can follow the same line of the MITC methods, for instance, [9,20], to show the optimal convergence in the energy norm defined as in (3.3):

Theorem 3.1. *Let* $(\boldsymbol{\phi}, w, \boldsymbol{\gamma})$ *and* $(\boldsymbol{\phi}_h, w_h, \boldsymbol{\gamma}_h)$ *be the solutions to the problems* (1.1) *and* (2.8) *with the discrete spaces in* (3.1) *and* (3.2). *Then,*

$$\begin{aligned} \mathfrak{E}(\boldsymbol{\phi} - \boldsymbol{\phi}_h, w - w_h, \boldsymbol{\gamma} - \boldsymbol{\gamma}_h) \lesssim & h^{k-1} \big(|\boldsymbol{\phi}|_{H^k(\Omega)} + |w|_{H^k(\Omega)} + t|\boldsymbol{\gamma}|_{H^{k-1}(\Omega)} \\ & + t^2 |\operatorname{rot}\boldsymbol{\gamma}|_{H^{k-1}(\Omega)} + |\boldsymbol{\gamma}|_{H^{k-2}(\Omega)} \big). \end{aligned} \tag{3.9}$$

Remark 3.1. Compared to the error estimates in [1], the energy norm analyzed herein contains two more terms $t^4\|\operatorname{rot}(\boldsymbol{\gamma} - \boldsymbol{\gamma}_h)\|^2_{L^2(\Omega)}$ and $\|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|^2_{H^{-1}(\Omega)}$.

3.2 The a posteriori error analysis

This subsection considers the a posteriori error analysis for this family of discontinuous Galerkin methods. Define the a posteriori error estimator

$$\eta_h^2 := \sum_{K\in\mathcal{T}_h} \eta_K^2 + \sum_{E\in\mathcal{E}(\Omega)} \eta_E^2 + \sum_{E\in\mathcal{E}} h_E^{-1} \|[\boldsymbol{\phi}_h]_S\|^2_{L^2(E)}, \tag{3.10}$$

with the factors η_K and η_E defined as

$$\eta_K = \sqrt{t^2 + h_K^2} h_K \|\operatorname{div} \boldsymbol{\gamma}_h + g\|_{L^2(K)} + h_K \|\operatorname{div} \mathcal{C}\varepsilon_h(\boldsymbol{\phi}_h) + \boldsymbol{\gamma}_h\|_{L^2(K)}, \tag{3.11a}$$

$$\eta_E = \sqrt{t^2 + h_E^2} h_E^{\frac{1}{2}} \|[\boldsymbol{\gamma}_h] \cdot \boldsymbol{\nu}_E\|_{L^2(E)} + h_E^{\frac{1}{2}} \|[\mathcal{C}\varepsilon_h(\boldsymbol{\phi}_h)] \cdot \boldsymbol{\nu}_E\|_{L^2(E)}. \tag{3.11b}$$

Since $\boldsymbol{\gamma}_h \subset H_0(\text{rot}, \Omega)$ and $W_h \subset H_0^1(\Omega)$ and the conditions (H1)-(H3) of [18] hold for this class of discontinuous Galerkin methods, similar arguments of [18, Section 5] prove that

$$\eta_h - \text{osc}(g) \lesssim \mathfrak{E}(\boldsymbol{\phi} - \boldsymbol{\phi}_h, w - w_h, \boldsymbol{\gamma} - \boldsymbol{\gamma}_h) \lesssim \eta_h, \tag{3.12}$$

with the oscillation $\text{osc}(g)$ defined as

$$\text{osc}(g)^2 := \sum_{K \in \mathcal{T}_h} (t^2 + h_K^2) h_K^2 \|g - g_K\|_{L^2(K)}^2, \tag{3.13}$$

here g_K denotes the projection of g onto $P_\ell(K)$ with some nonnegative integer ℓ.

