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A Modified Helmholtz Equation with Impedance Boundary Conditions

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Abstract. Here considered is the problem of transient electromagnetic scattering from overfilled cavities embedded in an impedance ground plane. An artificial boundary condition is introduced on a semicircle enclosing the cavity that couples the fields from the infinite exterior domain to those fields inside. A Green's function solution is obtained for the exterior domain, while the interior problem is solved using finite element method. Well-posedness of the associated variational formulation is achieved and convergence and stability of the numerical scheme confirmed. Numerical experiments show the accuracy and robustness of the method.

AMS subject classifications: 65N30, 65N15

Key words: Helmholtz equation, impedance boundary conditions, finite element method.

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1 Introduction

The phenomenon of electromagnetic scattering by cavity-backed apertures has been an area of intense research in recent years. There is a wealth of results reported in both the engineering literature (see, for example, [1–4]) and mathematical journals (see [5–8] and the references therein). It is a common simplifying assumption that the cavity opening coincides with the aperture on a perfect electric conducting (PEC) ground plane. For over-filled cavities we further mention the works [9–13]. We note that most of the published work deals with either cavities with PEC ground planes or time-harmonic problems. The only mathematical treatment of transient problem with overfilled cavities appears to be reported in [9]. We are not aware of any work in that framework, transient and overfilled, under the more prevalent impedance boundary condition (IBC). The aim of this paper is to fill this gap by extending the results of [9] for PEC boundary conditions to IBC. Specifically we develop a hybrid integral

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equation/finite element method that is mathematically wellposed and numerically robust. This model is clearly more applicable physically, yet mathematically more challenging. Specifically, the usual separation of variables approach associated with PEC boundary conditions, or Dirchlet and Newmann boundary conditions, are no longer valid. Our key approach is the development of Green's functions that serve as solutions for the infinite exterior domain. We also, for the first time, numerically implement the method under mixed boundary conditions.

The paper is organized as follows. In Section 2, we establish the mathematical formulation of the problem. Section 3 focuses on the exterior problem where Green's function is derived and boundary operator analyzed. Variational formulation is developed and proved wellposed in Section 4. The paper is concluded in Section 5 with results from some of our numerical experiments.

2 Mathematical formulation

Let $\Omega \subset \mathbb{R}^2$ be the cross-section of a z-invariant cavity in the infinite ground plane, such that its fillings, with material of relative permittivity $\varepsilon_r \geq 1$, protrude above the ground plane. We denote S as the cavity wall and Γ the cavity aperture so that $\partial\Omega = S \cup \Gamma$. The infinite ground plane excluding the cavity opening is denoted as Γ_{ext} and the infinite homogenous, isotropic region above the cavity as $\mathcal{U} = \mathbb{R}^2_+ \setminus \Omega$. Furthermore, let $\mathcal{B}_R$ be a semicircle of radius R, centered at the origin and surrounded by free space, large enough to completely enclose the overfilled portion of the cavity. We denote the region bounded by $\mathcal{B}_R$ and the cavity wall S as Ω_R, so that Ω_R consists of the cavity itself and the homogeneous part between $\mathcal{B}_R$ and Γ. Let $\mathcal{U}_R$ be the homogeneous region outside of Ω_R; that is, $\mathcal{U}_R = \{(r,\theta) : r > R,\ \ 0 < \theta < \pi\}$. Refer to Fig. 1 for the complete problem geometry.

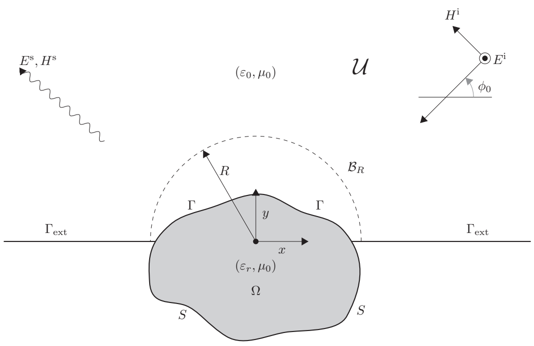


Figure 1: Problem Geometry-TM polarization depicted.

Giving the incident fields $(\boldsymbol{E}^{\rm i}, \boldsymbol{H}^{\rm i})$ impinging on the overfilled cavity, we wish to determine the resulting scattered fields $(\boldsymbol{E}^{\rm s}, \boldsymbol{H}^{\rm s})$. Due to the uniformity in the z-axis, the fields can be decomposed into two fundamental polarizations: transverse magnetic (TM) and transverse electric (TE). Here, for demonstration, we analyze the TM case. Thus assuming

$$\boldsymbol{E} = (0, 0, E_z), \quad \boldsymbol{H} = (H_x, H_y, 0),$$

we have, following [9], the following boundary value problem:

$$\begin{cases} -\Delta E_z + \varepsilon_r \dfrac{\partial^2 E_z}{\partial t^2} = 0, & \text{in } \Omega \cup \mathcal{U} \times (0, \infty), \\ \dfrac{\partial E_z}{\partial t} = -\dfrac{\eta}{\mu} \dfrac{\partial E_z}{\partial \boldsymbol{n}}, & \text{on } S \cup \Gamma_{\text{ext}} \times (0, \infty), \\ E_z|_{t=0} = E_0, \quad \dfrac{\partial E_z}{\partial t}\Big|_{t=0} = E_{t,0}, & \text{in } \Omega \cup \mathcal{U}, \end{cases} \tag{2.1}$$

where $\varepsilon_r = \varepsilon/\varepsilon_0$ is the relative electric permittivity, E_0 and $E_{t,0}$ are the given initial conditions and $\eta = \sqrt{\mu_r/\varepsilon_r}$ is the normalized intrinsic impedance of the infinite ground plane.

We assume non-dispersive material in the cavity, or that the permittivity is not a function of frequency, but could vary with respect to position. Hence, we assume that the impedance is constant in the time domain. Based on this assumption we can approximate impedance boundary conditions on the surface in the time domain using a first-order absorbing boundary condition, [7].

The homogeneous region $\mathcal{U}$ above the protruding cavity is assumed to be air and hence its permittivity is $\varepsilon_r = 1$. In $\mathcal{U}$, the total field can be decomposed as $E_z = E_z^i + E_z^s$ where E_z^i is the incident field and E_z^s the scattered field.

In what follows, we will discretize (2.1), obtain an integral representation of the solution, derive the Green's function for the half-plane, analyze the properties of the Steklov-Poincarè operator and recast the boundary value problem in order to solve it through a variational method.

