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Lower Bounds for Eigenvalues of the Stokes Operator

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Abstract. In this paper, we propose a condition that can guarantee the lower bound property of the discrete eigenvalue produced by the finite element method for the Stokes operator. We check and prove this condition for four nonconforming methods and one conforming method. Hence they produce eigenvalues which are smaller than their exact counterparts.

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Key words: Lower bound, eigenvalue, Stokes operator.

1 Introduction

We are interested in the lower bound property of the eigenvalue by the (conforming and nonconforming) finite element method for the Stokes operator. We propose a condition that can guarantee theoretically the lower bound property of the discrete eigenvalue for both conforming and nonconforming methods. We check and prove this condition for the nonconforming rotated Q_1 element [20], the enriched nonconforming rotated Q_1 element [16], the Crouzeix-Raviart element [9] and the enriched Crouzeix-Raviart element [11] and the conforming P_2-P_0 element.

The lower bound property of the eigenvalue by nonconforming methods of the Stokes eigenvalue problem was first analyzed in [17]. We here give a new error estimate for eigenvalues and eigenfunctions and slightly different analysis for the lower bound property. For the conforming element, we present the first analysis of the lower bound property of the discrete eigenvalue.

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The analysis herein will use some identity for the error of the eigenvalue. Such type of an identity was first analyzed in a remarkable paper Armentano and Duran [1] for the nonconforming linear element of the Laplace operator. The idea was independently extended to the Wilson element in Zhang et al. [25] and to the enriched nonconforming rotated Q_1 element in Li [14]. In those papers, canonical interpolation operators of these nonconforming elements were performed. For the nonconforming linear element of the Laplace operator, there is some special projection property for the canonical interpolation operator, namely, the interpolation is identical to the Galerkin projection. However, for the general case, one has not such a special projection property for the canonical interpolation operator, see Zhang, Yang et al. [25] and Li [14]. Therefore, that term will not be zero any more. From arguments in [1, 14, 25], it is straightforward to see that a similar identity holds for any function (not necessary the canonical interpolation) in the nonconforming finite element space, we refer interested readers to, Yang et al. [23] and Hu et al. [11] for more details. This idea was extended to the Stokes operator in Lin et al. [17], which will be used in the present paper.

We end this section by introducing necessary notation. We use the standard gradient operator:

$$\nabla r := \Big(\frac{\partial r}{\partial x}, \frac{\partial r}{\partial y}\Big).$$

Given any $2D$ vector function $\psi = (\psi_1, \psi_2)$, its divergence reads $\operatorname{div}\psi := \partial\psi_1/\partial x + \partial\psi_2/\partial y$. The spaces $H_0^1(\Omega)$ and $L_0^2(\Omega)$ are defined as usual,

$$\begin{aligned} &H_0^1(\Omega) := \big\{ v \in H^1(\Omega),\ v = 0 \text{ on } \partial\Omega \big\}, \\ &L_0^2(\Omega) := \Big\{ q \in L^2(\Omega),\ \int_\Omega dx = 0 \Big\}. \end{aligned}$$

Suppose that $\overline{\Omega}$ is covered exactly by a shape-regular triangulation $\mathcal{T}_h$ consisting of triangles in $2D$, see [8]. Let $\mathcal{E}_h$ be the set of all edges in $\mathcal{T}_h$, $\mathcal{E}_h(\Omega)$ the set of interior edges and $\mathcal{E}(K)$ the set of edges of any given element K in $\mathcal{T}_h$; $h_K = |K|^{1/2}$, the size of the element $K \in \mathcal{T}_h$, where $|K|$ is the area of element K. ω_K is the union of elements $K' \in \mathcal{T}_h$ that share an edge with K and ω_E is the union of elements that share a common edge E. Given any edge $E \in \mathcal{E}(\Omega)$ with the length h_E we assign one fixed unit normal $\nu_E := (\nu_1, \nu_2)$ and tangential vector $\tau_E := (-\nu_2, \nu_1)$. For E on the boundary we choose $\nu_E = \nu$ the unit outward normal to Ω. Once ν_E and τ_E have been fixed on E, in relation to ν_E one defines the elements $K_- \in \mathcal{T}_h$ and $K_+ \in \mathcal{T}_h$, with $E = K_+ \cap K_-$. Given $E \in \mathcal{E}(\Omega)$ and some $\mathbb{R}^d$-valued function v defined in Ω, with $d = 1,2$, we denote by $[v] := (v|_{K_+})|_e - (v|_{K_-})|_E$ the jump of v across E where $v|_K$ denote the restriction of v on K.

The paper is organized as follows. In the following section, we shall present the Stokes eigenvalue problem and its finite element methods in an abstract setting. In Section 3, based on two conditions on the discrete spaces, we analyze error estimates for both discrete eigenvalues and eigenfunctions. In Section 4, under one more condition, we prove an abstract result that eigenvalues produced by finite element methods are smaller than exact ones. In Sections 5 and 6, we check these conditions for four nonconforming element methods and one conforming methods. In the last section, we present some numerical example for the P_2-P_0 element of the Stokes eigenvalue problem.

2 The Stokes eigenvalue problem and FEMs

The Stokes eigenvalue problem is defined as follows: find $(\lambda,u,p)\in\mathbb{R}\times V\times Q:=\mathbb{R}\times(H_0^1(\Omega))^2\times L_0^2(\Omega)$ such that

$$a(u,v)+b(v,p)+b(u,q)=\lambda(u,v)_{L^2(\Omega)}, \tag{2.1a}$$
$$\|u\|_{L^2(\Omega)}=1 \text{ for any } (v,q)\in V\times Q, \tag{2.1b}$$

where the bilinear forms $a(u,v)$ and $b(v,q)$ are defined as, respectively,

$$a(u,v):=(\nabla u,\nabla v)_{L^2(\Omega)} \quad \text{and} \quad b(v,q):=-(\operatorname{div} v,q)_{L^2(\Omega)}. \tag{2.2}$$

