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Numerical Solutions of the System of Singular Integro-Differential Equations in Classical Hölder Spaces

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Abstract. New numerical methods based on collocation methods with the mechanical quadrature rules are proposed to solve some systems of singular integro-differential equations that are defined on arbitrary smooth closed contours of the complex plane. We carry out the convergence analysis in classical Hölder spaces. A numerical example is also presented.

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1 Introduction

Singular integral equations (SIE) and singular integro-differential equations (SIDE) are used to model many application problems in electronics, aerodynamics, mechanics, thermo-elasticity, and queuing analysis, see for example, [1–7] and the literature cited therein. The general theory of SIE and SIDE has been studied in depth in the last decades, see for example, [8–10]. Analytic solutions of SIE and SIDE are rarely available. It is necessary to find approximate solutions for SIE and SIDE. There is rich literature about approximate solutions of SIE and SIDE using collocation methods with mechanical quadrature rules. Most of existing methods are designed for two cases: One is that the contour Γ is a unit circle; and the other case is that Γ is an interval on the real axis, see for example, [12–16]. In this paper, we study the case for which

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the contour of integration Γ can be an arbitrary smooth closed curve on the complex plane. It should be noted that conformal mapping from an arbitrary smooth closed contour with origin to the unit circle is not trivial and may make the problem more complicated due to the following reasons:

- The coefficients, kernel, etc. may become more complicated after the conformal mapping.
- The convergence analysis may be more difficult after the transform.
- Many existing numerical methods may not work for the transformed problem.

We note that the collocation method with the mechanical quadrature rules for one dimensional SIDE in generalized Hölder spaces has been investigated in [23–25]; and for SIE in classical Hölder spaces has been investigated in [20–22].

The convergence analysis of the collocation method with the mechanical quadrature rules for systems of SIDE in classical Hölder spaces has not been investigated when the equations are defined on an arbitrary smooth closed contour.

In this article we study the collocation method with mechanical quadrature rules for approximate solutions of systems of SIDE. The theoretical background of our proposed methods is based on Krikunov's integral representation and V. Zolotarevski's results for approximate solutions of SIE published in [9, 18–21].

The paper is organized as follows. In Section 2 we introduce some definitions and notations. We describe the integro-differential equations in Section 3. We present our numerical schemes of the collocation method with the mechanical quadrature rules and carry out the convergence analysis in Section 4. We show numerical example in Section 5.

2 Main definitions and notations

Let Γ be an arbitrary smooth closed contour bounding a simply connected region F^+ of the complex plane; and $t = 0 \in F^+$ and $F^- = \mathbb{C} \setminus \{F^+ \cup \Gamma\}$, where $\mathbb{C}$ is the complex plane. Let $z = \psi(w)$ be a function, mapping conformably the outside of unit circle $\Gamma_0 (= |w| = 1)$ on F^- such that

$$\psi(\infty) = \infty, \quad \psi'(\infty) = c_0 > 0.^\dagger \tag{2.1}$$

By the Riemann mapping theorem the function $z = \psi(w)$ which satisfies the conditions (2.1) exists and it is unique [27]. There are a lot of such contours, see for example, Lyapunov contours [26] and others. We denote by Λ the set of the contours for which the relation (2.1) holds.

We denote by $[H_\beta(\Gamma)]_m$, $0 < \beta \leq 1$, the Banach space of m-dimensional vector-functions (v.f.), satisfying on Γ the Hölder condition with degree β. The norm is de-

†We use $c_0, c_1, \cdots$, to represents constants.

fined as

$$\begin{aligned}&\forall g(t)=\{g_1(t),\cdots,g_m(t)\},\quad \|g\|_\beta=\sum_{k=1}^m(\|g_k\|_C+H(g_k,\beta)),\\&\|g\|_C=\max_{t\in\Gamma}|g(t)|,\quad H(g;\beta)=\sup_{t'\neq t''}\{|t'-t''|^{-\beta}\cdot|g(t')-g(t'')|\},\quad t',t''\in\Gamma.\end{aligned}$$

We denote by $[H_\beta^{(q)}(\Gamma)]_m$ the space of functions $g(t)$ containing the derivatives of degree q. Thus $g^{(q)}(t)\in[H_\beta(\Gamma)]_m$.