As in [9], we discretize the TM equations in time by using the Newmark time-marching scheme, an implicit time-stepping method that offers the advantage of stability. The Newmark method is a two-step finite difference method in which there is a prediction of the answer followed by a correction of the predicted value. It is defined by the following: Let N be a positive integer, T be the time interval, $\Delta t = T/N$ be the temporal step size and $t_{n+1} = (n+1)\Delta t$ for $n = 0, 1, 2, \cdots, N-1$. Denote the following as approximations at $t = t_{n+1}$:

$$u^{n+1} \approx u, \qquad \dot{u}^{n+1} \approx \frac{\partial u}{\partial t}, \qquad \ddot{u}^{n+1} \approx \frac{\partial^2 u}{\partial t^2}.$$

These approximations are related by

$$\begin{aligned} u^{n+1} &= u^n + \Delta t \dot{u}^n + \frac{(\Delta t)^2}{2}\big[2\beta \ddot{u}^{n+1} + (1-2\beta)\ddot{u}^n\big], && 0 \le n \le N-1, \\ \dot{u}^{n+1} &= \dot{u}^n + \Delta t\big[\gamma \ddot{u}^{n+1} + (1-\gamma)\ddot{u}^n\big], && 0 \le n \le N-1, \end{aligned}$$

where γ and β are parameters to be determined to guarantee stability of the time-marching scheme [9].

For simplicity, we denote u^i as the incident field E_z^i, u the total field E_z and u^s the scattered field E_z^s. The semidiscrete problem is to find u^{n+1}, $n=0,1,\cdots,N$, such that:

Prediction

$$\tilde{u}^{n+1}=u^n+\Delta t\dot{u}^n+\frac{(\Delta t)^2}{2}(1-2\beta)\ddot{u}^n, \tag{2.2a}$$

$$\tilde{\dot{u}}^{n+1}=\dot{u}+\Delta t(1-\gamma)\ddot{u}^n, \tag{2.2b}$$

Solution

$$\begin{cases} -\Delta u^{n+1}+\alpha^2\varepsilon_r u^{n+1}=\alpha^2\varepsilon_r\tilde{u}^{n+1}, & \text{in } \Omega_R,\\ \dot{u}^{n+1}=-\dfrac{\eta}{\mu}\dfrac{\partial u^{n+1}}{\partial n}, & \text{on } S,\\ u^{n+1}=u^{s,n+1}+u^{i,n+1}, & \text{on } \mathcal{B}_R, \end{cases} \tag{2.3}$$

Correction

$$\ddot{u}^{n+1}=\alpha^2(u^{n+1}-\tilde{u}^{n+1}), \tag{2.4a}$$

$$\dot{u}^{n+1}=\tilde{\dot{u}}^{n+1}+\Delta t\gamma\ddot{u}^{n+1}, \tag{2.4b}$$

where $\alpha^2=1/(\Delta t)^2\beta$.

As seen in (2.3), the impedance boundary condition (IBC) on Γ_{ext} and S from (2.1) then becomes:

$$\dot{u}^{n+1}=-\frac{\eta}{\mu}\frac{\partial u^{n+1}}{\partial n}. \tag{2.5}$$

Utilizing the correction factor described above for $\ddot{u}^{n+1}$ and $\dot{u}^{n+1}$, we express the IBC in (2.5) for the total field as:

$$\Delta t\gamma\alpha^2u^{n+1}+\frac{\eta}{\mu}\frac{\partial u^{n+1}}{\partial n}=\Delta t\gamma\alpha^2\tilde{u}^{n+1}-\tilde{\dot{u}}^{n+1}.$$

Therefore, the scattered field $u^{s,n+1}$ satisfies the following *exterior problem*:

$$\begin{cases} -\Delta u^{s,n+1}+\alpha^2u^{s,n+1}=\alpha^2\tilde{u}^{s,n+1}, & \text{in } \mathcal{U}_R,\\ u^{s,n+1}(R,\theta)=g(R,\theta), & \text{on } \mathcal{B}_R,\\ \Delta t\gamma\alpha^2u^{s,n+1}+\dfrac{\eta}{\mu}\dfrac{\partial u^{s,n+1}}{\partial n}=\Delta t\gamma\alpha^2\tilde{u}^{s,n+1}-\tilde{\dot{u}}^{s,n+1}, & \text{on } \Gamma_{\text{ext}}, \end{cases} \tag{2.6}$$

where

$$g\overset{def}{=}u^{n+1}-u^{i,n+1}$$

and the radiation condition

$$\lim_{r\to\infty}\sqrt{r}\Big(\frac{\partial u^{s,n+1}}{\partial r}+\frac{1}{c}\dot{u}^{s,n+1}\Big)=0.$$

3 Exterior problem

In this section we develop Green's function solution for the exterior problem. We also derive and analyze a boundary operator that will be used to bridge the exterior and interior domains.

3.1 Integral representation of solution

For simplicity, we suppress the $n+1$ superscript from (2.6). We seek the solution for the nonhomogeneous modified Helmholtz equation:

$$-\Delta u(\boldsymbol{r}) + \alpha^2 u(\boldsymbol{r}) = f(\boldsymbol{r}), \tag{3.1}$$

where $\boldsymbol{r}$ denotes position and $f(\boldsymbol{r}) = \alpha^2 \tilde{u}^{s,n+1}(\boldsymbol{r})$, subject to nonhomogeneous boundary conditions of the form:

$$Au(\boldsymbol{r}_s) + B\frac{\partial u(\boldsymbol{r}_s)}{\partial \boldsymbol{n}} = h(\boldsymbol{r}_s),$$

where $\boldsymbol{r}_s$ is on the surface and $\boldsymbol{n}$ is the outward unit normal, A and B are constants defined as $A = \Delta t \gamma \alpha^2$ and $B = \eta/\mu$ and $h(\boldsymbol{r}_s) = \Delta t \gamma \alpha^2 \tilde{u}^{s,n+1} - \tilde{\tilde{u}}^{s,n+1}$.