The kernel space of the divergence operator reads

$$V_0:=\big\{v\in V,\ b(v,q)=0 \text{ for any } q\in Q\big\}. \tag{2.3}$$

Let (λ,u,p) be the solution of the problem (2.1), we have $u\in V_0$ and

$$a(u,v)=\lambda(u,v)_{L^2(\Omega)} \text{ for any } v\in V_0. \tag{2.4}$$

Then, we have that the eigenvalue problem (2.1) has a sequence of eigenvalues

$$0<\lambda_1\le\lambda_2\le\lambda_3\le\cdots\nearrow+\infty,$$

and corresponding eigenfunctions

$$(u_1,p_1),(u_2,p_2),(u_3,p_3),\cdots, \tag{2.5}$$

which can be chosen to satisfy

$$(u_i,u_j)_{L^2(\Omega)}=\delta_{ij}, \quad i,j=1,2,\cdots. \tag{2.6}$$

We define

$$E_\ell:=\operatorname{span}\{u_1,u_2,\cdots,u_\ell\}. \tag{2.7}$$

Then, eigenvalues and eigenfunctions satisfy the following well-known minimum-maximum principle:

$$\lambda_k=\min_{\dim V_k=k, V_k\subset V_0}\max_{v\in V_k}\frac{a(v,v)}{(v,v)_{L^2(\Omega)}}=\max_{u\in E_k}\frac{a(u,u)}{(u,u)_{L^2(\Omega)}}. \tag{2.8}$$

We shall be interested in approximating the eigenvalue problem (2.1) by finite element methods. Let $Q_h \subset Q$ and V_h be some (conforming and nonconforming) discrete spaces associated to $\mathcal{T}_h$. It is assumed that integral means of jumps of discrete functions vanish:

(A1) For all $v_h \in V_h$ it holds

$$\int_E [v_h] ds = 0 \quad \text{for} \quad E \in \mathcal{E}_h. \tag{2.9}$$

Moreover, let $a_h : (V+V_h) \times (V+V_h) \to \mathbb{R}$ and $b_h : Q \times (V+V_h) \to \mathbb{R}$ be some extensions of a and b in the sense that $a_h|_{V\times V} = a$ and $b_h|_{Q\times V} = b$. Furthermore, we let ∇_h and div_h denote the discrete gradient operator and the discrete divergence operator, which are defined in the elementwise way.

The discrete eigenvalue problem reads: find $(\lambda_h, u_h, p_h) \in \mathbb{R} \times V_h \times Q_h$ such that $\|u_h\|_{L^2(\Omega)} = 1$ and

$$a_h(u_h, v_h) + b_h(v_h, p_h) + b_h(u_h, q_h) = \lambda_h (u_h, v_h)_{L^2(\Omega)} \quad \text{for all} \quad (v_h, q_h) \in V_h \times Q_h. \tag{2.10}$$

We define the semi-norm over $V_h + V$ by $\|\cdot\|_h := a_h(\cdot,\cdot)^{1/2}$. It follows from Condition **(A1)** that $\|\cdot\|_h$ is a norm over the discrete velocity space V_h under consideration. Moreover, we assume:

(A2) There exists a (Fortin interpolation) operator $\Pi_F : V \to V_h$ with

$$b_h(v - \Pi_F v, q) = 0 \text{ for all } q \in Q_h \quad \text{and} \quad \|\Pi_F v\|_h \lesssim \|v\|_V \quad \text{for all} \quad v \in V. \tag{2.11}$$

Throughout the paper, an inequality $A \lesssim B$ replaces $A \leq CB$ with some multiplicative mesh-size independent constant $C > 0$ that depends only on the domain Ω, the shape (e.g., through the aspect ratio) of elements and possible some norm of eigenfunctions u. Finally, $A \approx B$ abbreviates $A \lesssim B \lesssim A$.

We define the kernel space of the discrete divergence operator by

$$V_{0,h} := \{v_h \in V_h, \ b_h(v_h, q_h) = 0 \quad \text{for any} \quad q_h \in Q_h\}. \tag{2.12}$$

Let (λ_h, u_h, p_h) be the solution of the problem (2.1), we have $u_h \in V_{0,h}$ and

$$a_h(u_h, v_h) = \lambda_h (u_h, v_h)_{L^2(\Omega)} \quad \text{for any} \quad v_h \in V_{0,h}. \tag{2.13}$$

Let $N := \dim V_{0,h}$. Under Conditions **(A1)** and **(A2)**, the discrete problem (2.10) admits a sequence of discrete eigenvalues

$$0 < \lambda_{1,h} \leq \lambda_{2,h} \leq \cdots \leq \lambda_{N,h},$$

and corresponding eigenfunctions

$$(u_{1,h}, p_{1,h}), (u_{2,h}, u_{2,h}), \cdots, (u_{N,h}, p_{N,h}).$$

In the case where (V_h,Q_h) is a conforming approximation in the sense $V_{0,h}\subset V_0$, it immediately follows from the minimum-maximum principle (2.8) that

$$\lambda_k\le\lambda_{k,h},\quad k=1,2,\cdots,N,$$

which indicates that $\lambda_{k,h}$ is an approximation above λ_k.

We define the discrete counterpart of E_ℓ by

$$E_{\ell,h}:=\text{span}\{u_{1,h},u_{2,h},\cdots,u_{\ell,h}\}. \tag{2.14}$$

Then, we have the following discrete minimum-maximum principle:

$$\lambda_{k,h}=\min_{\dim V_{k,h}=k,V_{k,h}\subset V_{0,h}}\max_{v\in V_{k,h}}\frac{a_h(v,v)}{(v,v)_{L^2(\Omega)}}=\max_{u\in E_{k,h}}\frac{a_h(u,u)}{(u,u)_{L^2(\Omega)}}. \tag{2.15}$$

3 Error estimates of eigenvalues and eigenfunctions

In this section, we shall analyze errors of discrete eigenvalues and eigenfunctions by nonconforming methods. For simplicity of presentation, we only consider the case where λ_ℓ is an eigenvalue of multiplicity 1. We follow the idea of [11] and give abstract error estimates, which will be specified for a fixed discrete method.

In order to analyze the error, we define the quasi-Ritz-projection $P_h'u_\ell\in V_{0,h}$ by, for an eigenfunction $u_\ell\in V$,

$$a_h(P_h'u_\ell,v_h)=\lambda_\ell(u_\ell,v_h)_{L^2(\Omega)}\ \text{ for any }\ v_h\in V_{0,h}. \tag{3.1}$$

Under conditions **(A1)** and **(A2)**, the Strang Lemma for nonconforming finite element methods [5,8,21] and the mixed finite element theory [6], prove

Lemma 3.1. *Suppose* $(\lambda_\ell,u_\ell,p_\ell)$ *be the solution of problem* (2.1) *and define the stress* $\sigma_\ell=\nabla u_\ell+p_\ell\text{id}$ *with the identity matrix* id. *It holds that*

$$\begin{aligned}\|u_\ell-P_h'u_\ell\|_h\lesssim&\inf_{v_h\in V_{0,h}}\|u_\ell-v_h\|_h+\inf_{q_h\in Q_h}\|p_\ell-q_h\|_{L^2(\Omega)}\\&+\sup_{v_h\in V_{0,h}}\frac{(\sigma_\ell,\nabla_h v_h)_{L^2(\Omega)}-\lambda_\ell(u_\ell,v_h)_{L^2(\Omega)}}{\|v_h\|_h}.\end{aligned} \tag{3.2}$$