4 The Arnold-Brezzi-Falk-Marini element II

This is a family of triangular elements where all variables are approximated by completely discontinuous polynomials [1]. The displacement, the rotation, and the shear force spaces read, respectively,

$$W_h := \{v \in L^2(\Omega),\ v|_K \in P_k(K),\ \text{for any}\ K \in \mathcal{T}_h\}, \tag{4.1a}$$

$$\boldsymbol{\Theta}_h := \{\boldsymbol{\psi} \in (L^2(\Omega))^2,\ \boldsymbol{\psi}|_K \in (P_{k-1}(K))^2,\ \text{for any}\ K \in \mathcal{T}_h\}, \tag{4.1b}$$

$$\boldsymbol{\Gamma}_h := \{\boldsymbol{\psi} \in (L^2(\Omega))^2,\ \boldsymbol{\psi}|_K \in (P_{k-1}(K))^2,\ \text{for any}\ K \in \mathcal{T}_h\}, \tag{4.1c}$$

for some integer $k \geq 2$. Again, no reduction integration is used in these methods, namely, $\boldsymbol{R}_h = I$. Defining a lifting operator $\mathcal{P} : H^1(\mathcal{T}_h) \to \boldsymbol{\Gamma}_h$ by the equation

$$(\mathcal{P}(v), \boldsymbol{\delta}) = \sum_{E \in \mathcal{E}} \int_E [v] \boldsymbol{\nu}_E \cdot \{\boldsymbol{\delta}\} ds, \quad \text{for any}\ \boldsymbol{\delta} \in \boldsymbol{\Gamma}_h. \tag{4.2}$$

Then the discrete shear force $\boldsymbol{\gamma}_h$ reads

$$\boldsymbol{\gamma}_h = \lambda t^{-2} (\nabla_h w_h - \boldsymbol{\phi}_h - \mathcal{P}(w_h)). \tag{4.3}$$

Let the energy norm of the error be defined as

$$\begin{aligned} &\mathfrak{E}(\boldsymbol{\phi} - \boldsymbol{\phi}_h, w - w_h, \boldsymbol{\gamma} - \boldsymbol{\gamma}_h)^2 \\ =& |||\boldsymbol{\phi} - \boldsymbol{\phi}_h|||_{\boldsymbol{\Theta}}^2 + |||w - w_h|||_W^2 + t^2 \|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_{L^2(\Omega)}^2 + \|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_{H^{-1}(\text{div},\Omega)}^2. \end{aligned} \tag{4.4}$$

Since the conditions (H1)-(H3) of [18] hold for this class of methods, we will use the theory established in [18, Theorem 3.3]. We choose α as a global constant independent of the meshsize and the plate thickness. To control the nonconformity, we let $\tilde{w}_h \in W$, and $\tilde{\boldsymbol{\phi}}_h \in \boldsymbol{\Theta}$ be arbitrary. Finally, we take

$$\tilde{\gamma}_h' = \gamma_h' = \beta^{-2}(\nabla_h w_h - \boldsymbol{\phi}_h - \mathcal{P}(w_h)). \tag{4.5}$$

Now we bound the terms of the abstract estimator $\tilde{\eta}_h$ in [18, Theorem 3.3]. It follows from the aforementioned choices that the volume and edge terms η_K and η_E can be expressed as

$$\eta_K = h_K \|\operatorname{div} \gamma_h + g\|_{L^2(K)} + h_K \|\operatorname{div} \mathcal{C}\varepsilon_h(\boldsymbol{\phi}_h) + \gamma_h\|_{L^2(K)}, \tag{4.6a}$$

$$\eta_E = h_E^{\frac{1}{2}} \|[\gamma_h] \cdot \boldsymbol{\nu}_E\|_{L^2(E)} + h_E^{\frac{1}{2}} \|[\mathcal{C}\varepsilon_h(\boldsymbol{\phi}_h)] \cdot \boldsymbol{\nu}_E\|_{L^2(E)}. \tag{4.6b}$$

Since $\boldsymbol{R}_h = I$, we get $\mu_h(\gamma_h) = 0$

$$\begin{aligned} \tilde{\eta}_R &= \Big\| \frac{1}{\alpha}(\gamma_h - \alpha^2(\nabla \tilde{w}_h - \tilde{\boldsymbol{\phi}}_h) - \tilde{\gamma}_h') \Big\|_{L^2(\Omega)} \\ &\lesssim \|\nabla_h(w_h - \tilde{w}_h) - \boldsymbol{\phi}_h + \tilde{\boldsymbol{\phi}}_h - \mathcal{P}(w_h)\|_{L^2(\Omega)}. \end{aligned} \tag{4.7}$$