The associated Green's function satisfies (where $\mathbf{r}'$ denotes source location):

$$\begin{cases} -\Delta G(\boldsymbol{r}|\mathbf{r}') + \alpha^2 G(\boldsymbol{r}|\mathbf{r}') = \delta(\boldsymbol{r} - \mathbf{r}'), & \text{in } \mathcal{U}, \\ AG(\boldsymbol{r}|\mathbf{r}') + B\dfrac{\partial G(\boldsymbol{r}|\mathbf{r}')}{\partial y} = 0, & \text{on } \{y=0\}. \end{cases} \tag{3.2}$$

We note that the Green's function for an impedance plane has been well studied by several authors such as [14–16] and more recently [17], which significantly extends the work of [14].

Denoting

$$R = \sqrt{(x'-x)^2 + (y'-y)^2} \quad \text{and} \quad R^* = \sqrt{(x'-x)^2 + (y'+y)^2},$$

it can be shown the Green's function is

$$G(\boldsymbol{r}|\mathbf{r}') = \frac{1}{2\pi}K_0(\alpha R) - \frac{1}{2\pi}K_0(\alpha R^*) - \frac{2B}{4\pi}\int_{-\infty}^{\infty} \frac{e^{-\sqrt{\xi^2+\alpha^2}(y'+y)}}{(A - B\sqrt{\xi^2+\alpha^2})} e^{i(x'-x)\xi} d\xi. \tag{3.3}$$

We observe that if the surface were a PEC, then $B = 0$, causing the third term in (3.3) to vanish. This is consistent with an assumed Dirichlet boundary condition on the surface. Furthermore, we note that the first two terms in (3.3) correspond to the "classical" wave behavior, whereas the third term incorporates the surface wave behavior expected of an impedance-type surface.

Using residue theory and observing an outgoing progressive surface wave, we can rewrite the third term in (3.3) as

$$
\begin{aligned}
&-i\Big[-\frac{A}{B}\frac{e^{-\frac{A}{B}(y'+y)}}{\sqrt{(\frac{A}{B})^2-\alpha^2}}\cos\Big(|x-x'|\sqrt{\Big(\frac{A}{B}\Big)^2-\alpha^2}\Big)\Big]\\
&+\frac{\alpha}{2\pi}e^{-\frac{A}{B}(y'+y)}\int_{-\infty}^{y'+y}K_1\Big(\alpha\sqrt{(x'-x)^2+\xi^2}\Big)\frac{\xi e^{\frac{A}{B}\xi}}{\sqrt{(x'-x)^2+\xi^2}}d\xi.
\end{aligned}
$$

Thus we arrive at the Green's function for our problem to be:

$$
\begin{aligned}
G(\boldsymbol{r}|\mathbf{r}') =&\frac{1}{2\pi}K_0(\alpha R)-\frac{1}{2\pi}K_0(\alpha R^*)-i\Big[-\frac{A}{B}\frac{e^{-\frac{A}{B}(y'+y)}}{\sqrt{(\frac{A}{B})^2-\alpha^2}}\cos\Big(|x-x'|\sqrt{\Big(\frac{A}{B}\Big)^2-\alpha^2}\Big)\Big]\\
&+\frac{\alpha}{2\pi}e^{-\frac{A}{B}(y'+y)}\int_{-\infty}^{y'+y}K_1\Big(\alpha\sqrt{(x'-x)^2+\xi^2}\Big)\frac{\xi e^{\frac{A}{B}\xi}}{\sqrt{(x'-x)^2+\xi^2}}d\xi. \qquad (3.4)
\end{aligned}
$$

Using a generalized Green's function method and applying Green's second identity then yields:

$$
u(\mathbf{r}')=\iint_{\mathcal{U}_R}f(\boldsymbol{r})G(\boldsymbol{r}|\mathbf{r}')dS+\int_C\big(G(\boldsymbol{r}|\mathbf{r}')\nabla u(\boldsymbol{r})-u(\boldsymbol{r})\nabla G(\boldsymbol{r}|\mathbf{r}')\big)\cdot\boldsymbol{n}d\ell, \qquad (3.5)
$$

where $C=\Gamma_{\rm ext}+\mathcal{B}_R+\Gamma_\infty$ is the contour enclosing the surface, $\mathcal{U}_R$ as shown in Fig. 2. We note that at Γ_∞, the radiation condition will cause the corresponding contour integral to vanish. Therefore, we need only analyze the boundary integral expression $\int\big(G(\boldsymbol{r}|\mathbf{r}')\nabla u(\boldsymbol{r})-u(\boldsymbol{r})\nabla G(\boldsymbol{r}|\mathbf{r}')\big)\cdot\boldsymbol{n}d\ell$ along $\Gamma_{\rm ext}$ and $\mathcal{B}_R$. Adding and subtracting the term $B\nabla u(\boldsymbol{r})\nabla G(\boldsymbol{r}|\mathbf{r}')/A$ in the boundary integral expression in (3.5) on $\Gamma_{\rm ext}$ gives

$$
\begin{aligned}
&\int_{\Gamma_{\rm ext}}\Big[G(\boldsymbol{r}|\mathbf{r}')\nabla u(\boldsymbol{r})-u(\boldsymbol{r})\nabla G(\boldsymbol{r}|\mathbf{r}')+\frac{B}{A}\nabla u(\boldsymbol{r})\nabla G(\boldsymbol{r}|\mathbf{r}')-\frac{B}{A}\nabla u(\boldsymbol{r})\nabla G(\boldsymbol{r}|\mathbf{r}')\Big]\cdot\boldsymbol{n}d\ell\\
=&\int_{\Gamma_{\rm ext}}\frac{1}{A}\big[(AG(\boldsymbol{r}|\mathbf{r}')+B\nabla G(\boldsymbol{r}|\mathbf{r}'))\nabla u(\boldsymbol{r})-\big(Au(\boldsymbol{r})+B\nabla u(\boldsymbol{r})\big)\nabla G(\boldsymbol{r}|\mathbf{r}')\big]\cdot\boldsymbol{n}d\ell\\
=&\int_{\Gamma_{\rm ext}}\frac{1}{A}\big[(0)\nabla u(\boldsymbol{r})-\big(h(\boldsymbol{r})\big)\nabla G(\boldsymbol{r}|\mathbf{r}')\big]\cdot\boldsymbol{n}d\ell\\
=&\int_{\Gamma_{\rm ext}}\frac{1}{A}\big(-h(\boldsymbol{r})\big)\big[\nabla G(\boldsymbol{r}|\mathbf{r}')\big]\cdot\boldsymbol{n}d\ell.
\end{aligned}
$$

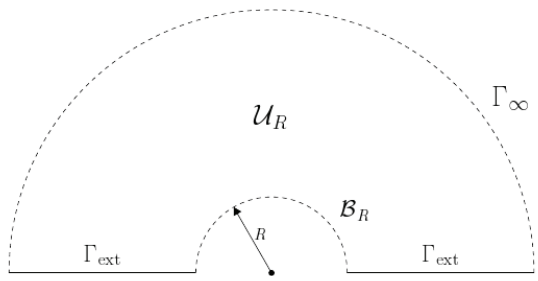


Figure 2: Exterior domain.