Let U_n be the Lagrange interpolation operator constructed by distinct points $\{t_j\}_{j=0}^{2n}$ (n is a natural number) for any continuous function on Γ :

$$(U_ng)(t)=\sum_{j=0}^{2n}g(t_j)\cdot l_j(t),\quad t\in\Gamma,$$

where

$$l_j(t)=\Big(\frac{t_j}{t}\Big)^n\prod_{(k=0,k\neq j)}^{2n}\frac{t-t_k}{t_j-t_k}\equiv\sum_{k=-n}^{n}\Lambda_k^{(j)}t^k,\quad t\in\Gamma,\quad j=0,\cdots,2n. \tag{2.2}$$

Let $G(t)=\{G_{k,l}(t)\}_{k,l=1}^m$ be a non-singular $m\times m$ matrix functions with $G_{k,l}(t)\in H_\beta(\Gamma)$.

Definition 2.1. *A factorization of non-singular matrix functions $G(t)$ relative to the contour Γ is a representation of G in the form*

$$G(t)=G^+(t)\Delta(t)G^-(t),$$

where $G^\pm(t)$ is a analytic matrix function and non-singular in $D^\pm$, satisfying $\det G^\pm(t)\neq 0$, *$\Delta(t)=diag\{t^{\kappa_1},t^{\kappa_2},\cdots,t^{\kappa_m}\}$ and $\kappa_1,\kappa_2,\cdots,\kappa_m$ are integers. The numbers $\kappa_1,\kappa_2,\cdots,\kappa_m$ are called left partial indexes [30].*

3 The problem formulation and some existing results

We consider the system of the SIDE in $[H_\beta(\Gamma)]_m$

$$\begin{aligned}(Mx\equiv)\sum_{r=0}^q\Big[&\tilde A_r(t)x^{(r)}(t)+\tilde B_r(t)\frac{1}{\pi i}\int_\Gamma\frac{x^{(r)}(\tau)}{\tau-t}d\tau\\&+\frac{1}{2\pi i}\int_\Gamma K_r(t,\tau)\cdot x^{(r)}(\tau)d\tau\Big]=f(t),\quad t\in\Gamma,\end{aligned} \tag{3.1}$$

where $\tilde A_r(t),\tilde B_r(t),K_r(t,\tau)$ $(r=0,\cdots,q)$ are known $m\times m$ matrix functions (m.f.), the elements of the m.f. belong to $H_\beta(\Gamma)$, $f(t)$ is a known m-dimensional vector-functions (v.f.) in $[H_\beta(\Gamma)]_m$, $x^{(0)}(t)=x(t)$ is an unknown v.f. in $[H_\beta(\Gamma)]_m$, $x^{(r)}(t)=d^r x(t)/dt^r$ $(r=1,\cdots,q)$, and q is a positive integer. We suppose that the v.f. $x^{(q)}(t)$ belongs to $[H_\beta(\Gamma)]_m$, that is,

$$x^{(k)}(t)\in[H_\beta(\Gamma)]_m,\quad k=0,\cdots,q-1.$$

We search for a solution of (3.1) in the class of v.f. satisfying the condition

$$\frac{1}{2\pi i}\int_\Gamma x(\tau)\tau^{-k-1}d\tau=0,\quad k=0,\cdots,q-1. \tag{3.2}$$

We note that the solution of SIDE (3.1) can differ by a constant [9,18,19]. In this case we cannot investigate the solution of SIDE (3.1) directly. That is why we introduce additional conditions (3.2) for function $x(t)$.

We denote the system (3.1) with conditions (3.2) as problem (3.1)-(3.2). Using the Riesz operators

$$P=\frac{1}{2}(I+S),$$

where I is the identity operator and

$$(Sx)(t)=\frac{1}{\pi i}\int_\Gamma\frac{x(\tau)}{\tau-t}d\tau$$

is the singular operator (with Cauchy kernel), we rewrite the system (3.1) in the following form:

$$\begin{aligned}(Mx)\equiv&\sum_{r=0}^{q}\Big[A_r(t)(Px^{(r)}(t)+B_r(t)(Qx^{(r)}(t)+\frac{1}{2\pi i}\int_\Gamma K_r(t,\tau)\cdot x^{(r)}(\tau)d\tau\Big]\\ =&f(t),\quad t\in\Gamma,\end{aligned}\tag{3.3}$$

where $A_r(t)=\tilde{A}_r(t)+\tilde{B}_r(t)$, $B_r(t)=\tilde{A}_r(t)-\tilde{B}_r(t)$, $r=0,\cdots,q$, are $m\times m$ m.f. The elements belong to $H_\beta(\Gamma)$.