To get the error estimate in the L^2 norm, we need the following dual problem: find $(w_d,r_d)\in V\times Q$ such that

$$a(w_d,v)+b(v,r_d)+b(w_d,q)=(u_\ell-P_h'u_\ell,v)_{L^2(\Omega)}\ \text{ for any }\ (v,q)\in V\times Q. \tag{3.3}$$

Then we have the following decomposition:

$$\begin{aligned}\|u_\ell-P_h'u_\ell\|^2_{L^2(\Omega)}=&\|u_\ell-P_h'u_\ell\|^2_{L^2(\Omega)}-a_h(w_d,u_\ell-P_h'u_\ell)-b_h(u_\ell-P_h'u_\ell,r_d)\\&+a_h(w_d,u_\ell-P_h'u_\ell)+b_h(u_\ell-P_h'u_\ell,r_d).\end{aligned} \tag{3.4}$$

The first term on the right-hand side of (3.4) is a consistency error, which can be expressed as

$$\begin{aligned}&\|u_\ell - P_h' u_\ell\|^2_{L^2(\Omega)} - a_h(w_d, u_\ell - P_h' u_\ell) - b_h(u_\ell - P_h' u_\ell, r_d)\\ =&\|u_\ell - P_h' u_\ell\|^2_{L^2(\Omega)} - (\sigma_d, \nabla_h(u_\ell - P_h' u_\ell))_{L^2(\Omega)},\end{aligned}$$

where $\sigma_d = \nabla w_d + r_d \,\mathrm{id}$. Since $b(w_d, q) = 0$ for any $q \in Q$, it follows from (3.1) that the second term on the right-hand side of (3.4) can be rewritten as, for any $v_h \in V_{0,h}$ and $q_h \in Q_h$,

$$\begin{aligned}a_h(w_d, u_\ell - P_h' u_\ell) =& a_h(w_d - v_h, u_\ell - P_h' u_\ell) - b_h(v_h - w_d, p_\ell - q_h)\\ &+ a_h(u_\ell, v_h - w_d) + b_h(v_h - w_d, p_\ell) - \lambda_\ell(u_\ell, v_h - w_d)_{L^2(\Omega)}.\end{aligned}$$

For the third term on the right–hand side of (3.4), it holds that

$$b_h(u_\ell - P_h' u_\ell, r_d) = b_h(u_\ell - P_h' u_\ell, r_d - s_h) \quad \text{for any} \quad s_h \in Q_h.$$

Let $\Pi_h^G : V_0 \to V_{0,h}$ be defined by

$$a_h(\Pi_h^G w, v_h) = a_h(w, v_h) \quad \text{for any} \quad V_{0,h}.$$

A summation of these identities, together with the Cauchy-Schwarz inequality, proves that

Lemma 3.2. *Suppose* $(\lambda_\ell, u_\ell, p_\ell)$ *be the solution of problem* (2.1). *It holds that*

$$\begin{aligned}&\|u_\ell - P_h' u_\ell\|^2_{L^2(\Omega)}\\ \lesssim&\big(\|w_d - \Pi_h^G w_d\|_h + \inf_{q_h \in Q_h} \|r_d - q_h\|_{L^2(\Omega)}\big)\big(\|u_\ell - P_h' u_\ell\|_h + \inf_{q_h \in Q_h} \|p_\ell - q_h\|_{L^2(\Omega)}\big)\\ &+ \big|(u_\ell - P_h' u_\ell, u_\ell - P_h' u_\ell)_{L^2(\Omega)} - (\sigma_d, \nabla_h(u_\ell - P_h' u_\ell))_{L^2(\Omega)}\big|\\ &+ \big|(\sigma_\ell, \nabla_h(w_d - \Pi_h^G w_d))_{L^2(\Omega)} - \lambda_\ell(u_\ell, w_d - \Pi_h^G w_d)_{L^2(\Omega)}\big|.\end{aligned} \tag{3.5}$$

In the sequel, we shall use $P_h' u_\ell \in V_{0,h}$ to estimate the L^2 norm of the error $u_\ell - u_{\ell,h}$. We have the following decomposition:

$$P_h' u_\ell = \sum_{j=1}^N (P_h' u_\ell, u_{j,h}) u_{j,h}. \tag{3.6}$$

For the projection operator P_h', we have the following important property

$$(\lambda_{j,h} - \lambda_\ell)(P_h' u_\ell, u_{j,h})_{L^2(\Omega)} = \lambda_\ell\big((u_\ell - P_h' u_\ell), u_{j,h}\big)_{L^2(\Omega)}. \tag{3.7}$$

In fact, we have

$$\lambda_{j,h}(P_h' u_\ell, u_{j,h})_{L^2(\Omega)} = a_h(u_{j,h}, P_h' u_\ell) = \lambda_\ell(u_\ell, u_{j,h})_{L^2(\Omega)}. \tag{3.8}$$

Suppose that $\lambda_\ell \neq \lambda_j$ if $\ell \neq j$. Then there exists a separation constant d_ℓ with

$$\frac{\lambda_\ell}{|\lambda_{j,h}-\lambda_\ell|} \leq d_\ell \quad \text{for any} \quad j \neq \ell, \tag{3.9}$$

provided that the meshsize h is small enough.

Theorem 3.1. *Let u_ℓ and $u_{\ell,h}$ be eigenfunctions of* (2.1) *and* (2.10), *respectively. Suppose that* **(A1)** *and* **(A2)** *hold. Then,*

$$\|(u_\ell - u_{\ell,h})\|_{L^2(\Omega)} \leq 2(1+d_\ell)\|(u_\ell - P_h' u_\ell)\|_{L^2(\Omega)}. \tag{3.10}$$

Proof. This lemma can be proved by following the same line of Theorem 3.2 in [11]. For readers' convenience, we give details. We denote the key coefficient $(P_h' u_\ell, u_{\ell,h})_{L^2(\Omega)}$ by β_ℓ. The rest can be bounded as follows:

$$\begin{aligned}\|(P_h' u_\ell - \beta_\ell u_{\ell,h})\|^2_{L^2(\Omega)} &= \sum_{j\neq\ell}(P_h' u_\ell, u_{j,h})^2_{L^2(\Omega)} \leq d_\ell^2 \sum_{j\neq\ell}((u_\ell - P_h' u_\ell), u_{j,h})^2_{L^2(\Omega)} \\ &\leq d_\ell^2 \|(u_\ell - P_h' u_\ell)\|^2_{L^2(\Omega)}. \end{aligned} \tag{3.11}$$