As a result, (3.5) reduces to:

$$u(\mathbf{r}') = \iint_{\mathcal{U}_R} f(\boldsymbol{r})G(\boldsymbol{r}|\mathbf{r}')dS + \int_{\mathcal{B}_R} \big(G(\boldsymbol{r}|\mathbf{r}')\nabla u(\boldsymbol{r}) - u(\boldsymbol{r})\nabla G(\boldsymbol{r}|\mathbf{r}')\big)\cdot \boldsymbol{n}d\ell \\ - \frac{1}{A}\int_{\Gamma_{\rm ext}} h(\boldsymbol{r})\big[\nabla G(\boldsymbol{r}|\mathbf{r}')\big]\cdot \boldsymbol{n}d\ell$$

and hence

$$u(\boldsymbol{r}) = \iint_{\mathcal{U}_R} f(\mathbf{r}')G(\boldsymbol{r}|\mathbf{r}')dS' + \int_{\mathcal{B}_R} \Big(G(\boldsymbol{r}|\mathbf{r}')\frac{\partial u(\boldsymbol{r})}{\partial n'} - u(\mathbf{r}')\frac{\partial G(\boldsymbol{r}|\mathbf{r}')}{\partial n'}\Big)d\ell' \\ - \frac{1}{A}\int_{\Gamma_{\rm ext}} h(\mathbf{r}')\Big[\frac{\partial G(\boldsymbol{r}|\mathbf{r}')}{\partial n'}\Big]d\ell'. \tag{3.6}$$

3.2 Steklov-Poincaré operator analysis

Armed with the Green's function, we now aim to develop a boundary operator that serves to bridge the analytic solution of the exterior domain with the numerical solution of the interior domain and prove the resulting variational problem wellposed. From Eq. (3.5), we shift $\boldsymbol{r}$ onto the artificial boundary $\mathcal{B}_R$ to obtain

$$\frac{1}{2}u(\boldsymbol{r}) = \overbrace{\iint_{\mathcal{U}_R} f(\mathbf{r}')G(\boldsymbol{r}|\mathbf{r}')dS'}^{\text{Newton potential}} - \frac{1}{A}\int_{\Gamma_{\rm ext}} h(\mathbf{r}')\frac{\partial G(\boldsymbol{r}|\mathbf{r}')}{\partial y'}dx' \\ + \int_{\mathcal{B}_R}\Big(\underbrace{G(\boldsymbol{r}|\mathbf{r}')\frac{\partial u(\mathbf{r}')}{\partial \boldsymbol{n}_{\mathbf{r}'}}}_{\text{Single-layer potential}} - \underbrace{u(\mathbf{r}')\frac{\partial G(\boldsymbol{r}|\mathbf{r}')}{\partial \boldsymbol{n}_{\mathbf{r}'}}}_{\text{Double-layer potential}}\Big)d\theta' \boldsymbol{r}\in\mathcal{B}_R. \tag{3.7}$$

Taking a normal derivative $\partial/\partial n$ of (3.7) along $\mathcal{B}_R$ yields the hypersingular boundary integral equation

$$\frac{1}{2}\frac{\partial u(\boldsymbol{r})}{\partial \boldsymbol{n}_{\boldsymbol{r}}} = \overbrace{\frac{\partial}{\partial n}\Big(\iint_{\mathcal{U}_R} f(\mathbf{r}')G(\boldsymbol{r}|\mathbf{r}')dS'}^{\text{Normal Derivative of Newton potential}} - \frac{1}{A}\int_{\Gamma_{\rm ext}} h(\mathbf{r}')\frac{\partial G(\boldsymbol{r}|\mathbf{r}')}{\partial y'}dx'\Big) \\ + \int_{\mathcal{B}_R}\Big(\underbrace{\frac{\partial G(\boldsymbol{r}|\mathbf{r}')}{\partial \boldsymbol{n}_{\boldsymbol{r}}}\frac{\partial u(\mathbf{r}')}{\partial \boldsymbol{n}_{\mathbf{r}'}}}_{\text{Adjoint Double-layer potential}} - \underbrace{u(\mathbf{r}')\frac{\partial}{\partial \boldsymbol{n}_{\boldsymbol{r}}}\frac{\partial G(\boldsymbol{r}|\mathbf{r}')}{\partial \boldsymbol{n}_{\mathbf{r}'}}}_{\text{Hypersingular operator}}\Big)d\theta' \boldsymbol{r}\in\mathcal{B}_R. \tag{3.8}$$

We define our boundary operator, known as the Steklov-Poincarè operator

$$\mathcal{T}_R : H^{\frac{1}{2}}(\mathcal{B}_R) \to H^{-\frac{1}{2}}(\mathcal{B}_R)$$

to be

$$\mathcal{T}_R = S^{-1}\Big(\frac{1}{2}I + D\Big) = \Big(\frac{1}{2}I + A\Big)S^{-1}\Big(\frac{1}{2}I + D\Big) + H,$$

where

$$\begin{aligned}
(S\varphi)(\boldsymbol{r}) &= \int_{\mathcal{B}_R} G(\boldsymbol{r}|\mathbf{r}')\varphi(\mathbf{r}')d\theta', && S: H^{-\frac{1}{2}}(\mathcal{B}_R) \to H^{\frac{1}{2}}(\mathcal{B}_R),\\
(Du)(\boldsymbol{r}) &= \int_{\mathcal{B}_R} u(\mathbf{r}')\frac{\partial G(\boldsymbol{r}|\mathbf{r}')}{\partial r'}d\theta', && D: H^{\frac{1}{2}}(\mathcal{B}_R) \to H^{\frac{1}{2}}(\mathcal{B}_R),\\
(A\varphi)(\boldsymbol{r}) &= \int_{\mathcal{B}_R} \frac{\partial G(\boldsymbol{r}|\mathbf{r}')}{\partial r}\varphi(\mathbf{r}')d\theta', && A: H^{-\frac{1}{2}}(\mathcal{B}_R) \to H^{-\frac{1}{2}}(\mathcal{B}_R),\\
(Hu)(\boldsymbol{r}) &= -\int_{\mathcal{B}_R} u(\mathbf{r}')\frac{\partial}{\partial r}\frac{\partial G(\boldsymbol{r}|\mathbf{r}')}{\partial r'}d\theta', && H: H^{\frac{1}{2}}(\mathcal{B}_R) \to H^{-\frac{1}{2}}(\mathcal{B}_R).
\end{aligned}$$

We need the following lemma for the wellposedness proof in the next section. Similar theorems and their proofs can be found in [18] and [19].