The vector functions $d^q(Px)(t)/dt^q$ and $d^q(Qx)(t)/dt^q$ can be represented by integrals of Cauchy type with the same density $v(t)$:

$$\left.\begin{aligned}\frac{d^q(Px)(t)}{dt^q}&=\frac{1}{2\pi i}\int_\Gamma\frac{v(\tau)}{\tau-t}d\tau,\quad t\in F^+,\\ \frac{d^q(Qx)(t)}{dt^q}&=\frac{t^{-q}}{2\pi i}\int_\Gamma\frac{v(\tau)}{\tau-t}d\tau,\quad t\in F^-.\end{aligned}\right\}\tag{3.4}$$

Using the integral representation (3.4) we reduce the problem (3.1)-(3.2) to an equivalent system of SIE (in terms of solvability):

$$(\Upsilon v\equiv)C(t)v(t)+\frac{D(t)}{\pi i}\int_\Gamma\frac{v(\tau)}{\tau-t}d\tau+\frac{1}{2\pi i}\int_\Gamma h(t,\tau)v(\tau)d\tau=f(t),\quad t\in\Gamma, \tag{3.5}$$

for unknowns $v(t)$ where

$$C(t)=\frac{1}{2}[A_q(t)+t^{-q}B_q(t)],\quad D(t)=\frac{1}{2}[A_q(t)-t^{-q}B_q(t)], \tag{3.6a}$$

$$\begin{aligned}h(t,\tau)=&\frac{1}{2}[K_q(t,\tau)+K_q(t,\tau)\tau^{-n}]-\frac{1}{2\pi i}\int_\Gamma[K_q(t,\bar{t})-K_q(t,\bar{t})\bar{t}^{-n}]\frac{d\bar{t}}{\bar{t}-\tau}\\&+\sum_{j=0}^{q-1}\Big[A_j(t)\tilde{M}_j(t,\tau)+\int_\Gamma K_j(t,\bar{t})\tilde{M}_j(\bar{t},\tau)d\bar{t}\Big]\\&-\sum_{j=0}^{q-1}\Big[B_j(t)\tilde{N}_j(t,\tau)+\int_\Gamma K_j(t,\bar{t})\tilde{N}_j(\bar{t},\tau)d\bar{t}\Big],\end{aligned} \tag{3.6b}$$

where $\tilde{M}_j(t,\tau)$, $\tilde{N}_j(t,\tau)$, $j=0,\cdots,q$ are known Hölder continuous matrix functions. An explicit form for these functions is given in [18]. By virtue of the properties of the m.f. $\tilde{M}_j(t,\tau)$, $\tilde{N}_j(t,\tau)$, $K_j(t,\tau)$, $A_j(t)$, $B_j(t)$, $j=0,\cdots,q$, we obtain that the m.f. $h(t,\tau)$ is a Hölder continuous function. Note that the right hand sides in (3.5) and (3.1) coincide by conditions (3.2).

Lemma 3.1. *The system of SIE* (3.5) *and problem* (3.1)-(3.2) *are equivalent in terms of solvability. That is, for each solution $v(t)$ of system of SIE* (3.5)*, there is a solution of problem* (3.1)-(3.2)*, determined by the formulae*

$$(Px)(t)=\frac{(-1)^q}{2\pi i(q-1)!}\int_\Gamma v(\tau)\Big[(\tau-t)^{q-1}\log\Big(1-\frac{t}{\tau}\Big)+\sum_{k=1}^{q-1}\alpha_k\tau^{q-k-1}t^k\Big]d\tau, \tag{3.7a}$$

$$(Qx)(t)=\frac{(-1)^q}{2\pi i(q-1)!}\int_\Gamma v(\tau)\tau^{-q}\Big[(\tau-t)^{q-1}\log\Big(1-\frac{\tau}{t}\Big)+\sum_{k=1}^{q-2}\beta_k\tau^{q-k-1}t^k\Big]d\tau, \tag{3.7b}$$

where α_k, $k=1,\cdots,q-1$, and β_k, $k=1,\cdots,q-2$ are vector numbers. On the other hand, for each solution $x(t)$ of the problem (3.1)-(3.2) *there is a solution $v(t)$*

$$v(t)=\frac{d^q(Px)(t)}{dt^q}+t^q\frac{d^q(Qx)(t)}{dt^q}$$

of system of SIE (3.5)*. Furthermore, for linearly independent solutions of* (3.5)*, there are corresponding linearly independent solutions of problem* (3.1)-(3.2) *from* (3.7b) *and vice versa.*

In formula (3.7b), for both $\log(1-t/\tau)$ and $\log(1-\tau/t)$, and a given τ, there are branches that vanish at $t=0$ and $t=\infty$, respectively.