This leads to

$$\begin{aligned}\|(u_\ell - \beta_\ell u_{\ell,h})\|_{L^2(\Omega)} &\leq \|(u_\ell - P_h' u_\ell)\|_{L^2(\Omega)} + \|(P_h' u_\ell - \beta_\ell u_{\ell,h})\|_{L^2(\Omega)} \\ &\leq (1+d_\ell)\|(u_\ell - P_h' u_\ell)\|_{L^2(\Omega)}, \end{aligned} \tag{3.12a}$$

$$\begin{aligned}\|u_\ell\|_{L^2(\Omega)} - \|(u_\ell - \beta_\ell u_{\ell,h})\|_{L^2(\Omega)} &\leq \|\beta_\ell u_{\ell,h}\|_{L^2(\Omega)} \\ &\leq \|u_\ell\|_{L^2(\Omega)} + \|(u_\ell - \beta_\ell u_{\ell,h})\|_{L^2(\Omega)}. \end{aligned} \tag{3.12b}$$

Since both u_ℓ and $u_{\ell,h}$ are unit vectors, we can choose them such that $\beta_\ell \geq 0$. Hence we have $|\beta_\ell - 1| \leq \|(u_\ell - \beta_\ell u_{\ell,h})\|_{L^2(\Omega)}$. Thus, we obtain

$$\begin{aligned}\|(u_\ell - u_{\ell,h})\|_{L^2(\Omega)} &\leq \|(u_\ell - \beta_\ell u_{\ell,h})\|_{L^2(\Omega)} + |\beta_\ell - 1|\|u_{\ell,h}\|_{L^2(\Omega)} \\ &\leq 2\|(u_\ell - \beta_\ell u_{\ell,h})\|_{L^2(\Omega)} \leq 2(1+d_\ell)\|(u_\ell - P_h' u_\ell)\|_{L^2(\Omega)}. \end{aligned} \tag{3.13}$$

This completes the proof. □

To analyze the error of the eigenvalue, we define $(\tilde{u}_{\ell,h}, \tilde{p}_{\ell,h}) \in V \times Q$ by

$$a(\tilde{u}_{\ell,h}, v) + b(\tilde{u}_{\ell,h}, q) + b(v, \tilde{p}_{\ell,h}) = \lambda_{\ell,h}(u_{\ell,h}, v)_{L^2(\Omega)} \quad \text{for any} \quad (v,q) \in V \times Q. \tag{3.14}$$

Since $(u_{\ell,h}, p_{\ell,h})$ is the finite element approximation of $(\tilde{u}_{\ell,h}, \tilde{p}_{\ell,h}) \in V \times Q$, a similar argument of (3.2) and (3.5) proves

Lemma 3.3. *Let the stress* $\tilde{\sigma}_{\ell,h}=\nabla\tilde{u}_{\ell,h}+\tilde{p}_{\ell,h}\mathrm{id}$. *It holds that*

$$\begin{aligned}&\|\tilde{u}_{\ell,h}-u_{\ell,h}\|_h+\|\tilde{p}_{\ell,h}-p_{\ell,h}\|_{L^2(\Omega)}\\ \lesssim&\inf_{v_h\in V_{0,h}}\|\tilde{u}_{\ell,h}-v_h\|_h+\inf_{q_h\in Q_h}\|\tilde{p}_{\ell,h}-q_h\|_{L^2(\Omega)}\\ &+\sup_{v_h\in V_{0,h}}\frac{(\tilde{\sigma}_{\ell,h},\nabla_h v_h)_{L^2(\Omega)}-\lambda_{\ell,h}(u_{\ell,h},v_h)_{L^2(\Omega)}}{\|v_h\|_h}.\end{aligned}\tag{3.15}$$

In order to estimate the L^2 error, let $(\tilde{w}_d,\tilde{r}_d)$ be the solution of the following dual problem: find $(\tilde{w}_d,\tilde{r}_d)\in V\times Q$ such that

$$a(\tilde{w}_d,v)+b(v,\tilde{r}_d)+b(\tilde{w}_d,q)=(\tilde{u}_{\ell,h}-u_{\ell,h},v)_{L^2(\Omega)}\quad\text{for any}\quad(v,q)\in V\times Q.\tag{3.16}$$

Lemma 3.4. *Let* $\tilde{\sigma}_d=\mu\nabla\tilde{w}_d+\tilde{r}_d\mathrm{id}$. *It holds that*

$$\begin{aligned}&\|\tilde{u}_{\ell,h}-u_{\ell,h}\|^2_{L^2(\Omega)}\\ \lesssim&\big(\|\tilde{u}_{\ell,h}-u_{\ell,h}\|_h+\|\tilde{p}_{\ell,h}-p_{\ell,h}\|_{L^2(\Omega)}\big)\big(\|\tilde{w}_d-\Pi_h^G\tilde{w}_d\|_h+\inf_{q_h\in Q_h}\|\tilde{r}_d-q_h\|_{L^2(\Omega)}\big)\\ &+\big|(\tilde{u}_{\ell,h}-u_{\ell,h},\tilde{u}_{\ell,h}-u_{\ell,h})_{L^2(\Omega)}-(\tilde{\sigma}_d,\nabla_h(\tilde{u}_{\ell,h}-u_{\ell,h}))_{L^2(\Omega)}\big|\\ &+\big|(\tilde{\sigma}_{\ell,h},\nabla_h(\tilde{w}_d-\Pi_h^G\tilde{w}_d))_{L^2(\Omega)}-\lambda_{\ell,h}(u_{\ell,h},\tilde{w}_d-\Pi_h^G\tilde{w}_d)_{L^2(\Omega)}\big|.\end{aligned}\tag{3.17}$$

Proof. A similar argument of (3.5) shows the desired result. □

Theorem 3.2. *It holds that*

$$|\lambda_{\ell,h}-\lambda_\ell|\lesssim\|\tilde{u}_{\ell,h}-u_{\ell,h}\|_{L^2(\Omega)}.\tag{3.18}$$

Proof. It follows from (2.1) and (3.14) that

$$\begin{aligned}&((\tilde{u}_{\ell,h}-u_{\ell,h}),u_\ell)_{L^2(\Omega)}\\ =&\lambda_\ell^{-1}\lambda_{\ell,h}(u_{\ell,h},u_\ell)_{L^2(\Omega)}-(u_{\ell,h},u_\ell)_{L^2(\Omega)}\\ =&\frac{(\lambda_{\ell,h}-\lambda_\ell)(u_{\ell,h},u_\ell)_{L^2(\Omega)}}{\lambda_\ell}.\end{aligned}$$