Lemma 3.1. *The Steklov-Poincarè operator, $\mathcal{T}_R$ is bounded, its principal part $\mathcal{T}_{R,P}$, satisfies the coercivity condition*

$$-\langle \mathcal{T}_{R,P}u, u\rangle \geq C\,\|u\|^2_{H^{\frac{1}{2}}(\mathcal{B}_R)}$$

for some $C > 0$, such that the difference $\mathcal{T}_R - \mathcal{T}_{R,P}$ is a compact operator from $H^{1/2}(\mathcal{B}_R) \to H^{-1/2}(\mathcal{B}_R)$.

Proof. The boundedness of the operator, $\mathcal{T}_R$, is well established, (see, for example, [18] and [19]). We define the principal part of $\mathcal{T}_R$ to be $\mathcal{T}_{R,P}$, which corresponds to the Laplacian portion of the operator. That is, instead of (3.1), we would be solving $-\Delta u(\boldsymbol{r}) = 0$ and investigating the corresponding mapping properties of $\mathcal{T}_{R,P} : u \to \partial u/\partial n$ on the semicircle $\mathcal{B}_R$. By standard continuous extension results this can be extended to the whole disk, denoted $\mathcal{B}_{\text{DISK}}$. Assuming u a radiating solution outside a disk $\mathcal{B}_{\text{DISK}}$ of radius R, we obtain

$$u(r,\theta) = \sum_{-\infty}^{\infty} a_n r^{-n} e^{in\theta}, \qquad r \geq R \quad \text{and} \quad 0 \leq \theta \leq 2\pi.$$

Thus

$$u|_{\partial\mathcal{B}_{\text{DISK}}} = \sum_{-\infty}^{\infty} b_n e^{in\theta},$$

where $b_n = a_n R^{-n}$. Let

$$T_{R,P} : u \to \frac{\partial u}{\partial n}$$

be such that

$$T_{R,P}u = -\sum_{-\infty}^{\infty} \frac{n}{R} b_n e^{in\theta}.$$

It then follows that

$$-\int_{\partial\mathcal{B}_{\text{DISK}}} T_{R,P}u\bar{u}ds \geq C\,\|u\|^2_{H^{\frac{1}{2}}(\partial\mathcal{B}_{\text{DISK}})}.$$

Standard extension argument then gives

$$-\int_{\partial\mathcal{B}_R} T_{R,P} u\bar{u} ds \geq C \|u\|^2_{H^{\frac{1}{2}}(\partial\mathcal{B}_R)}.$$

It follows that $\mathcal{T}_R - \mathcal{T}_{R,P}$ is a compact operator from $H^{1/2}(\mathcal{B}_R) \to H^{-1/2}(\mathcal{B}_R)$. $\square$

Employing the field continuity conditions across the artificial boundary, $\mathcal{B}_R$, we recast the boundary value problem (2.6) in terms of $\mathcal{T}_{\mathcal{R}}$:

$$\begin{cases} -\Delta u^{n+1} + \alpha^2 \varepsilon_r u^{n+1} = \alpha^2 \varepsilon_r \tilde{u}^{n+1}, & \text{in } \Omega_R, \\ \Delta t \gamma \alpha^2 u^{n+1} + \dfrac{\eta}{\mu} \dfrac{\partial u^{n+1}}{\partial n} = \Delta t \gamma \alpha^2 \tilde{u}^{n+1} - \tilde{u}^{n+1}, & \text{on } S, \\ \dfrac{\partial u^{n+1}}{\partial r} - \mathcal{T}_R u^{n+1} = \dfrac{\partial u^i}{\partial r} - (\Psi_R \tilde{u}^{s,n+1}) - \mathcal{T}_R(u^i), & \text{on } \mathcal{B}_R, \end{cases} \tag{3.9}$$

where

$$\Psi_R f(\mathbf{r}) = \int\int_{\mathcal{U}_{\mathcal{R}}} f(\mathbf{r}') G(\mathbf{r}|\mathbf{r}') dS' - \frac{1}{A} \int_{\Gamma_{ext}} h(\mathbf{r}') \frac{\partial G(\mathbf{r}|\mathbf{r}')}{\partial y'} dx'.$$

In what follows we solve (3.9) via a variational formulation and establish its wellposedness.

4 Interior problem

The aim of this section is to derive a variational formulation for our problem and prove its wellposedness. In a separate subsection we will provide the background and state the convergence and stability results reported in [9] that continue to hold true for the current framework with impedance boundary conditions.

4.1 Variational formulation

Instead of enforcing the boundary conditions on the test function space V as in [9], we choose to define the subspace V simply as $H^1(\Omega_R)$. The variational problem for (3.9) is then find $u \in V$ such that

$$b_{TM}(u, v) = F(v), \quad \forall v \in V. \tag{4.1}$$

Multiplying (3.9) by a test function $v \in V$ yields

$$-\int_{\Omega_R} \Delta u \bar{v} dx dy + \alpha^2 \int_{\Omega_R} \varepsilon_r u \bar{v} dx dy = \alpha^2 \int_{\Omega_R} \varepsilon_r \tilde{u} \bar{v} dx dy, \tag{4.2a}$$

$$\Delta t \gamma \alpha^2 \int_S u \bar{v} d\ell + \frac{\eta}{\mu} \int_S \frac{\partial u}{\partial n} \bar{v} d\ell = \Delta t \gamma \alpha^2 \int_S \tilde{u} \bar{v} d\ell - \int_S \tilde{u} \bar{v} d\ell, \tag{4.2b}$$

$$\int_{\mathcal{B}_R} \frac{\partial u}{\partial r} \bar{v} d\ell - \int_{\mathcal{B}_R} \mathcal{T}_R u \bar{v} d\ell = \int_{\mathcal{B}_R} \frac{\partial u^i}{\partial r} \bar{v} d\ell - \int_{\mathcal{B}_R} \Psi_R \tilde{u}^s \bar{v} d\ell - \int_{\mathcal{B}_R} \mathcal{T}_R(u^i) \bar{v} d\ell. \tag{4.2c}$$