The following theorems are the theoretical justification for our collocation method with mechanical quadrature rules for a system of SIE [20,21].

Theorem 3.1. *Assume that the following conditions are true:*

a) The matrix functions $a(t), b(t), h(t,\tau)$ are in $[H_\alpha^{(r)}(\Gamma)]_m$ for both of the variables, and vector functions $f(t)$ is in $[H_\alpha^{(r)}(\Gamma)]_m$, $r\geq 0,1,\cdots$, $\alpha\in(0,1]$, where $\Gamma\in\Lambda$, $t\in\Gamma$;

b) $Det(a(t)) \neq 0$, $Det(b(t)) \neq 0$;

c) *left partial indices of* $a(t) * b^{-1}(t)$ *are equal to zero;*

d) *operator* $\Upsilon = aP + bQ + H$ *is invertible in* $[H_\beta(\Gamma)]_m$, $0 < \beta < \alpha < 1$; *where* H *is integral operator with kernel* $h(t,\tau)$, P *and* Q *are Riesz projectors* $P = (I+S)/2$, $Q = (I-S)/2$, *and* S *is a singular operator with Cauchy kernel;*

e) *the points* t_j $(j = 0,\cdots,2n)$ *form a system of Fejér nodes [17,28] on* Γ*:*

$$t_j = \psi\Big[\exp\Big(\frac{2\pi i}{2n+1}(j-n)\Big)\Big], \quad j = 0,\cdots,2n;$$

f) $0 < \beta < \alpha < 1$.

Then the operator of the collocation method

$$A_n = U_n[aP + bQ + H]U_n$$

of operator $\Upsilon v = f$ *for numbers* $(n \geq n_0)$ *large enough is invertible in the subspace* $U_n[H_\beta(\Gamma)]_m$ *and* $v_n(t) = \Upsilon_n^{-1}U_n f$ *converges to the function* $v = \Upsilon^{-1}f$. *The following rate of convergence holds:*

$$\|v - v_n\|_\beta^{(m)} \leq \frac{c_1 + c_2\log n}{n^{r+\alpha-\beta}} H(v^{(r)};\alpha), \tag{3.8}$$

where $a(t) = C(t) + D(t)$, $b(t) = C(t) - D(t)$.

Theorem 3.2. *Let the conditions a)-f) of Theorem* 3.1 *be satisfied. Then the operator of the mechanical quadrature rule*

$$B_n = U_n[aP + bQ + \Delta_n]U_n,$$

where

$$\Delta(\cdot) = \frac{1}{2\pi i}\int_\Gamma \frac{1}{\tau} U_n^\tau[\tau \cdot h(t,\tau)](\cdot)(\tau)d\tau$$

for the equation $\Upsilon v = f$ *and for numbers* n $(\geq n_1 \geq n_0)$ *large enough is invertible in* $U_n H_\beta(\Gamma)$. *The approximate solution* $v_n(t) = B_n^{-1}U_n f$ *converges to the function* $v(x) = \Upsilon^{-1}f$. *The following rate for the convergence speed holds:*

$$\|v - v_n\|_\beta^{(m)} \leq \frac{c_3 + c_4\log n + c_5\log^2 n}{n^{r+\alpha-\beta}} H(v^{(r)};\alpha). \tag{3.9}$$

4 New collocation methods with the mechanical quadrature rules

We search for the approximate solution of problem (3.1)-(3.2) in the form

$$x_n(t) = \sum_{k=0}^{n} \xi_k^{(n)} t^{k+q} + \sum_{k=-n}^{-1} \xi_k^{(n)} t^k, \quad t \in \Gamma, \tag{4.1}$$

where $\xi_k^{(n)} = \xi_k = \xi_{k,1}, \xi_{k,2}, \cdots, \xi_{k,m}$, $k = -n, \cdots, n$, are m-dimensional unknowns. Note that the function $x_n(t)$, constructed using the formula (4.1), satisfies condition (3.2).