Thus we have

$$\lambda_{\ell,h}-\lambda_\ell=\frac{\lambda_\ell((\tilde{u}_{\ell,h}-u_{\ell,h}),u_\ell)_{L^2(\Omega)}}{(u_{\ell,h},u_\ell)_{L^2(\Omega)}}.\tag{3.19}$$

It follows from (3.10) that there exists some positive constant C such that

$$C\le(u_{\ell,h},u_\ell)_{L^2(\Omega)}.$$

This completes the proof. □

Theorem 3.3. *It holds that*

$$\begin{aligned}\|u_\ell-u_{\ell,h}\|_h\lesssim &\inf_{v_h\in V_{0,h}}\|u_\ell-v_h\|_h+\inf_{q_h\in Q_h}\|p_\ell-q_h\|_{L^2(\Omega)}\\&+\sup_{v_h\in V_{0,h}}\frac{(\sigma_\ell,\nabla_h v_h)_{L^2(\Omega)}-\lambda_\ell(u_\ell,v_h)_{L^2(\Omega)}}{\|v_h\|_h}\\&+\|(u_\ell-u_{\ell,h})\|_{L^2(\Omega)}+|\lambda_{\ell,h}-\lambda_\ell|^{\frac{1}{2}}.\end{aligned}\tag{3.20}$$

Proof. We can use the following formulation:

$$\begin{aligned}&a_h(u_\ell-u_{\ell,h},u_\ell-u_{\ell,h})\\=&a(u_\ell,u_\ell)+a_h(u_{\ell,h},u_{\ell,h})-2a_h(u_\ell,u_{\ell,h})\\=&\lambda_\ell\|(u_\ell-u_{\ell,h})\|^2_{L^2(\Omega)}+\lambda_{\ell,h}-\lambda_\ell+2b_h(u_{\ell,h}-u_\ell,p_\ell-q_h)\\&+2\lambda_\ell(u_\ell,u_{\ell,h}-u_\ell)-2b_h(u_{\ell,h}-u_\ell,p_\ell)-2a_h(u_\ell,u_{\ell,h}-u_\ell),\end{aligned}\tag{3.21}$$

for any $q_h\in Q_h$. Then the desired result follows. □

Under Condition **(A2)**, the discrete inf-sup condition holds uniformly, see [6], namely,

$$\|q_h\|_{L^2(\Omega)}\lesssim\sup_{v_h\in V_h}\frac{b_h(v_h,q_h)}{\|v_h\|_h}\quad\text{for any}\quad q_h\in Q_h.\tag{3.22}$$

Then the mixed theory of [6] states that

$$\inf_{v_h\in V_{0,h}}\|w-v_h\|_h\lesssim\inf_{v_h\in V_h}\|w-v_h\|_h\quad\text{for any}\quad w\in V_0,\tag{3.23}$$

such an inequality is frequently used in the error estimate. Finally, it follows from the discrete inf-sup condition that

Theorem 3.4. *It holds that*

$$\begin{aligned}\|p_\ell-p_{\ell,h}\|_{L^2(\Omega)}\lesssim&\inf_{q_h\in Q_h}\|p_\ell-q_h\|_{L^2(\Omega)}+\sup_{v_h\in V_h}\frac{(\sigma_\ell,\nabla_h v_h)_{L^2(\Omega)}-\lambda_\ell(u_\ell,v_h)_{L^2(\Omega)}}{\|v_h\|_h}\\&+|\lambda_{\ell,h}-\lambda_\ell|+\|u_\ell-u_{\ell,h}\|_{L^2(\Omega)}+\|u_\ell-u_{\ell,h}\|_h,\end{aligned}\tag{3.24}$$

provided that $\|v_h\|_{L^2(\Omega)}\lesssim\|v_h\|_h$ *for any* $0\neq v_h\in V_h$.

4 Lower bounds for eigenvalues: an abstract theory

This section proposes a condition on the finite element method and proves that it is sufficient to guarantee the method to yield lower bounds for eigenvalues of the operators.

Lemma 4.1. *Let* (λ,u,p) *and* (λ_h,u_h,p_h) *be solutions of problems* (2.1) *and* (2.10)*, respectively. For any* $v_h\in V_h$*, we have the following identity:*

$$\lambda-\lambda_h=\|u-u_h\|_h^2-\lambda_h\|(v_h-u_h)\|^2_{L^2(\Omega)}+\lambda_h(\|v_h\|^2_{L^2(\Omega)}-\|u\|^2_{L^2(\Omega)})\\ +2a_h(u-v_h,u_h)-2b_h(v_h,p_h). \tag{4.1}$$

Proof. Such an identity can be actually established by following the idea of [1,11,14,23–25], see [17] for the detailed proof. □

The sufficient condition that guarantees the lower bound property of the discrete eigenvalue can be expressed as

(A3) Let (u,p) and (u_h,p_h) be eigenfunctions of problems (2.1) and (2.10), respectively. We assume that there exists an interpolation $\Pi_h u\in V_{0,h}$ with the following properties:

$$a_h(u-\Pi_h u,u_h)=0, \tag{4.2a}$$

$$\big|\|u\|^2_{L^2(\Omega)}-\|\Pi_h u\|^2_{L^2(\Omega)}\big|\lesssim h^{2(k+s-1)+\triangle s}, \tag{4.2b}$$

$$\|(\Pi_h u-u)\|^2_{L^2(\Omega)}\lesssim h^{2(k+s-1)+\triangle \mathcal{S}}, \tag{4.2c}$$

when $u\in V_0\cap(H^{k+s}(\Omega))^2$ with $0<s\le 1$, $k\ge 1$ and two constants $0<\triangle s$ and $0<\triangle\mathcal{S}$.

From the abstract error estimate (3.10) we have

$$\|(u-u_h)\|^2_{L^2(\Omega)}\lesssim h^{2(k+s-1)+\triangle S}. \tag{4.3}$$

Hence the triangle inequality and **(A3)** show that the second and third terms on the right-hand side of (4.1) are of higher order than the first term. Finally, the last two terms

$$a_h(u-\Pi_h u,u_h)=b_h(\Pi_h u,p_h)=0.$$

This actually proves the following theorem:

Theorem 4.1. *Let* (λ,u,p) *and* (λ_h,u_h,p_h) *be solutions of problems* (2.1) *and* (2.10)*, respectively. Assume that* $(u,p)\in V\cap\in(H^{k+s}(\Omega))^2\times Q\cap H^{k-1+s}(\Omega)$ *and that* $h^{(k+s-1)}\lesssim\|u-u_h\|_h$ *with* $0<s\le 1$*. If the three assumptions* **(A1)-(A3)** *hold, then*

$$\lambda_h\le\lambda, \tag{4.4}$$

provided that h is small enough.