Green's identity

$$\int_{\Omega_R} \nabla u \cdot \nabla \bar{v} dxdy = \int_{\mathcal{B}_R \cup S} \frac{\partial u}{\partial n} \bar{v} d\ell - \int_{\Omega_R} \Delta u \bar{v} dxdy,$$

then simplifies (4.2a) to

$$\int_{\Omega_R} \nabla u \cdot \nabla \bar{v} dxdy - \Big(\int_{\mathcal{B}_R} \frac{\partial u}{\partial n} \bar{v} d\ell + \int_S \frac{\partial u}{\partial n} \bar{v} d\ell \Big) + \alpha^2 \int_{\Omega_R} \varepsilon_r u \bar{v} dxdy = \alpha^2 \int_{\Omega_R} \varepsilon_r \tilde{u} \bar{v} dxdy.$$

Now substituting the known values from (4.2b) and (4.2c) and letting

$$J = \frac{\partial u^i}{\partial r}\Big|_{r=R} - \mathcal{T}_R u^i,$$

we obtain for (4.2a):

$$\begin{aligned} \int_{\Omega_R} \nabla u \cdot \nabla \bar{v} dxdy - \Big[\int_{\mathcal{B}_R} \mathcal{T}_R u \bar{v} d\ell + \int_{\mathcal{B}_R} J \bar{v} d\ell - \int_{\mathcal{B}_R} \Psi \tilde{u}^s \bar{v} d\ell \\ + \frac{\mu}{\eta} \Delta t \gamma \alpha^2 \int_S \tilde{u} \bar{v} d\ell - \frac{\mu}{\eta} \int_S \tilde{\tilde{u}} \bar{v} d\ell - \frac{\mu}{\eta} \Delta t \gamma \alpha^2 \int_S u \bar{v} d\ell \Big] \\ + \alpha^2 \int_{\Omega_R} \varepsilon_r u \bar{v} dxdy = \alpha^2 \int_{\Omega_R} \varepsilon_r \tilde{u} \bar{v} dxdy. \end{aligned}$$

Define the sesquilinear term

$$b_{TM}(u,v) = \int_{\Omega_R} \nabla u \cdot \nabla \bar{v} dxdy - \int_{\mathcal{B}_R} \mathcal{T}_R u \bar{v} d\ell + \frac{\mu}{\eta} \Delta t \gamma \alpha^2 \int_S u \bar{v} d\ell + \alpha^2 \int_{\Omega_R} \varepsilon_r u \bar{v} dxdy \tag{4.3}$$

and the conjugate linear functional term

$$F(v) = \int_{\mathcal{B}_R} J \bar{v} d\ell - \int_{\mathcal{B}_R} \Psi_R \tilde{u}^s \bar{v} d\ell + \frac{\mu}{\eta} \Delta t \gamma \alpha^2 \int_S \tilde{u} \bar{v} d\ell - \frac{\mu}{\eta} \int_S \tilde{\tilde{u}} \bar{v} d\ell + \alpha^2 \int_{\Omega_R} \varepsilon_r \tilde{u} \bar{v} dxdy. \tag{4.4}$$

Following [19], we rewrite $b_{TM}(u,v) = b_{TM1}(u,v) + b_{TM2}(u,v)$, where

$$\begin{aligned} b_{TM1}(u,v) &= \int_{\Omega_R} (\nabla u \cdot \nabla \bar{v} + u \bar{v}) dxdy - \int_{\mathcal{B}_R} \mathcal{T}_{R,P} u \bar{v} d\ell, \\ b_{TM2}(u,v) &= (-1) \int_{\Omega_R} u \bar{v} dxdy - \int_{\mathcal{B}_R} (\mathcal{T}_R - \mathcal{T}_{R,P}) u \bar{v} d\ell + \frac{\mu}{\eta} \Delta t \gamma \alpha^2 \int_S u \bar{v} d\ell + \alpha^2 \int_{\Omega_R} \varepsilon_r u \bar{v} dxdy. \end{aligned}$$

The problem then becomes finding $u \in V$ such that

$$b_{TM1}(u,v) + b_{TM2}(u,v) = F(v), \quad \forall v \in V. \tag{4.5}$$

The motivation for splitting up the sesquilinear term $b_{TM}(u,v)$ (to include splitting up the Steklov-Poincarè operator, $\mathcal{T}_R$) is that the entire term itself is not strictly coercive, but a portion of it (i.e., $b_{TM1}(u,v)$) can be shown to be. We will also show that $b_{TM2}(u,v)$ is a compact operator.

We are now ready to prove our main theorem.

Theorem 4.1. *The variational problem* (4.1) *has a unique solution $u \in V$ and there exists a constant $C > 0$, such that*

$$\|u\| \le C\big[\|u^i\| + \|\tilde{u}^s\| + \|\tilde{u}\| + \|\tilde{\tilde{u}}\| + \|\varepsilon_r \tilde{u}\|\big].$$

Proof. Using trace theory, for constants $D_1, D_2 > 0$, we have

$$\begin{aligned} b_{TM1}(u,u) =& \int_{\Omega_R} (\nabla u \cdot \nabla \bar{u} + u\bar{u}) dxdy - \int_{\mathcal{B}_R} \mathcal{T}_{R,P} u\bar{u} d\ell \\ \ge& \|u\|^2_{H^1(\Omega_R)} + D_1 \|u\|^2_{H^{\frac{1}{2}}(\mathcal{B}_R)} \\ \ge& \|u\|^2_{H^1(\Omega_R)} + D_2 \|u\|^2_{H^1(\Omega_R)} \\ \ge& (1+D_2)\|u\|^2_{H^1(\Omega_R)}. \end{aligned}$$