According to the collocation method, we determine the unknowns ξ_k, $k = -n, \cdots, n$ from the condition

$$(Mx_n)(t_j) - f(t_j) = 0 \tag{4.2}$$

at $2n+1$ different points $t_j \in \Gamma$, $j = 0, \cdots, 2n$. Using the formula in [25]

$$(Px)^{(r)}(t) = (Px^(r))(t), \quad (Qx)^{(r)}(t) = (Qx^{(r)})(t)$$

and the relations

$$(t^{(k+q)})^{(r)} = \frac{(k+q)!}{(k+q-r)!} t^{k+q-r}, \qquad k = 0, \cdots, n, \tag{4.3a}$$

$$(t^{(-k)})^{(r)} = (-1)^r \frac{(k+r-1)!}{(k-1)!} t^{-k-r}, \qquad k = 1, \cdots, n, \tag{4.3b}$$

$$S(\tau^k) = \begin{cases} t^k, & \text{when } k \geq 0, \\ -t^k, & \text{when } k < 0, \end{cases} \tag{4.3c}$$

we obtain a system of linear algebraic equations (SLAE):

$$\begin{aligned}
&\sum_{r=0}^{q} \Big\{ A_r(t_j) \sum_{k=0}^{n} \frac{(k+q)!}{(k+q-r)!} t_j^{k+q-r} \xi_k + B_r(t_j) \sum_{k=1}^{n} (-1)^r \frac{(k+r-1)!}{(k-1)!} t_j^{-k-r} \xi_{-k} \\
&\quad + \frac{1}{2\pi i} \cdot \sum_{k=0}^{n} \frac{(k+q)!}{(k+q-r)!} \int_\Gamma K_r(t_j, \tau) \tau^{k+q-r} d\tau \cdot \xi_k \\
&\quad + \sum_{k=1}^{n} (-1)^r \frac{(k+r-1)!}{(k-1)!} \cdot \frac{1}{2\pi i} \int_\Gamma h_r(t_j, \tau) \tau^{-k-r} d\tau \cdot \xi_{-k} \Big\} \\
=& f(t_j), \quad j = 0, \cdots, 2n.
\end{aligned} \tag{4.4}$$

If the problem (3.1)-(3.2) is solved by the mechanical quadrature method, we replace the integrals by the quadrature rule:

$$\frac{1}{2\pi i} \int_\Gamma g(\tau) \tau_k^{l+k} d\tau \cong \frac{1}{2\pi i} \int_\Gamma U_n[\tau^{l+1} g(\tau)] \tau^{k-1} d\tau, \quad k = 0, \pm 1, \cdots, \pm n, \tag{4.5}$$

where $k=0,\cdots,n$ for $l=0,1,2,\cdots$, and $k=-1,\cdots,-n$ for $l=-1,-2,\cdots$, and U_n the Lagrange interpolation operator defined by (2.2). Since Γ is an arbitrary smooth closed contour, for all integer numbers r the following relation holds:

$$\frac{1}{2\pi i}\int_\Gamma \tau^k d\tau = \begin{cases} 0, & \text{when } r\neq -1, \\ 1, & \text{when } r=-1. \end{cases}$$

Using the previous relation and (4.5) from (4.4) we obtain the following SLAE for the mechanical quadrature method:

$$\begin{aligned}
&\sum_{r=0}^{q}\Big\{A_r(t_j)\sum_{k=0}^{n}\frac{(k+q)!}{(k+q-r)!}t_j^{k+q-r}\xi_k + B_r(t_j)\sum_{k=1}^{n}(-1)^r\frac{(k+r-1)!}{(k-1)!}t_j^{-k-r}\xi_{-k}\\
&\quad+\sum_{k=0}^{n}\frac{(k+q)!}{(k+q-r)!}\sum_{s=0}^{2n}K_r(t_j,t_s)t_s^{1+q-r}\Lambda_{-k}^{(s)}\xi_k\\
&\quad+\sum_{k=1}^{n}(-1)^r\frac{(k+r-1)!}{(k-1)!}\sum_{s=0}^{2n}K_r(t_j,t_s)t_s^{-1-r}\Lambda_k^{(s)}\xi_{-k}\Big\}\\
&=f(t_j),\quad j=0,\cdots,2n,
\end{aligned}\tag{4.6}$$

where t_j, $j=0,\cdots,2n$, are distinct points on Γ. To find the numbers $\Lambda_k^{(s)}$ we use the relation (2.2). We note that the SLAE (4.4) and (4.6) represent the system of m-dimensional equations. It contains $m(2n+1)$ equations and $m(2n+1)$ unknowns.