From the error analysis in the previous section, we can find that the error $\|u-u_h\|_h$ usually consists of three parts: the approximation error of the velocity space V_h, the approximation error of the pressure space Q_h and the consistency error of the velocity space V_h. Note that the convergence rate of $\|(u-\Pi_h u)\|_h$ is only dependent on the approximation property of the velocity space V_h. Hence it follows from the condition **(A3)** and Theorem 4.1 that the lower bound property of the discrete eigenvalue will be guaranteed for the following two cases:

- The local approximation property of the discrete velocity space is better than the global continuity property for nonconforming finite element methods;
- The approximation property of the discrete velocity space is better than the approximation property of the discrete pressure space for both the conforming finite element method and the nonconforming finite element method.

The above remark partially explains the lower bound property of the eigenvalues by the Bernadi-Raugel element, which was first reported in [15].

5 Lower order nonconforming finite elements

In this section, we shall present some nonconforming schemes with Conditions **(A1)-(A3)**. In all methods under consideration, we take Q_h as the piecewise constant space with respect to the triangulation $\mathcal{T}_h$. Furthermore, for all of these spaces V_h, the conditions **(A1)** and **(A2)** follows immediately from their own definitions.

5.1 The nonconforming rotated Q_1 element

This is a rectangular element. Denote by $Q_{RQ}(K)$ the nonconforming rotated Q_1 element space on the element $K\in\mathcal{T}_h$ which reads [20]

$$Q_{RQ}(K):=P_1(K)+\text{span}\{x_1^2-x_2^2\}, \tag{5.1}$$

with the space $P_1(K)$ of polynomials of degree ≤ 1 over the element K. For any $v\in H^1(K)$, we define the following edge functional

$$\mathcal{F}_E(v):=\frac{1}{h_E}\int_E v ds \tag{5.2}$$

with $E\subset\partial K$ and the diameter h_E of the edge E. The nonconforming rotated Q_1 element space V_h is then defined by

$$\begin{aligned} V_h:=\big\{&v\in(L^2(\Omega))^2,\ v|_K\in(Q_{RQ}(K))^2 \text{ for each } K\in\mathcal{T}_h,\ v \text{ continuous with respect} \\ &\text{to } \mathcal{F}_E \text{ for all internal edges } E \text{ and } \mathcal{F}_E(v)=0 \text{ for all } f \text{ on } \partial\Omega\big\}. \end{aligned}$$

For the nonconforming rotated Q_1 element, we define the interpolation operator $\Pi_h:V\to V_h$ by

$$\int_E \Pi_h v ds=\int_E v ds \text{ for any } v\in V,\ E\in\mathcal{E}_h\ . \tag{5.3}$$

Since

$$\int_E (u-\Pi_h u)ds=0$$

for any edge E of K, the Poincare inequality states

Lemma 5.1. *(see* [1]*) It holds that*

$$\|u-\Pi_h u\|_{L^2(K)} \lesssim h^{1+s}|u|_{H^{1+s}(K)} \tag{5.4}$$

for any $u\in(H^{1+s}(K))^2$ *with* $0<s<1$ *and* $K\in\mathcal{T}_h$.

Lemma 5.2. *For the nonconforming rotated* Q_1 *element, it holds the condition* **(A3)** *when* $u\in V\cap(H^{1+s}(\Omega))^2$ *with* $0<s<1$.

Proof. Since $\triangle_h v_h=0$ with the operator $\triangle_h$ defined elementwise, we use the integration by parts to prove that $a_h(u-\Pi_h u,v_h)=0$ for any $v_h\in V_h$. Furthermore, $\Pi_h u\in V_{0,h}$ since $u\in V_0$. Then, the desired result follows immediately from Lemma 5.1. ☐

In the case the eigenfunction is singular in the sense that $(u,p)\in V\cap(H^{1+s}(\Omega))^2\times Q\cap H^s(\Omega)$ with $0<s<1$, it is proved in [11] that $h^s\lesssim\|u-u_h\|_h$. Therefore, we have that the result in Theorem 4.1 holds for this class of elements.

Remark 5.1. The extension of the analysis and results herein to the Crouzeix-Raviart element [9] is straightforward.

5.2 The enriched nonconforming rotated Q_1 element

This is also a rectangular element. Denote by $Q_{EQ}(K)$ the enriched nonconforming rotated Q_1 element space defined by [16]

$$Q_{EQ}(K):=Q_{RQ}(K)+\text{span}\{x_1^2+x_2^2\}. \tag{5.5}$$

The enriched nonconforming rotated Q_1 element space V_h is then defined by

$$\begin{aligned}V_h:=\big\{&v\in(L^2(\Omega))^2,\ v|_K\in(Q_{EQ}(K))^2 \text{ for each } K\in\mathcal{T}_h, v \text{ continuous with respect}\\ &\text{to } \mathcal{F}_E \text{ for all internal edges } E \text{ and } \mathcal{F}_E(v)=0 \text{ for all } E \text{ on } \partial\Omega\big\}.\end{aligned}$$

For the enriched nonconforming rotated Q_1 element, we define the interpolation operator $\Pi_h: V\to V_h$ by

$$\int_E \Pi_h v ds=\int_E v ds \ \text{ for any } \ v\in V,\ E\in\mathcal{E}_h\ , \tag{5.6a}$$

$$\int_K \Pi_h v dx=\int_K v dx \ \text{ for any } \ K\in\mathcal{T}_h. \tag{5.6b}$$

For this interpolation operator, we have

Lemma 5.3. *(see [14]) It holds that*

$$\|u-\Pi_h u\|_{L^2(K)}\lesssim h^2|u|_{H^2(K)} \ \textit{for any} \ u\in(H^2(K))^2 \ \textit{and} \ K\in\mathcal{T}_h, \tag{5.7a}$$

$$\|u-\Pi_h u\|_{L^2(K)}\lesssim h^{1+s}|u|_{H^{1+s}(K)} \ \textit{for any}\ u\in(H^{1+s}(K))^2 \ \textit{with} \ 0<s<1 \ \textit{and}\ K\in\mathcal{T}_h. \tag{5.7b}$$

Proof. Since $u-\Pi_h u$ has vanishing mean on each element K, it follows from the Poincare inequality that

$$\|u-\Pi_h u\|_{L^2(K)} \lesssim h_K \|\nabla(u-\Pi_h u)\|_{L^2(K)}.$$

Then the desired result follows from the usual interpolation theory and the interpolation space theory for the singular case $u\in(H^{1+s}(K))^2$. □

Lemma 5.4. *For the enriched nonconforming rotated Q_1 element, it holds the condition* **(A3)**.

