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# Application of Homotopy Analysis Method for Solving Systems of Volterra Integral Equations

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**Abstract.** In this paper, we prove the convergence of homotopy analysis method (HAM). We also apply the homotopy analysis method to obtain approximate analytical solutions of systems of the second kind Volterra integral equations. The HAM solutions contain an auxiliary parameter which provides a convenient way of controlling the convergence region of series solutions. It is shown that the solutions obtained by the homotopy-perturbation method (HPM) are only special cases of the HAM solutions. Several examples are given to illustrate the efficiency and implementation of the method.

**AMS subject classifications**: 54A40, 26E50

**Key words**: Homotopy analysis method, homotopy perturbation method, systems of Volterra integral equations.

## 1 Introduction

Differential equations, integral equations or combinations of them, integro-differential equations, are obtained in modeling of real-life engineering phenomena that are inherently nonlinear with variable coefficients. Most of these types of equations do not have an analytical solution. Therefore, these problems should be solved by using numerical or semi-analytical techniques. In numeric methods, computer codes and more powerful processors are required to achieve accurate results. Acceptable results are obtained via semi-analytical methods which are more convenient than numerical methods. The main advantage of semi-analytical methods, compared with other methods, is based

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on the fact that they can be conveniently applied to solve various complicated problems. Several analytical methods including the linear superposition technique [14], the exp-function method [16], the Laplace decomposition method [8], the matrix exponential method [15], the homotopy perturbation method [7], variational iteration methods [2] and the Adomian decomposition method [12] have been developed for solving linear or nonlinear non-homogeneous partial differential equations. One of these semi-analytical solution methods is the Homotopy analysis method (HAM). In 1992, Liao [9] employed the basic ideas of the homotopy in topology to propose a general analytic method for nonlinear problems, namely the Homotopy analysis method, [5, 6, 9–11]. In recent years, homotopy analysis method has been used in obtaining approximate solutions of a wide class of differential, integral and integro-differential equations. The method provides the solution in a rapidly convergent series with components that are elegantly computed. The main advantage of the method is that it can be used directly without using assumptions or transformations. In this work, we aim to implement this reliable technique to solving systems of Volterra integral equations. A system of integral equations of the second kind can be presented as

$$f(t) = g(t) + \int_a^t K(s,t,(f(s))ds,$$

where

$$\begin{aligned} &f(t) = \big(f_1(t),\cdots,f_n(t)\big)^T, \qquad g(t) = \big(g_1(t),\cdots,g_n(t)\big)^T, \\ &K(s,t,(f(s)) = \big(K_1(s,t,(f(s)),\cdots,K_n(s,t,(f(s))\big)^T. \end{aligned}$$

## 2 Basic idea of HAM

We consider the following differential equation

$$\mathcal{N}[u(\tau)] = 0, \tag{2.1}$$

where $\mathcal{N}$ is a nonlinear operator, $\tau$ denotes independent variable, $u(\tau)$ is an unknown function, respectively. For simplicity, we ignore all boundary or initial conditions, which can be treated in the similar way. By means of generalizing the traditional homotopy method, Liao [11] construct the so-called zero-order deformation equation

$$(1-p)\mathcal{L}[\phi(\tau;p) - u_0(\tau)] = p\hbar\mathcal{H}(\tau)\mathcal{N}[\phi(\tau;p)], \tag{2.2}$$

where $p \in [0,1]$ is the embedding parameter, $h \neq 0$ is a non-zero auxiliary parameter, $\mathcal{H}(\tau) \neq 0$ is an auxiliary function, $u_0(\tau)$ is an initial guess of $u(\tau)$ and $\phi(\tau;p)$ is an unknown function and $\mathcal{L}$ an auxiliary linear operator with the property

$$\mathcal{L}[f(\tau)] = 0 \quad \text{when} \ \ f(\tau) = 0. \tag{2.3}$$

It is important, that one has great freedom to choose auxiliary things in HAM. Obviously, when $p=0$ and $p=1$, it holds

$$\phi(\tau;0)=u_0(\tau), \qquad \phi(\tau;1)=u(\tau), \tag{2.4}$$

respectively. Thus, as $p$ increases from 0 to 1, the solution $\phi(\tau;p)$ varies from the initial guess $u_0(\tau)$ to the solution $u(\tau)$. Expanding $\phi(\tau;p)$ in Taylor series with respect to $p$, we have