Let $[\overset{o}{H}{}_\beta^{(q)}(\Gamma)]_m$ be subspace of the space $[H_\beta^{(q)}(\Gamma)]_m$ that consists of functions satisfying condition (3.2). The norm is given as in $[H_\beta^{(q)}(\Gamma)]_m$.

The convergence of the collocation method with the mechanical quadrature rules is given in the following theorems.

Theorem 4.1. *Assume that $\Gamma\in\Lambda$ and the following conditions be satisfied:*

1. the m.f. $A_k(t)$, $B_k(t)$, $h_k(t,\tau)$, $(k=0,\cdots,q)$ and v.f. $f(t)$ belong to the space $[H_\alpha^{(r)}(\Gamma)]_m$; $0<\alpha<1, r\geq 0$;
2. $det(A_q(t))\neq 0$, $det(B_q(t))\neq 0$, $t\in\Gamma$;
3. the left partial indices of m.f. $t^qB_q^{-1}(t)A_q(t)$ are equal to zero;
4. the operator $M:[\overset{o}{H}{}_\beta^{(q)}(\Gamma)]_m\rightarrow[H_\beta(\Gamma)]_m$ is linear and invertible;
5. the points t_j $(j=0,\cdots,2n)$ form a system of Fejér nodes [17, 28] on Γ
$$t_j=\psi\Big[\exp\Big(\frac{2\pi i}{2n+1}(j-n)\Big)\Big],\quad j=0,\cdots,2n;$$
6. $0<\beta<\alpha<1$.

Then, there exists $n \geq n_2$, SLAE (4.4) *has a unique solution ξ_k, $k = -n, \cdots, n$. The approximate solution $x_n(t)$, constructed using formula* (4.1) *converges when $n \to \infty$ in the norm of the space $[H_\beta(\Gamma)]_m$ to the exact solution $x(t)$ of the problem* (3.1)-(3.2)*. The following estimate for the convergence speed holds:*

$$\|x - x_n\|_{\beta,q}^{(m)} = \frac{c_6 + c_7 \log n}{n^{r+\alpha-\beta}} H(x^{(r)}, \alpha). \tag{4.7}$$

Proof. It is obvious that the SLAE(4.4) is equivalent to the following operator equation:

$$U_n M U_n x_n = U_n f, \tag{4.8}$$

where M is an operator given in (3.1). We should show that for numbers $n \geq n_2$ large enough, the operator is invertible. The operator M acts from the subspace

$$[\overset{o}{X}_n]_m = t^q P[X_n]_m + Q[X_n]_m,$$

the norm defined as in $[H_\beta^{(q)}]_m(\Gamma)$, to the space $[X_n]_m = U_n[H_\beta(\Gamma)]_m$ of m-dimensional functions of the form $\sum_{k=-n}^{n} r_k t^k$ (the norm as in $[H_\beta(\Gamma)]_m$, where r_k are arbitrary complex vector numbers.

In the similar way, using the formulae (3.4), the v.f. $d^q(Px_n)(t)/dt^q$ and $d^q(Qx_n)(t)/dt^q$ can be represented by Cauchy type integrals with the same density $v_n(t)$:

$$\left.\begin{aligned} \frac{d^q(Px_n)(t)}{dt^q} &= \frac{1}{2\pi i}\int_\Gamma \frac{v_n(\tau)}{\tau - t} d\tau, \quad t \in F^+, \\ \frac{d^q(Qx_n)(t)}{dt^q} &= \frac{t^{-q}}{2\pi i}\int_\Gamma \frac{v_n(\tau)}{\tau - t} d\tau, \quad t \in F^-. \end{aligned}\right\} \tag{4.9}$$

Using the integral representations (4.9) and the relations

$$(Px)^{(r)}(t) = P(x^{(r)})(t), \quad (Qx)^{(r)}(t) = Q(x^{(r)})(t), \tag{4.10}$$

we obtain

$$v_n(t) = \sum_{k=0}^{n} \frac{(k+q)!}{k!} t^k \xi_k + (-1)^q \sum_{k=1}^{n} \frac{(k+q-1)!}{(k-1)!} t^{-k} \xi_{-l}$$

and so $v_n(t) \in [X_n]_m$.