Proof. First one can prove that $a_h(u-u_h,v_h)=0$ for any $v_h\in V_h$ by following the line of [14]. Second we have that $\Pi_h u\in V_{0,h}$ since $u\in V_0$. Finally It follows from the definition of the interpolation operator Π_h that

$$\begin{aligned}&\|\Pi_h u\|^2_{L^2(\Omega)}-\|u\|^2_{L^2(\Omega)}\\ =&((\Pi_h u-u),\Pi_h u+u)_{L^2(\Omega)}\\ =&((\Pi_h u-u),\Pi_h u+u-\Pi_0(\Pi_h u+u))_{L^2(\Omega)},\end{aligned} \tag{5.8}$$

where Π_0 is the piecewise constant projection operator. This completes the proof of **(A3)** with $k=1$, $\triangle s=1$ and $\triangle S=2s$ provided that $(u,p)\in V\cap(H^{1+s}(\Omega))^2\times Q\cap H^s(\Omega)$ for some $0<s\le 1$. □

It is proved in [11] that $h\lesssim\|u-u_h\|_h$ when $(u,p)\in V\cap(H^2(\Omega))^2\times Q\cap H^1(\Omega)$ and that $h^s\lesssim\|u-u_h\|_h$ when $(u,p)\in V\cap(H^{1+s}(\Omega))\times Q\cap H^s(\Omega)$ with $0<s<1$. Thus, we have that the result in Theorem 4.1 holds for this class of elements.

5.3 The enriched Crouzeix-Raviart element

This is a triangle element. Denote by $Q_{ECR}(K)$ the enriched Crouzeix-Raviart element space defined by [11,17]

$$Q_{ECR}(K):=P_1(K)+\text{span}\{x_1^2+x_2^2\}. \tag{5.9}$$

The enriched Crouzeix-Raviart element space V_h is then defined by

$$\begin{aligned}V_h:=\big\{&v\in(L^2(\Omega))^2,\ v|_K\in(Q_{ECR}(K))^2 \text{ for each } K\in\mathcal{T}_h,\ v \text{ continuous with respect}\\ &\text{to } \mathcal{F}_E \text{ for all internal edges } E \text{ and } \mathcal{F}_E(v)=0 \text{ for all edges } E \text{ on } \partial\Omega\big\}.\end{aligned} \tag{5.10}$$

For the enriched Crouzeix-Raviart element, we define the interpolation operator $\Pi_h: V\to V_h$ by

$$\int_E \Pi_h v ds=\int_E v ds \text{ for any } v\in V \text{ for any edge } E, \tag{5.11a}$$

$$\int_K \Pi_h v dx=\int_K v dx \text{ for any } K\in\mathcal{T}_h. \tag{5.11b}$$

For this interpolation operator, a similar argument of Lemma 5.3 leads to:

Lemma 5.5. *It holds that*

$$\|u-\Pi_h u\|_{L^2(K)} \lesssim h^2 |u|_{H^2(K)} \ \textit{for any } u\in (H^2(K))^2 \textit{ and } K\in\mathcal{T}_h, \tag{5.12a}$$

$$\|u-\Pi_h u\|_{L^2(K)} \lesssim h^{1+s} |u|_{H^{1+s}(K)} \ \textit{for any } u\in (H^{1+s}(K))^2 \textit{ with } 0<s<1 \textit{ and } K\in\mathcal{T}_h. \tag{5.12b}$$

Lemma 5.6. *For the enriched Crouzeix-Raviart element, it holds the condition* **(A3)***.*

Proof. We first prove $a_h(u-\Pi_h u, u_h)=0$. Let $u=(u_1,u_2)$. We only need to consider the first component u_1 since the analysis holds for the second component u_2. We define the space

$$Q_K := \begin{pmatrix} a_{11}+a_{12}x_1 \\ a_{21}+a_{12}x_2 \end{pmatrix}$$

with free parameters a_{11}, a_{21},a_{12}. From the definition of the operator Π_h, we have

$$(\nabla(u_1-\Pi_h u_1),\boldsymbol{\psi})_{L^2(K)}=0 \ \text{ for any } \ \boldsymbol{\psi}\in Q_K. \tag{5.13}$$

Indeed, we integrate by parts to get

$$\begin{aligned}&\big(\nabla(u_1-\Pi_h u_1),\boldsymbol{\psi}\big)_{L^2(K)}\\ =&-(u_1-\Pi_h u_1,\operatorname{div}\boldsymbol{\psi})_{L^2(K)}+\sum_{E\subset\partial K}\int_E (u_1-\Pi_h u_1)\boldsymbol{\psi}\cdot\nu_E ds.\end{aligned}$$

Since $\operatorname{div}\boldsymbol{\psi}$ and $\boldsymbol{\psi}\cdot\nu_E$ (on each edge E) are constant, then (5.13) follows from (5.11). Since $\nabla_h\Pi_h u_1|_K\in Q_K$, the identity (5.13) leads to

$$(\nabla_h\Pi_h u_1)|_K := P_K(\nabla u_1|_K) \tag{5.14}$$

with the L^2 projection operator P_K from $L^2(K)$ onto Q_K. This proves

$$(\nabla_h(u_1-\Pi_h u_1),\nabla_h u_{1,h})=0$$

with $u_{1,h}$ the first component of u_h.

It remains to show the estimate in **(A3)**. Then, it follows from the definition of the interpolation operator Π_h that

$$\begin{aligned}&\|\Pi_h u\|^2_{L^2(\Omega)}-\|u\|^2_{L^2(\Omega)}\\ =&((\Pi_h u-u),\Pi_h u+u)_{L^2(\Omega)}\\ =&((\Pi_h u-u),\Pi_h u+u-\Pi_0(\Pi_h u+u))_{L^2(\Omega)}.\end{aligned} \tag{5.15}$$

This completes the proof of **(A3)** with $k=1$, $\triangle s=1$ and $\triangle S=2s$ provided that $(u,p)\in V\cap(H^{1+s}(\Omega))^2\times Q\cap H^s(\Omega)$ for some $0<s\leq 1$. □

We establish in [11] that $h\lesssim\|u-u_h\|_h$ when $(u,p)\in V\cap(H^2(\Omega))^2\times Q\cap H^1(\Omega)$ and that $h^s\lesssim\|u-u_h\|_h$ when $(u,p)\in V\cap(H^{1+s}(\Omega))\times Q\cap H^s(\Omega)$ with $0<s<1$. This implies that we have that the result in Theorem 4.1 holds for this class of elements.