Lemma 4.1. *With the residual*

$$\tilde{\boldsymbol{r}}_h = \nabla_h w_h - \boldsymbol{\phi}_h - \mathcal{P}(w_h) - (\nabla \tilde{w}_h - \tilde{\boldsymbol{\phi}}_h),$$

it holds that

$$\begin{aligned} \tilde{\eta}(\tilde{\boldsymbol{r}}_h) \lesssim & \Big(\sum_{K \in \mathcal{T}_h} \min\Big(\frac{1}{h_K^2}, \frac{1}{t^2}\Big) \|\boldsymbol{\phi}_h - \tilde{\boldsymbol{\phi}}_h\|_{L^2(K)}^2 \Big)^{\frac{1}{2}} \\ & + \Big(\sum_{E \in \mathcal{E}} \min\Big(\frac{1}{t}, \frac{h_E}{t^2}\Big) \|[\nabla_h w_h - \boldsymbol{\phi}_h - \mathcal{P}(w_h)] \cdot \boldsymbol{\tau}_E\|_{L^2(E)}^2 \Big)^{\frac{1}{2}} \\ & + \Big(\sum_{K \in \mathcal{T}_h} \min\Big(1, \frac{h_K}{t}\Big)^2 \|\operatorname{rot}(\boldsymbol{\phi}_h + \mathcal{P}(w_h) - \tilde{\boldsymbol{\phi}}_h)\|_{L^2(K)}^2 \Big)^{\frac{1}{2}} \\ & + \|\nabla_h w_h - \boldsymbol{\phi}_h - \mathcal{P}(w_h) - (\nabla \tilde{w}_h - \tilde{\boldsymbol{\phi}}_h)\|_{L^2(\Omega)}. \end{aligned} \tag{4.8}$$

Proof. By the definition of the residual $\tilde{\boldsymbol{r}}_h$, we have

$$(\tilde{\boldsymbol{r}}_h, \operatorname{curl} \mathcal{J}p)_{L^2(\Omega)} = \big(\nabla_h w_h - \boldsymbol{\phi}_h - \mathcal{P}(w_h) - (\nabla \tilde{w}_h - \tilde{\boldsymbol{\phi}}_h), \operatorname{curl} \mathcal{J}p\big)_{L^2(\Omega)}. \tag{4.9}$$

Integrating by parts and applying the definition of $\mathcal{P}(w_h)$ leads to

$$\big(\nabla_h w_h - \mathcal{P}(w_h), \operatorname{curl} \mathcal{J}p\big)_{L^2(\Omega)} = 0. \tag{4.10}$$

This and the inverse estimate together with

$$\|p - \mathcal{J}p\|_{L^2(K)} \lesssim \min\big(\|p\|_{L^2(\omega_K)}, h_K \|\nabla p\|_{L^2(\omega_K)}\big),$$

yield

$$\sup_{0\neq p\in\hat{H}^1(\Omega)}\frac{(\tilde{\boldsymbol{r}}_h,\operatorname{curl}\mathcal{J}p)_{L^2(\Omega)}}{\|p\|_{L^2(\Omega)}+\|t\nabla p\|_{L^2(\Omega)}}\lesssim\Big(\sum_{K\in\mathcal{T}_h}\min\Big(\frac{1}{h_K^2},\frac{1}{t^2}\Big)\|\boldsymbol{\phi}_h-\tilde{\boldsymbol{\phi}}_h\|_{L^2(K)}^2\Big)^{\frac{1}{2}}. \tag{4.11}$$

The other terms can be easily bounded. □

Define the a posteriori error estimator

$$\begin{aligned}\eta_h^2=&\sum_{K\in\mathcal{T}_h}\eta_K^2+\sum_{E\in\mathcal{E}(\Omega)}\eta_E^2+\sum_{E\in\mathcal{E}}h_E^{-1}\big(\|[w_h]\|_{L^2(E)}^2+\|[\boldsymbol{\phi}_h]\|_{L^2(E)}^2\big)\\&+\sum_{E\in\mathcal{E}}\min\Big(\frac{1}{t},\frac{h_E}{t^2}\Big)\|[\nabla_h w_h-\boldsymbol{\phi}_h-\mathcal{P}(w_h)]\cdot\boldsymbol{\tau}_E\|_{L^2(E)}^2\\&+\sum_{K\in\mathcal{T}_h}\min\Big(1,\frac{h_K}{t}\Big)^2\|\operatorname{rot}\mathcal{P}(w_h)\|_{L^2(K)}^2.\end{aligned} \tag{4.12}$$