Using the previous relation (3.4) we reduce the operator equation (4.8) to an equivalent system of equations, in the sense of solvability,

$$U_n Y U_n v_n = U_n f, \tag{4.11}$$

which is treated as an equation in the subspace $[X_n]_m$. Obviously, the system (4.11) is the system of the collocation method for the system of SIE (3.5). Using the conditions of Theorem 4.1 and Lemma 3.1 from (3.4) and $v_n(t) \in [X_n]_m$, we conclude that if the v.f. $v_n(t)$ is the solution of (4.11), then v.f. $y_n(t)$ is the discrete solution of $U_n M U_n x_n = U_n f$ and vice versa. We can determine the discrete solution $y_n(t)$ from (3.7b) by replacing

$y_n(t)$ with $x(t)$. As was mentioned above, the v.f. $y_n(t)$ is determined through $v_n(t)$ from (3.7b) uniquely. It follows that if (4.11) has a unique solution $v_n(t)$ in subspace $[X_n]_m$, then the following relation

$$y_n(t) = x_n(t) \tag{4.12}$$

holds. We should show that all conditions of Theorem 3.1 for the collocation method are satisfied. From the condition 2 of Theorem 4.1 and from (3.6a) we obtain the condition b) of Theorem 3.1. From the equality

$$[C(t) - D(t)]^{-1}[C(t) + D(t)] = t^q B_q^{-1} A_q(t),$$

we conclude that all the left partial indices of m.f. of $[C(t) - D(t)]^{-1}[C(t) + D(t)]$ are equal to zero, which coincides with the condition c) of Theorem 3.1. Other conditions of Theorem 3.1 and Theorem 4.1 are coincide. From $h(t,\tau) \in [H_\alpha(\Gamma)]_m$ we conclude that the exact v.f. $v(t) \in [H_\alpha(\Gamma)]_m$. Conditions of Theorem 4.1 provide the validity of all conditions of Theorem 3.1. Therefore when $n \geq n_2$, (4.11) is uniquely solvable. The approximate solutions $v_n(t)$ of (4.11) converge to the exact solution $v(t)$ of the system of SIE (3.5) in the norm of the space $[H_\beta(\Gamma)]_m$ as $n \to \infty$. Therefore the operator equation (4.11) has a unique solution for $n \geq n_2$. From Theorem 3.1, we know that the relation (3.8) holds. From (3.7b), (4.12) and from the definition of the norm in Höder spaces we obtain

$$\|x - x_n\|_{\beta,q}^{(m)} \leq c\|v - v_n\|_{\beta}^{(m)}.$$

From the last relation and from (3.8) we have (4.7). $\square$

Theorem 4.2. *Assume that all conditions in the previous theorem are satisfied. Then, for $n \geq n_3$ $(\geq n_2)$, SLAE* (4.6) *has a unique solution α_k, $k = -n, \cdots, n$. The approximate solution $x_n(t)$ constructed by the formula* (4.1)*, converges when $n \to \infty$ according to the norm of $[H_\beta(\Gamma)]_m$ to the exact solution $x(t)$ of problem* (3.1)-(3.2)*. The following estimate for the convergence speed is true*

$$\|x - x_n\|_{\beta,q}^{(m)} \leq \frac{c_8 + c_9 \log n + c_{10} \log^2 n}{n^{r+\alpha-\beta}} H(x^{(r)}; \alpha). \tag{4.13}$$

Proof. Using (4.10) and the quadrature formula (4.5), we can easily verify that SLAE (4.6) is equivalent to the system of equation

$$U_n M_0 U_n x_n + \sum_{r=0}^{q} \frac{1}{2\pi i} \int_\Gamma \frac{1}{\tau} U_n^\tau [\tau \cdot K_r(t,\tau)] x_n^{(r)}(\tau) d\tau = U_n f, \tag{4.14}$$

where M_0 is characteristic part of (3.1). After applying the integral representation (3.4), we obtain that (4.14) is equivalent, in sense of solvability, to the operator equations

$$U_n Y_0 U_n v_n + \frac{1}{2\pi i} \int_\Gamma \frac{1}{\tau} U_n^\tau [\tau \cdot h(t,\tau)] v_n(\tau) d\tau = U_n f, \tag{4.15}$$