6 The P_2-P_0 element

This is a triangle element where Q_h is the piecewise constant space and the discrete velocity space reads

$$V_h:=\{v\in V,\, v|_K\in(P_2(K))^2 \text{ for any } K\in\mathcal{T}_h\}, \tag{6.1}$$

where $P_2(K)$ is the space of polynomials of degree ≤ 2 over K. For this element, we have

$$\inf_{q_h\in Q_h}\|p-q_h\|_{L^2(\Omega)}\lesssim h|p|_{H^1(\Omega)} \text{ for any } P\in H^1(\Omega), \tag{6.2}$$

$$\inf_{v_h\in V_h}\|\nabla(u-v_h)\|_{L^2(\Omega)}\lesssim h^{1+\mathfrak{s}}|u|_{H^{2+\mathfrak{s}}(\Omega)} \text{ for any } u\in V\cap H^{2+\mathfrak{s}}(\Omega),\ 0<\mathfrak{s}\leq 1. \tag{6.3}$$

Let (λ,u,p) and (λ_h,u_h,p_h) be solutions of the problems (2.1) and (2.10), respectively. Assume that $(u,p)\in(H^{2+\mathfrak{s}}(\Omega))^2\times H^1(\Omega)$ with $0<\mathfrak{s}\leq 1$. Then, from the error analysis in Section 3, we have

$$\|\nabla(u-u_h)\|_{L^2(\Omega)}+\|p-p_h\|_{L^2(\Omega)}\lesssim h(|u|_{H^2(\Omega)}+|p|_{H^1(\Omega)}). \tag{6.4}$$

Compared to the approximation property of the velocity space in (6.3), only sub-optimal error estimates can be guaranteed theoretically for the velocity.

We have the following saturation condition

Lemma 6.1. *It holds that*

$$h\|\nabla p\|_{L^2(\Omega)}\lesssim\|p-p_h\|_{L^2(\Omega)}, \tag{6.5}$$

provided that h is small enough.

Proof. The result follows from the abstract theory Theorem A.1 in [11] by choosing the canonical interpolation operator of (5.6) as the local interpolation operator of Theorem A.1 in [11], see [13] for more details. □

In the sequel, we shall prove the condition **(A3)** for this element. Let Π_h denote the projection operator from $V_0\rightarrow V_{0,h}$ in the sense that

$$a(\Pi_h u,v_h)=a(u,v_h) \text{ for any } v_h\in V_{0,h}. \tag{6.6}$$

Then we have

$$\begin{aligned}\|\nabla(u-\Pi_h u)\|_{L^2(\Omega)}&\leq\inf_{v_h\in V_{0,h}}\|\nabla(u-v_h)\|_{L^2(\Omega)}\\ &\lesssim\inf_{v_h\in V_h}\|\nabla(u-v_h)\|_{L^2(\Omega)}\lesssim h^{1+\mathfrak{s}}|u|_{H^{2+\mathfrak{s}}(\Omega)}.\end{aligned} \tag{6.7}$$

To estimate the error in the L^2 norm, we need the following dual problem: find $(w_d,p_d)\in V\times Q$ such that

$$a(w_d,v)+b(v,p_d)+b(w_d,q)=(u-\Pi_h u,v) \text{ for any } (v,q)\in V\times q. \tag{6.8}$$

Assume the domain Ω is convex, we have

$$\|w_d\|_{H^2(\Omega)}+\|p_d\|_{H^1(\Omega)}\lesssim\|u-\Pi_h u\|_{L^2(\Omega)}. \tag{6.9}$$

Then we have

$$\begin{aligned}\|u-\Pi_h u\|^2_{L^2(\Omega)}\leq&\|\nabla(w_d-\Pi_h w_d)\|_{L^2(\Omega)}\|\nabla(u-\Pi_h u)\|_{L^2(\Omega)}\\&+\|\nabla(u-\Pi_h u)\|_{L^2(\Omega)}\inf_{q_h\in Q_h}\|p_d-q_h\|_{L^2(\Omega)}.\end{aligned} \tag{6.10}$$

We use the regularity of (w_d,p_d) and the approximation properties of V_h and Q_h to obtain

$$\|u-\Pi_h u\|_{L^2(\Omega)}\lesssim h^{2+\mathfrak{s}}|u|_{H^{2+\mathfrak{s}}(\Omega)}. \tag{6.11}$$

This proves the condition **(A3)** with $k=s=1$, $\triangle s=\mathfrak{s}$ and $\triangle S=2$.

7 Numerical results

In this section, we present some numerical results for the P_2-P_0 element; cf. [17] for the numerical examples for nonconforming elements.

In the example, we take $\Omega=[0,1]^2$ and partition it into uniform triangles by first dividing Ω into $N\times N$ sub-squares and then decomposing each sub-square into two triangles. The first five discrete eigenvalues are listed in Table 1. In the second example, we take $\Omega=[-1,1]^2/[0,1][-1,0]$. The first five eigenvalues are reported in Table 2.

We observe that the discrete eigenvalues converge monotonically from below to the exact ones when the meshsize is small enough.

Table 1: The discrete eigenvalues.

h	1/4	1/8	1/16	1/32	1/64
$\lambda_{1,h}$	52.0198	52.0911	52.2610	52.3216	52.3390
$\lambda_{2,h}$	87.7118	90.9887	91.7959	92.0366	92.1017
$\lambda_{3,h}$	94.5117	91.7991	91.9498	92.0721	92.1105
$\lambda_{4,h}$	128.1250	126.9650	127.6905	128.0574	128.1691
$\lambda_{5,h}$	147.5175	152.7796	153.5726	153.9652	154.0832

Table 2: The discrete eigenvalues.

h	1/4	1/8	1/16	1/32	1/64
$\lambda_{1,h}$	32.5780	31.9251	32.0209	32.0952	32.1209
$\lambda_{2,h}$	33.3472	36.1769	36.7291	36.9286	36.9917
$\lambda_{3,h}$	42.5339	41.7788	41.8092	41.8988	41.9289
$\lambda_{4,h}$	46.3741	48.4381	48.7087	48.8970	48.9595
$\lambda_{5,h}$	51.6742	55.2502	55.1355	55.3184	55.3880

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