$$\phi(\tau;p)=u_0(\tau)+\sum_{m=1}^{+\infty}u_m(\tau)p^m, \tag{2.5}$$

where

$$u_m(\tau)=\Big[\frac{1}{m!}\frac{\partial^m\phi(\tau;p)}{\partial p^m}\Big]_{p=0}. \tag{2.6}$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter $\hbar$, and the auxiliary function are so properly chosen, the series (2.4) converges at $p=1$, then we have

$$u(\tau)=u_0(\tau)+\sum_{m=1}^{+\infty}u_m(\tau), \tag{2.7}$$

which must be one of solutions of original nonlinear equation, as proved by [11]. As $\hbar=-1$ and $\mathcal{H}(\tau)=1$, Eq. (2.2) becomes

$$(1-p)\mathcal{L}[\phi(\tau;p)-u_0(\tau)]+p\mathcal{N}[\phi(\tau;p)]=0, \tag{2.8}$$

which is used mostly in the homotopy perturbation method [13], where as the solution obtained directly, without using Taylor series [7]. According to the definition (2.5), the governing equation can be deduced from the zero-order deformation equation (2.2). Define the vector

$$\overrightarrow{u}_n=\{u_0(\tau),u_1(\tau),\cdots,u_n(\tau)\}.$$

Differentiating Eq. (2.2) $m$ times with respect to the embedding parameter $p$ and then setting $p=0$ and finally dividing them by $m!$, we have the so-called $m$ th-order deformation equation

$$\mathcal{L}[u_m(\tau)-\chi_m u_{m-1}(\tau)]=\hbar\,\mathcal{H}(\tau)\mathcal{R}_m(\overrightarrow{u}_{m-1}), \tag{2.9}$$

where

$$\mathcal{R}_m(\overrightarrow{u}_{m-1})=\Big[\frac{1}{(m-1)!}\frac{\partial^{m-1}\mathcal{N}[\phi(\tau;p)]}{\partial p^{m-1}}\Big]_{p=0}, \tag{2.10}$$

and

$$\chi_m=\begin{cases}0, & m\le 1,\\ 1, & m>1.\end{cases} \tag{2.11}$$

It should be emphasized that $u_m(\tau)$ for $m\ge 1$ is governed by the linear equation (2.8) under the linear boundary conditions that come from original problem, which can be easily solved by symbolic computation software such as Matlab. For the convergence of the above method we refer the reader to Liao's work [11] . If Eq. (2.1) admits unique solution, then this method will produce the unique solution. If Eq. (2.1) does not possess unique solution, the HAM will give a solution among many other (possible) solutions.

# 3 Series solutions when convergent

In this section, we will prove that, as long as the solution series (2.7) given by the homotopy analysis method is convergent, it must be the solution of the considered nonlinear problem.

**Theorem 3.1.** *As long as the series*

$$u_0(t)+\sum_{m=1}^{+\infty} u_m(t),$$

*is convergent, where $u_m(t)$ is governed by the high-order deformation equation* (2.9) *under the definitions* (2.10) *and* (2.11), *it must be a solution of Eq.* (2.1).

*Proof.* Let

$$s(t)=u_0(t)+\sum_{m=1}^{+\infty} u_m(t),$$

denote the convergent series. Using (2.9) and (2.11), we have

$$\begin{aligned}&\hbar\mathcal{H}(t)\sum_{m=1}^{+\infty}\Re_m(u_{m-1})\\ =&\sum_{m=1}^{+\infty}\mathcal{L}\big[u_m(t)-\chi_m u_{m-1}(t)\big]=\mathcal{L}\Big[\sum_{m=1}^{+\infty}u_m(t)-\sum_{m=1}^{+\infty}\chi_m u_{m-1}(t)\Big]\\ =&\mathcal{L}\Big[(1-\chi_2)\sum_{m=1}^{+\infty}u_m(t)\Big]=\mathcal{L}\big[(1-\chi_2)(s(t)-u_0(t))\big],\end{aligned}$$

which gives, since $\hbar\neq 0$, $\mathcal{H}(t)\neq 0$ and from (2.3),

$$\sum_{m=1}^{+\infty}\Re_m(u_{m-1})=0. \tag{3.1}$$

On the other side, we have according to the definition (2.10), that

$$\sum_{m=1}^{+\infty}\Re_m(u_{m-1})=\sum_{m=1}^{+\infty}\frac{1}{(m-1)!}\frac{\partial^{m-1}\mathcal{N}[\phi(t;q)]}{\partial q^{m-1}}\Big|_{q=0}=0. \tag{3.2}$$