Theorem 4.1 (The reliability of the estimator)**.** *There holds that*

$$\mathfrak{E}(\boldsymbol{\phi}-\boldsymbol{\phi}_h,w-w_h,\gamma-\gamma_h)\lesssim\eta_h. \tag{4.13}$$

Proof. Given any $\tilde{w}_h\in W$ and $\tilde{\boldsymbol{\phi}}_h\in\boldsymbol{\Theta}$, we define

$$\begin{aligned}\tilde{\eta}_h(\boldsymbol{\phi}_h)=&\|\varepsilon_h(\boldsymbol{\phi}_h-\tilde{\boldsymbol{\phi}}_h)\|_{L^2(\Omega)}+\Big(\sum_{K\in\mathcal{T}_h}\min\Big(1,\frac{h_K}{t}\Big)^2\|\operatorname{rot}(\boldsymbol{\phi}_h-\tilde{\boldsymbol{\phi}}_h)\|_{L^2(K)}^2\Big)^{\frac{1}{2}}\\&+\Big(\sum_{K\in\mathcal{T}_h}\min\Big(\frac{1}{h_K^2},\frac{1}{t^2}\Big)\|\boldsymbol{\phi}_h-\tilde{\boldsymbol{\phi}}_h\|_{L^2(K)}^2\Big)^{\frac{1}{2}}.\end{aligned} \tag{4.14}$$

It follows from [18, Theorem 3.3] and the estimates (4.6)-(4.8) that

$$|||\boldsymbol{\phi}-\tilde{\boldsymbol{\phi}}_h,w-\tilde{w}_h|||+\|\gamma'-\tilde{\gamma}_h'\|_Q\lesssim\tilde{\eta}_h\lesssim\eta_h+\|\nabla_h(w_h-\tilde{w}_h)\|_{L^2(\Omega)}+\tilde{\eta}_h(\boldsymbol{\phi}_h), \tag{4.15}$$

herein we use the fact that

$$\|\mathcal{P}(w_h)\|_{L^2(\Omega)}^2\lesssim\sum_{E\in\mathcal{E}}h_E^{-1}\|[w_h]\|_{L^2(E)}^2.$$

Then the triangle inequality leads to

$$|||\boldsymbol{\phi}-\boldsymbol{\phi}_h,w-w_h|||_h+\|\gamma'-\tilde{\gamma}_h'\|_Q\lesssim\eta_h+\|\nabla_h(w_h-\tilde{w}_h)\|_{L^2(\Omega)}+\tilde{\eta}_h(\boldsymbol{\phi}_h). \tag{4.16}$$

The terms $\sum_{E\in\mathcal{E}}h_E^{-1}\|[w_h]\|_{L^2(E)}^2$ and $\sum_{E\in\mathcal{E}}h_E^{-1}\|[\boldsymbol{\phi}_h]\|_{L^2(E)}^2$ are already contained in the estimator. An application of [18, Lemma 2.1] ends the proof. □

Now we define

$$\operatorname{osc}(g)^2:=\sum_{K\in\mathcal{T}_h}h_K^2\|g-g_K\|_{L^2(K)}^2, \tag{4.17}$$

here g_K denotes the projection of g onto $P_\ell(K)$ with some nonnegative integer ℓ. Then we have the the following efficiency for the estimator:

Theorem 4.2. *It holds that*

$$\eta_h \lesssim \mathfrak{E}(\boldsymbol{\phi}-\boldsymbol{\phi}_h, w-w_h, \boldsymbol{\gamma}-\gamma_h)+\operatorname{osc}(g). \tag{4.18}$$

Proof. The efficiency for the first two and the sixth terms can be verified by a similar argument in [18, Theorem 4.5], and the efficiency of the fourth and the fifth terms is straightforward. We only need to consider the seventh term. Given $K\in\mathcal{T}_h$ with the center M, let B_K be the usual element bubble function with $B_K(M)=1$. Define

$$\rho_K = B_K \operatorname{rot}\mathcal{P}(w_h),$$

we come to

$$\begin{aligned} &\|\operatorname{rot}\mathcal{P}(w_h)\|^2_{L^2(K)} \lesssim (\operatorname{rot}\mathcal{P}(w_h),\rho_K)_{L^2(K)}\\ =&(\operatorname{rot}(\mathcal{P}(w_h)-\nabla_h w_h+\boldsymbol{\phi}_h+\nabla w-\boldsymbol{\phi}),\rho_K)_{L^2(K)}-(\operatorname{rot}(\boldsymbol{\phi}_h-\boldsymbol{\phi}),\rho_K)_{L^2(K)}\\ \lesssim& t^2\|\gamma_h-\boldsymbol{\gamma}\|_{L^2(K)}\|\operatorname{curl}\rho_K\|_{L^2(K)}+\|\operatorname{rot}(\boldsymbol{\phi}_h-\boldsymbol{\phi})\|_{L^2(K)}\|\rho_K\|_{L^2(K)}. \end{aligned}\tag{4.19}$$