Also, for constants $D_3, D_4 > 0$, we have

$$\begin{aligned} \Big|b_{TM1}(u,v)\Big| =& \Big|\int_{\Omega_R} \Big(\nabla u \cdot \nabla \bar{v} + u\bar{v}\Big) dxdy - \int_{\mathcal{B}_R} \mathcal{T}_{R,P} u\bar{v} d\ell\Big| \\ \le& \Big|\int_{\Omega_R} \Big(\nabla u \cdot \nabla \bar{v} + u\bar{v}\Big) dxdy\Big| + \Big|\int_{\mathcal{B}_R} \mathcal{T}_{R,P} u\bar{v} d\ell\Big| \\ \le& \int_{\Omega_R} |\nabla u \cdot \nabla \bar{v}| dxdy + \int_{\Omega_R} |u\bar{v}| dxdy + \Big|\int_{\mathcal{B}_R} \mathcal{T}_{R,P} u\bar{v} d\ell\Big| \\ \le& \|\nabla u\|_{L^2} \|\nabla v\|_{L^2} + \|u\|_{L^2}\|v\|_{L^2} + D_3 \|u\|_{H^{1/2}(\mathcal{B}_R)} \|v\|_{H^{1/2}(\mathcal{B}_R)} \\ \le& \|u\|_{H^1(\Omega_R)} \|v\|_{H^1(\Omega_R)} + D_4 \|u\|_{H^1(\Omega_R)} \|v\|_{H^1(\Omega_R)} \\ =& (1+D_4)\|u\|_{H^1(\Omega_R)} \|v\|_{H^1(\Omega_R)}. \end{aligned}$$

Thus, the Lax-Milgram Lemma applies and there exists a bijective bounded linear operator $B_1 : V \to V$ with bounded inverse such that

$$b_{TM1}(u,v) = \langle B_1 u, v\rangle, \qquad \forall v \in V.$$

Similarly, to analyze $b_{TM2}(u,v)$, we further split up the individual terms and show that each is a compact operator. We introduce the bounded linear operator $B_2 : V \to V$:

$$\begin{aligned} b_{TM2}(u,v) = \langle B_2 u, v\rangle =& (-1)\langle u,v\rangle_{L^2(\Omega_R)} - \langle(\mathcal{T}_R - \mathcal{T}_{R,P})u, v\rangle + \Big(\frac{\mu}{\eta}\Delta t \gamma \alpha^2\Big)\langle u,v\rangle \\ &+ \alpha^2 \langle \varepsilon_r u, v\rangle_{L^2(\Omega_R)}, \qquad \forall v \in V. \end{aligned}$$

The portion of the operator representing $\langle u,v\rangle_{L^2(\Omega_R)}$ is compact, based on the fact that the injection of $H^1(\Omega_R)$ into $L^2(\Omega_R)$ is compact. It was established earlier in Theorem 3.1 that $\mathcal{T}_R - \mathcal{T}_{R,P}$ is a compact operator. Finally, as in [20], we define the space $L^2_{\varepsilon_r}(\Omega_R)$ with inner product

$$\langle u,v\rangle_{L^2_{\varepsilon_r}(\Omega_R)} = \langle \varepsilon_r u, v\rangle, \quad \forall u,v \in L^2_{\varepsilon_r}(\Omega_R).$$

Then the norm on $L^2_{\varepsilon_r}(\Omega_R)$ is equivalent to the standard $L^2(\Omega_R)$ norm. This establishes that the term $\alpha^2\langle\varepsilon_r u, v\rangle_{L^2(\Omega_R)}$ is compact. Therefore, we can deduce that the operator B_2, a linear combination of compact operators, is compact. It can also be shown that $F(v)$ as defined in (4.4), is bounded:

$$\Big|F(v)\Big| \le \big[C_1\|u^i\| + C_2\,\|\tilde{u}^s\| + C_3\,\|\tilde{u}\| + C_4\,\|\tilde{\tilde{u}}\| + C_5\,\|\varepsilon_r\tilde{u}\|\,\big]\,\|v\|\,,$$

which implies

$$\|F\| \le \tilde{C}\big[\|u^i\| + \|\tilde{u}^s\| + \|\tilde{u}\| + \|\tilde{\tilde{u}}\| + \|\varepsilon_r\tilde{u}\|\,\big]$$

for some $\tilde{C}$. By Riesz Representation Theorem, there exists a unique $w \in V$ such that $F(v) = \langle w, v\rangle$. We now recast (4.5) to be finding $u \in V$ such that

$$B_1u + B_2u = (B_1 + B_2)u = w.$$

Since B_1 has a bounded inverse and B_2 is compact, Fredholm Alternative holds and uniqueness implies existence.

To prove uniqueness, $b_{TM}(u,u) = 0$ implies

$$\begin{aligned} b_{TM}(u,u) &= \int_{\Omega_R}\nabla u\cdot\nabla\bar{u}dxdy - \int_{\mathcal{B}_R}\mathcal{T}_R u\bar{u}d\ell + \frac{\mu}{\eta}\Delta t\gamma\alpha^2\int_S u\bar{u}d\ell + \alpha^2\int_{\Omega_R}\varepsilon_r u\bar{u}dxdy = 0\\ &= \int_{\Omega_R}|\nabla u|^2 + \alpha^2\varepsilon_r|u|^2dxdy - \int_{\mathcal{B}_R}\mathcal{T}_R u\bar{u}d\ell + \frac{\mu}{\eta}\Delta t\gamma\alpha^2\int_S|u|^2d\ell = 0, \end{aligned} \tag{4.6a}$$

$$\int_{\mathcal{B}_R}\mathcal{T}_R u\bar{u}d\ell = \int_{\Omega_R}|\nabla u|^2 + \alpha^2\varepsilon_r|u|^2dxdy + \frac{\mu}{\eta}\Delta t\gamma\alpha^2\int_S|u|^2d\ell. \tag{4.6b}$$

Assuming $\Re(\varepsilon_r) > 0$ and $\Im(\varepsilon_r) \le 0$ and on the surface S, $\mu = 1$ and $\Im(\eta) = 0$, we have

$$\Im\Big(\int_{\mathcal{B}_R}\mathcal{T}_R u\bar{u}d\ell\Big) \le 0, \tag{4.7}$$

hence $u \equiv 0$. Finally, the Fredholm Alternative also implies the boundedness of the inverse of $(B_1 + B_2)$: there exist constants D_5, $C > 0$, such that

$$\begin{aligned} \|u\| \le& D_5\,\|F\| \le D_5\big[\tilde{C}\big(\|u^i\| + \tilde{u}^s + \|\tilde{u}\| + \|\tilde{\tilde{u}}\| + \|\varepsilon_r\tilde{u}\|\,\big)\big]\\ =& C\big(\|u^i\| + \|\tilde{u}^s\| + \|\tilde{u}\| + \|\tilde{\tilde{u}}\| + \|\varepsilon_r\tilde{u}\|\,\big). \end{aligned}$$

The proof is completed. □

4.2 Numerical analysis

Error and stability results of [9] hold for this problem and proofs are similar. Here we state the results and the necessary definitions for completeness.