In general, $\phi(t;q)$ does not satisfy the original nonlinear equation (2.1). Let

$$\epsilon(t;q) = \mathcal{N}[\phi(t;q)],$$

denote the residual error of Eq. (2.1). Clearly,

$$\epsilon(t;q) = 0.$$

Corresponds to the exact solution of the original equation (2.1). According to the above definition, the Maclaurin series of the residual error $\epsilon(t;q)$ about the embedding parameter $q$ is

$$\sum_{m=0}^{+\infty} \frac{1}{m!} \frac{\partial^m \epsilon(t;q)}{\partial q^m} q^m|_{q=0} = \sum_{m=0}^{+\infty} \frac{1}{m!} \frac{\partial^m \mathcal{N}[\phi(t;q)]}{\partial q^m} q^m|_{q=0}.$$

When $q = 1$, the above expression gives, using (3.2),

$$\epsilon(t;q) = \sum_{m=0}^{+\infty} \frac{1}{m!} \frac{\partial^m \epsilon(t;q)}{\partial q^m}\Big|_{q=0} = 0.$$

This means, according to the definition of $\epsilon(t;q)$, that we gain the exact solution of the original equation (2.1) when $q$. Thus, as long as the series

$$u_0(t) + \sum_{m=1}^{+\infty} u_m(t),$$

is convergent, it must be the solution of the original equation (2.1). This ends the proof. □

## 4 Applications

In order to assess the advantages and the accuracy of homotopy analysis method for solving system of integral equations of the second kind, we will consider the following four examples.

**Example 4.1.** Consider the following linear system of integral equations

$$f_1(t) = \cosh t + t \sin t - \int_0^t \big[e^{-(s-t)} f_1(s) + \cos(s-t) f_2(s)\big] ds, \tag{4.1a}$$

$$f_2(t) = 2\sin t + t(\sin^2 t + e^t) - \int_a^t \big[e^{(s+t)} f_1(s) + t \cos s f_2(s)\big] ds. \tag{4.1b}$$

The exact solutions [21] of (4.1) are given below:

$$f_1(t) = e^{-t}, \qquad f_2(t) = 2\sin t.$$

To solve the system (3.2) by means of homotopy analysis method, we choose the linear operators

$$\mathcal{L}_i[\phi_i(t;p)] = \phi_i(t;p), \quad i = 1,2. \tag{4.2}$$

We now define a nonlinear operators as

$$\mathcal{N}_1[\phi_1,\phi_2] = \phi_1(t;p) - (\cosh t + t\sin t) + \int_0^t \big[e^{-(s-t)}\phi_1(s;p) + \cos(s-t)\phi_2(s;p)\big]ds,$$
$$\mathcal{N}_2[\phi_1,\phi_2] = \phi_2(t;p) - (2\sin t + t(\sin^2 t + e^t)) + \int_0^t \big[e^{(s+t)}\phi_1(s;p) + t\cos s\phi_2(s;p)\big]ds.$$

Using above definition, we construct the zeroth-order deformation equations

$$(1-p)\mathcal{L}_1[\phi_1(t;p) - f_{1,0}(x)] = p\hbar_1\mathcal{H}_1(t)\mathcal{N}_1[\phi_1,\phi_2], \tag{4.3a}$$
$$(1-p)\mathcal{L}_2[\phi_2(t;p) - f_{2,0}(x)] = p\hbar_2\mathcal{H}_2(t)\mathcal{N}_2[\phi_1,\phi_2]. \tag{4.3b}$$

Thus, we obtain the $m$ th-order $(m \geq 1)$ deformation equations

$$\mathcal{L}_1[f_{1,m}(t) - \chi_m f_{1,m-1}(t)] = \hbar_1\mathcal{H}_1(t)\mathcal{R}_{1,m}(\vec{f}_{1,m-1}), \tag{4.4a}$$
$$\mathcal{L}_2[f_{2,m}(t) - \chi_m f_{2,m-1}(t)] = \hbar_2\mathcal{H}_2(t)\mathcal{R}_{2,m}(\vec{f}_{2,m-1}), \tag{4.4b}$$

where

$$\begin{aligned}\mathcal{R}_{1,m}(\vec{f}_{1,m-1},\vec{f}_{2,m-1}) &= f_{1,m-1}(t) + \int_0^t \big[e^{-(s-t)}f_{1,m-1}(s) + \cos(s-t)f_{2,m-1}(s)\big]ds \\ &\quad - (1-\chi_m)(\cosh t + t\sin t), \\ \mathcal{R}_{2,m}(\vec{f}_{1,m-1},\vec{f}_{2,m-1}) &= f_{2,m-1}(t) + \int_0^t \big[e^{(s+t)}f_{1,m-1}(s) + t\cos s f_{2,m-1}(s)\big]ds \\ &\quad - (1-\chi_m)(2\sin t + t(\sin^2 t + e^t)).\end{aligned}$$