where Υ_0 is a characteristic part of the system (3.5) and the m.f. $h(t,\tau)$ is a Hölder m.f. by both variables. Equation (4.15) represents an equation of the mechanical quadrature method for (3.5). It is easy to verify that the conditions of Theorem 4.2 provide the validity of all conditions of Theorem 3.2. The Eq. (4.15) represents the equation of the mechanical quadrature method. Moreover, the approximate solutions $v_n(t) \in [X_n]_m$ converge to the exact solution $v(t)$ of the system of SIE (3.5) in the norm $[H_\beta(\Gamma)]_m$ as $n \to \infty$. The v.f. $x_n(t)$ can be expressed via the v.f. $v_n(t)$ by formula (3.7b). Using the definition of the norm in the space $[H_\beta(\Gamma)]_m$, the relations (4.12) and (3.9), we obtain (4.13). Thus the theorem is proved. □

5 A numerical experiment

Let us consider an example of SIDE (3.1) with $m = 1$. In this example the exact solution is

$$x(t) = \frac{1}{t-1}.$$

The coefficients are chosen as follows:

$$\begin{aligned}
\tilde{A}_r(t) &= \frac{1}{2}\Big(t + \frac{1}{2} - \frac{1}{t}\Big)\Big(\frac{1}{t} + 1\Big), \\
\tilde{B}_r(t) &= \frac{1}{2}\Big(t + \frac{1}{2} - \frac{1}{t}\Big)\Big(\frac{1}{t} - 1\Big), \\
K_r(t,\tau) &= \frac{(t+r+1)}{\tau}, \quad r = 0, \cdots, 1.
\end{aligned}$$

The contour Γ is the Limaçon

$$\tilde{x} = \frac{a}{2} + b\cos(\theta) + \frac{a}{2}\cos(2*\theta), \quad \tilde{y} = b\sin(\theta) + \frac{a}{2}\sin(2*\theta).$$

We take, for example, $a = 2.3$ and $b = 5$.

It is easy to verify that the conditions of Theorem 4.2 are satisfied. To calculate the index of the function $t^q B_q^{-1}(t) A_q(t)$ we use the numerical algorithm from [9]. To construct the Fejér points we build the conformal mapping function (2.1) using the numerical algorithm from [29]. A similar approach for construction of Fejér points can be found in [28]. In our test, the selected points are obtained from the following formula

$$\begin{aligned}
&\tilde{t}_j = \tilde{x}_j + i * \tilde{y}_j, && \tilde{x}_j = \frac{a}{2} + b\cos(\theta_j) + \frac{a}{2}\cos(2*\theta_j), \\
&\tilde{y}_j = b\sin(\theta_j) + \frac{a}{2}\sin(2*\theta_j), && \theta_j = \frac{2\pi*(j-1)}{k} + \frac{\pi}{16},
\end{aligned}$$

where k is an integer and $j = 1, \cdots, k$.

We list a grid refinement analysis result in Table 1. We also observed that we should take enough collocation points to guarantee the convergence.

Table 1: The error between the exact solution $x(\tilde{t}_j)$ and the approximate solution $x_n(\tilde{t}_j)$ at selected points $\tilde{t}_j$, $j = 0, \cdots, k+1$, for different number $2n$ of the collocation points is given in this table. The error is the largest error in the magnitude of all selected points.

nr of collocation points	Error
13	0.0027
17	1.4373e-04
21	2.7587e-05
27	2.7084e-06
35	1.7235e-07

Remark 5.1. Similar numerical results have been obtained when the contour Γ is one of the following:

- Epitrochoid:

$$\begin{aligned} &x = (R+r) * \cos(\varphi) - d * \cos\Big(\frac{(R+r)}{r}\varphi\Big), \\ &y = (R+r) * \sin(\varphi) - d * \sin\Big(\frac{(R+r)}{r}\varphi\Big), \\ &R = 3, \quad d = 0.5, \quad r = 1; \end{aligned}$$

- Ellipse

$$x = R\cos(\varphi), \quad y = r\sin(\varphi), \quad R = 4, \quad r = 3.$$

6 Conclusions

In this paper we have developed the collocation method with the mechanical quadrature rules for solving a system of SIDE. The equations are defined on an arbitrary smooth closed contour. The convergence of these method was proved in classical Hölder spaces. A numerical example illustrates the performance of the collocation with the mechanical quadrature rules.

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