Let $\tau_h = \{K\}$ be the partition of Ω_R where each $K \in \tau_h$ represents a triangle. These finite elements form an exact partition of Ω_R; that is, $\Omega_R = \bigcup_{K\in\tau_h} K$. For an arbitrary triangle K, we denote $h_K = \text{diam}(K) = \max\{\|\mathbf{x}-\mathbf{y}\| \,|\mathbf{x},\mathbf{y} \in K\}$, the mesh size $h = \max_{K\in\tau_h} h_K$, and s_K the diameter of the largest circle inscribed in K. Following [21], we have the finite-dimensional subspace V_h of the test space V:

$$V_h = \{v_h \in H^1(\Omega_R) : v_h|_K \in P_1, K \in \tau_h\},$$

where $\{\phi_j^h(x)\}_{j=1}^N$ is a linear nodal basis of V_h. Each $v_h \in V_h$ is expressed as

$$v_h = \sum_{j=1}^{N} v_j \phi_j^h(x).$$

We note that V_h is closed in V and $V_h \to V$ as $h \to 0$. The fully discrete problem is then finding $u_h^n \in V_h$, $n = 1, 2, \cdots, N$, such that

$$b_{TM}(u_h^n, v_h) = F^n(v_h), \quad \forall v_h \in V_h, \tag{4.8}$$

where $b_{TM}(u_h^n, v_h)$ and $F^n(v_h)$ are defined as in (4.3) and (4.4), respectively. We have the following convergence result:

Theorem 4.2. *Let $u^n \in V$ and $u_h^n \in V_h$ be the solutions to* (4.1) *and* (4.8)*, respectively, for $F^n \in V'$. Given $\epsilon > 0$, there exists an $h_0 = h_0(\epsilon)$ such that for all $0 < h < h_0$, then*

$$\|u^n - u_h^n\|_{L^2(\Omega_R)} \le \epsilon \|u^n - u_h^n\|_V . \tag{4.9}$$

Furthermore, given $\epsilon > 0$, there exists an $h_1 = h_1(\epsilon)$ such that for all $0 < h < h_1$,

$$\|u^n - u_h^n\|_V \le C\epsilon \|F^n\|_{L^2(\Omega_R)} \tag{4.10}$$

for some positive constant C independent of h. It follows that

$$\|u^n - u_h^n\|_{L^2(\Omega_R)} \le C\epsilon^2 \|F^n\|_{L^2(\Omega_R)} . \tag{4.11}$$

Following a very similar procedure as in [9] we have the following stability result:

Theorem 4.3. *The Newmark scheme for the TM variational problem* (4.1) *is unconditionally stable for arbitrary $\delta t > 0$ satisfying*

$$2\beta \ge \gamma > \frac{1}{2}.$$

5 Numerical implementation

We consider an over-filled cavity of 1m deep by 0.5m wide as shown in Fig. 3. We use an incident Gaussian pulse with $\delta t = 0.0625$, $\epsilon_r = 2$, $\mu_0 = 1$ and Newmark parameters

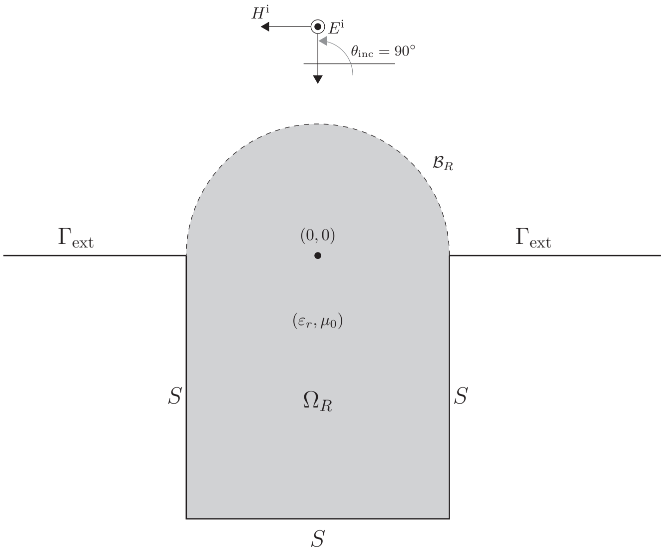
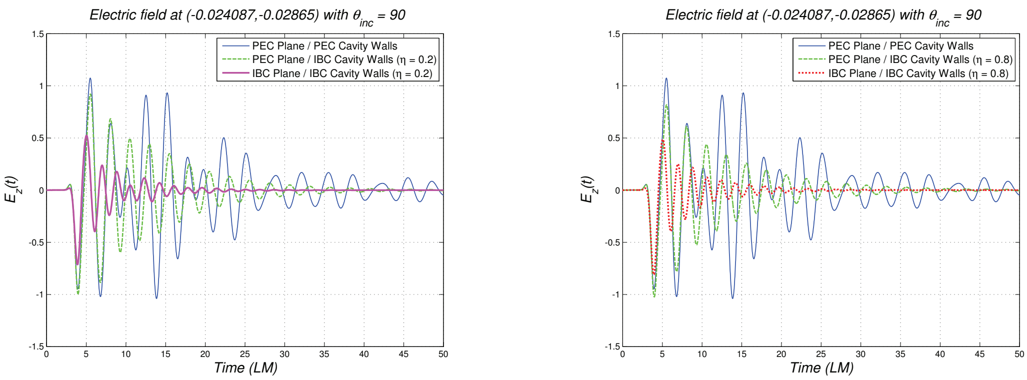


Figure 3: Overfilled deep cavity.



(a) (b)

Figure 4: Electric field depiction.

$\gamma = 0.95$ and $\beta = 0.5256$ for stability. We ran three separate cases: 1.) PEC plane and PEC cavity walls; 2.) PEC plane and IBC cavity walls; 3.) IBC plane and IBC cavity walls. We present here an observation point inside the cavity, Fig. 4.

We observe that the scattered fields become more attenuated as the boundary conditions for the plane and cavity walls approach an IBC surface. The observations are truncated at 50 LM for scaling but the simulations exhibit the same stability beyond this point. We also observe that as $\eta \to 0$ the fields manifest the characteristics of a PEC surface, as seen in the more oscillatory behavior of the IBC plane and IBC cavity walls condition.

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