Now the solution of the $m$ th-order $(m \geq 1)$ deformation equations (4.3) becomes

$$\begin{aligned} f_{1,m}(t) &= \chi_m f_{1,m-1}(t) + \hbar_1\mathcal{H}_1(t)\mathcal{R}_{1,m}(\vec{f}_{1,m-1},\vec{f}_{2,m-1}), \\ f_{2,m}(t) &= \chi_m f_{2,m-1}(t) + \hbar_2\mathcal{H}_2(t)\mathcal{R}_{2,m}(\vec{f}_{1,m-1},\vec{f}_{2,m-1}). \end{aligned}$$

By start with an initial approximations

$$f_{1,0}(t) = \cosh t + t\sin t, \qquad f_{2,0}(t) = 2\sin t + t(\sin^2 t + e^t),$$

and by choose $\mathcal{H}_i = 1$ $(i = 1,2)$ we suppose

$$f_1(t) \approx \sum_{m=0}^{5} f_{1,m} \quad \text{and} \quad f_2(t) \approx \sum_{m=0}^{5} f_{2,m}.$$

The comparison of the results of the HAM and the HPM [21] are presented in Table 1.

Table 1: The absolute error between the $HAM$ ($h=-0.98$), the $HPM$ ($h=-1$) and the exact solution.

| $t_i$ | $e(f_{1(HAM)})$ | $e(f_{1(HPM)})$ | $e(f_{2(HAM)})$ | $e(f_{2(HPM)})$ |
|---|---|---|---|---|
| 0.0 | 0 | $8.8818E-16$ | $1.3878E-17$ | $4.7878E-16$ |
| 0.1 | $2.0163E-09$ | $2.3210E-08$ | $3.1167E-09$ | $1.7048E-08$ |
| 0.2 | $1.1357E-07$ | $1.9188E-06$ | $4.2050E-08$ | $1.6652E-06$ |
| 0.3 | $6.2312E-06$ | $2.8336E-05$ | $6.0365E-06$ | $2.8771E-05$ |
| 0.4 | $7.3007E-05$ | $2.0695E-04$ | $8.5015E-05$ | $2.4354E-04$ |
| 0.5 | $4.6483E-04$ | $1.0278E-03$ | $6.2664E-04$ | $1.3895E-03$ |

**Example 4.2.** Let us solve the following non-linear system of two integral equations:

$$f_1(t) = \sin t - t + \int_0^t \big[f_1^2(s) + f_2^2(s)\big]ds, \tag{4.5a}$$

$$f_2(t) = \cos t - \frac{1}{2}\sin^2 t + \int_0^t f_1(s) f_2(s) ds. \tag{4.5b}$$

With the exact solutions [21]

$$f_1(t) = \sin t, \qquad f_2(t) = \cos t.$$

To solve the system (4.4) by means of homotopy analysis method, we choose the linear operators

$$\mathcal{L}_i[\phi_i(t;p)] = \phi_i(t;p), \quad i = 1,2. \tag{4.6}$$

We now define a nonlinear operators as

$$\begin{aligned}
\mathcal{N}_1[\phi_1,\phi_2] &= \phi_1(t;p) - (\sin t - t) - \int_0^t \big[\phi_1^2(s;p) + \phi_2^2(s;p)\big]ds,\\
\mathcal{N}_2[\phi_1,\phi_2] &= \phi_2(t;p) - \Big(\cos t - \frac{1}{2}\sin^2 t\Big) + \int_0^t \big[\phi_1(s;p)\phi_2(s;p)\big]ds.
\end{aligned}$$

Using above definition, we construct the zeroth-order deformation equations

$$(1-p)\mathcal{L}_1[\phi_1(t;p) - f_{1,0}(x)] = p\,\hbar_1 \mathcal{H}_1(t)\mathcal{N}_1[\phi_1,\phi_2], \tag{4.7a}$$

$$(1-p)\mathcal{L}_2[\phi_2(t;p) - f_{2,0}(x)] = p\,\hbar_2 \mathcal{H}_2(t)\mathcal{N}_2[\phi_1,\phi_2]. \tag{4.7b}$$