Applying the inverse estimate, this gives

$$\min\Big(1,\frac{h_K}{t}\Big)\|\operatorname{rot}\mathcal{P}(w_h)\|_{L^2(K)} \lesssim t\|\gamma_h-\boldsymbol{\gamma}\|_{L^2(K)}+\|\operatorname{rot}(\boldsymbol{\phi}_h-\boldsymbol{\phi})\|_{L^2(K)}, \tag{4.20}$$

which completes the proof. □

5 The Arnold-Brezzi-Marini element

This section uses the notation of the previous section. In this family of triangular elements, the displacement space W_h and the shear force space $\boldsymbol{\Gamma}_h$ are the same as those defined in the previous section, what difference is the rotation space, which reads [2]

$$\boldsymbol{\Theta}_h := W_h \times W_h. \tag{5.1}$$

The reduction integration operator $\boldsymbol{R}_h$ is the L^2 projection operator from $L^2(\Omega)^2$ onto $\boldsymbol{\Gamma}_h$.

Let the lifting operator $\mathcal{P}$ be defined as in (4.2). The discrete shear force reads

$$\gamma_h = \lambda t^{-2}\boldsymbol{R}_h\big(\nabla_h w_h-\boldsymbol{\phi}_h-\mathcal{P}(w_h)\big). \tag{5.2}$$

To establish the robust a posteriori error analysis, we need to specify the choices in [18, Theorem 3.3]: we choose α as a global constant independent of the meshsize and the plate thickness, let $\tilde{w}_h\in W$ and $\tilde{\boldsymbol{\phi}}_h\in\boldsymbol{\Theta}$ be arbitrary, and take

$$\tilde{\gamma}'_h = \gamma'_h = \beta^{-2}\boldsymbol{R}_h\big(\nabla_h w_h-\boldsymbol{\phi}_h-\mathcal{P}(w_h)\big). \tag{5.3}$$

Define the estimator as

$$\begin{aligned}\eta_h^2 =& \sum_{K\in\mathcal{T}_h}\eta_K^2+\sum_{E\in\mathcal{E}(\Omega)}\eta_E^2+\sum_{E\in\mathcal{E}}h_E^{-1}\big(\|[w_h]\|^2_{L^2(E)}+\|[\boldsymbol{\phi}_h]\|^2_{L^2(E)}\big)\\&+\sum_{E\in\mathcal{E}}\min\Big(\frac{1}{t},\frac{h_E}{t^2}\Big)\|[\boldsymbol{R}_h(\nabla_h w_h-\boldsymbol{\phi}_h-\mathcal{P}(w_h))]\cdot\boldsymbol{\tau}_E\|^2_{L^2(E)}\\&+\sum_{K\in\mathcal{T}_h}\min\Big(1,\frac{h_K}{t}\Big)^2\big(\|\operatorname{rot}\mathcal{P}(w_h)\|^2_{L^2(K)}+\|\operatorname{rot}(\boldsymbol{R}_h-I)(\nabla_h w_h-\boldsymbol{\phi}_h-\mathcal{P}(w_h))\|^2_{L^2(K)}\big)\\&+\|(\boldsymbol{R}_h-I)\big(\nabla_h w_h-\boldsymbol{\phi}_h-\mathcal{P}(w_h)\big)\|^2_{L^2(\Omega)}.\end{aligned}\tag{5.4}$$

A similar procedure of the previous section proves:

Theorem 5.1. *The energy norm of the error and the estimator η_h are equivalent in the sense*

$$\eta_h-\operatorname{osc}(g)\lesssim\mathfrak{E}(\boldsymbol{\phi}-\boldsymbol{\phi}_h,w-w_h,\boldsymbol{\gamma}-\boldsymbol{\gamma}_h)\lesssim\eta_h,\tag{5.5}$$

with the energy norm from (4.4) *and the oscillation* $\operatorname{osc}(g)$ *of* (4.17).

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