Thus, we obtain the $m$ th-order ($m \geq 1$) deformation equations

$$\mathcal{L}_1[f_{1,m}(t) - \chi_m f_{1,m-1}(t)] = \hbar_1 \mathcal{H}_1(t)\mathcal{R}_{1,m}(\vec{f}_{1,m-1}), \tag{4.8a}$$

$$\mathcal{L}_2[f_{2,m}(t) - \chi_m f_{2,m-1}(t)] = \hbar_2 \mathcal{H}_2(t)\mathcal{R}_{2,m}(\vec{f}_{2,m-1}), \tag{4.8b}$$

Table 2: The absolute error between the $HAM$ $(h=-1)$, the $HPM$ and the exact solution.

| $t_i$ | $e(f_{1(HAM)})$ | $e(f_{1(HPM)})$ | $e(f_{2(HAM)})$ | $e(f_{2(HPM)})$ |
|---|---|---|---|---|
| 0.0 | 0 | 0 | $5.0E-8$ | $5.0E-8$ |
| 0.1 | $1.4E-07$ | $1.4E-07$ | $3.2E-7$ | $3.2E-7$ |
| 0.2 | $3.5E-06$ | $3.5E-06$ | $1.1E-5$ | $1.1E-5$ |
| 0.3 | $5.5E-05$ | $5.5E-05$ | $1.2E-4$ | $1.2E-4$ |
| 0.4 | $3.8E-04$ | $3.8E-04$ | $6.3E-4$ | $6.3E-4$ |
| 0.5 | $1.6E-03$ | $1.6E-03$ | $2.2E-3$ | $2.2E-3$ |

where

$$\begin{aligned}\mathcal{R}_{1,m}(\vec{f}_{1,m-1},\vec{f}_{2,m-1}) = f_{1,m-1}(t) - \int_0^t \Big[\sum_{i=0}^{m-1} f_{1,i}(s)f_{1,m-1-i}(s) + f_{2,i}(s)f_{2,m-1-i}(s)\Big]ds \\ -(1-\chi_m)(\sin t - t), \\ \mathcal{R}_{2,m}(\vec{f}_{1,m-1},\vec{f}_{2,m-1}) = f_{2,m-1}(t) + \int_0^t \Big[\sum_{i=0}^{m-1} f_{1,i}(s)f_{2,m-1-i}(s)\Big]ds \\ -(1-\chi_m)\Big(\cos t - \frac{1}{2}\sin^2 t\Big).\end{aligned}$$

Now the solution of the $m$ th-order $(m \geq 1)$ deformation equations (4.7) becomes

$$\begin{aligned}f_{1,m}(t) = \chi_m f_{1,m-1}(t) + \hbar_1 \mathcal{H}_1(t)\mathcal{R}_{1,m}(\vec{f}_{1,m-1},\vec{f}_{2,m-1}), \\ f_{2,m}(t) = \chi_m f_{2,m-1}(t) + \hbar_2 \mathcal{H}_2(t)\mathcal{R}_{2,m}(\vec{f}_{1,m-1},\vec{f}_{2,m-1}).\end{aligned}$$

By start with an initial approximations

$$f_{1,0}(t) = \sin t - t, \qquad f_{2,0}(t) = \cos t - \frac{1}{2}\sin^2 t,$$

and by choose $\mathcal{H}_i = 1$ $(i=1,2)$ we suppose

$$f_1(t) \approx \sum_{m=0}^{5} f_{1,m} \quad \text{and} \quad f_2(t) \approx \sum_{m=0}^{5} f_{2,m}.$$

The comparison of the results of the HAM and the HPM [21] are presented in Table 2.

# 5 Conclusions

In this paper, the HAM was used to obtain the analytic solutions of systems of linear and non-linear Volterra integral equations of the second kind. The comparison between the HAM and HPM was made and it was found that HAM is more effective than HPM. Hence, it may be concluded that this method is a powerful and an efficient technique in finding the analytic solutions for wide classes of problems. Furthermore, the advantage of this method is the fast convergence of the solutions by means of the auxiliary parameter h and the freedom of choosing $\hbar$ for HAM gives us more accuracy than HPM. It is also worth mentioning to this end that for the example considered, we have shown that HPM are special case of HAM. The computations associated with the example in this paper were performed using Matlab 7.

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