Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **5**, No. 1, pp. 78-89

DOI: 10.4208/aamm.11-m11168
February 2013

Convergent Overdetermined-RBF-MLPG for Solving Second Order Elliptic PDEs

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Received 14 November 2011; Accepted (in revised version) 17 June 2012

Available online 25 January 2013

Abstract. This paper deals with the solvability and the convergence of a class of unsymmetric Meshless Local Petrov-Galerkin (MLPG) method with radial basis function (RBF) kernels generated trial spaces. Local weak-form testings are done with step-functions. It is proved that subject to sufficiently many appropriate testings, solvability of the unsymmetric RBF-MLPG resultant systems can be guaranteed. Moreover, an error analysis shows that this numerical approximation converges at the same rate as found in RBF interpolation. Numerical results (in double precision) give good agreement with the provided theory.

AMS subject classifications: 35J25, 65N12, 65D30

Key words: Local integral equations, meshless methods, radial basis functions, overdetermined systems, solvability, convergence.

1 Introduction

In recent years, there is a rapid growth in research of different variants of meshless methods. Generally speaking, meshless methods for solving partial differential equations (PDEs) can be classified into two groups: one uses the strong-form collocation while another group uses the weak formulation for testing the PDEs. The meshless local Petrov-Galerkin method (MLPG), which was first proposed by Atluri and colleagues in 1998 [1, 2], belongs to the latter group. Since then, the MLPG method has been successfully applied to solve a wide range of problems in engineering and science; see also references [3–6] therein. To see some general properties of the unsymmetric meshless

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kernel based methods see [7]. For a brief introduction to the original MLPG method, let us consider a PDE in the form of:

$$Lu = f \text{ in } \Omega, \quad \text{and} \quad u = g \text{ on } \partial\Omega, \tag{1.1}$$

where $\partial\Omega$ denotes the boundary of the bounded domain Ω in $\mathbb{R}^d$. Both given functions f and $g : \mathbb{R}^d \to \mathbb{R}$ are sufficiently smooth. For any set of N scattered nodal points in the domain and on the boundary represented by $\Xi = \{\xi_k\}_{k=1}^N$. The unknown solution u is approximated via

$$u(x) \approx u_N(x) = \sum_{\boldsymbol{\xi}_i \in \Xi} \lambda_i \Phi_i(x), \tag{1.2}$$

where $\Phi_i(x)$, i=1,2,$\cdots$,N are called shape functions constructed on the set of nodal points Ξ and λ_i is the unknown coefficient at node i to be determined. For certain shape functions, e.g., the moving least-squares basis [8–10], we have weights $\lambda_i \approx u(x_i)$ approximating the solution values. To solve for the N unknowns $\lambda_1,\cdots,\lambda_N$, the "local" weak equations constructed on subdomains surrounding each node are as follows:

$$\int_{\Omega_{s^i}} (Lu) v dx = \int_{\Omega_{s^i}} f v dx \Rightarrow \sum_{j=1}^{N} \Big(\int_{\Omega_{s^i}} L\Phi_j(x) v dx \Big) \lambda_j = \int_{\Omega_{s^i}} f v dx. \tag{1.3}$$

This yields N equations for the N unknown coefficients. In (1.3), Ω_{s^i} denotes a (relatively small) subdomain in Ω surrounding the node x_i and v is a locally supported test function. Employing different test functions v results in different kinds of MLPG methods; see [1, 2]. One possible class of the test functions, that is of our interest in this work, is the step functions. In this paper, we study the solvability and convergence of an MLPG method using radial basis functions (RBF) as shape functions [11–13] and step functions as test functions. Some numerical demonstrations are given to show the exponential convergence (under double-precision computations) of the RBF-MLPG method.

2 Sufficient condition for solvability

In the original MLPG method, the sets of test and trial nodes are identical. Such linkage between these two sets of nodes will be decoupled in the RBF-MLPG method due to the requirement for solvability given in this section. Moreover, we use more test equations (denoted by M) than the number of basis in expansion (denoted by N) to yield overdetermined MLPG systems.

We assume that the differential equation (1.1) has an exact solution u^* lying in some infinite dimensional trial spaces $\mathcal{U}$. To obtain a numerical procedure, we first discretize the trial space $\mathcal{U}$ by some finite dimensional subspaces $\mathcal{U}_N$ generated by a set RBF kernel Φ centered at a set of N scattered nodes (or RBF centers) $\Xi_N := \{\boldsymbol{\xi}_i\}_1^N$. Any numerical approximations are of the form

$$u_N(x) = \sum_{\boldsymbol{\xi}_i \in \Xi} \lambda_i \Phi(x, \boldsymbol{\xi}_i) \in \mathcal{U}_N := \Big\{ v : v(x) = \sum_{\boldsymbol{\xi}_i \in \Xi} \lambda_i \Phi(x, \boldsymbol{\xi}_i), \ \lambda \in \mathbb{R}^N \Big\},$$

where $\lambda:=(\lambda_1,\cdots,\lambda_N)^T$ are the unknown coefficients to be determined.

Next, we consider an infinite set of test functions as follows:

$$\mathcal{V}:=\{v\in L^1(\Omega):\|v\|_1=1\}. \tag{2.1}$$

We test the boundary condition in strong-form and so it holds:

$$\int_\Omega vLu=\int_\Omega fv,\quad v\in\mathcal{V},\qquad \text{and}\qquad u(\mathbf{x}_i)=g(\mathbf{x}_i),\quad \mathbf{x}_i\in\partial\Omega.$$

To make the problem numerically accessible, the test space has to be discretized. Consider a large set of M_1 linearly independent test functions $v_i\in\mathcal{V}$ for $i=1,2,\cdots,M_1$. Also consider a large set of M_2 nodal points on the boundaries $\mathbf{x}_i$ for $i=1,2,\cdots,M_2$. We get the following discretized equation:

$$\begin{aligned}&\int_\Omega v_iLud\mathbf{x}=\int_\Omega fv_id\mathbf{x},\quad v_i\in\mathcal{V}, && i=1,2,\cdots,M_1,\\ &u(\mathbf{x}_i)=g(\mathbf{x}_i), && i=1,2,\cdots,M_2.\end{aligned}$$

Setting $M=M_1+M_2$ with $(M\gg N)$, the $M\times N$ overdetermined MLPG system, denoted as $A\lambda=b$, can be solved by appropriate means.

2.1 Example: MLPG for the Poisson equation

Consider the Poisson equation $\Delta u=f$ in $\Omega\subset\mathbb{R}^2$ with Dirichlet boundary condition $u=g$ on $\partial\Omega$. We use the multiquadrics (MQ) kernel $\Phi_c(r)=\sqrt{c^2+r^2}$, where $r\in R$ and $c>0$ is a positive constant, to generate the trial space. The unknown solution u is approximated by a linear combination in the form of

$$u(x)\approx u_N(x)=\sum_{j=1}^N\lambda_j\sqrt{c^2+\|x-\xi_j\|^2},$$

where $\Xi_N=\{\xi_j\}_{j=1}^N$ is an indexed set of the N RBF centers. For discretization on the test side, we consider $M=M_1+M_2$ test nodes $(M\geq N)$, M_1 of them located on the boundary and M_2 on the domain of the problem. Note that these test nodes and trial centers in Ξ may or may not share common points. For test nodes located on the boundaries, $\mathbf{x}^i\in\partial\Omega$, the Dirichlet boundary condition is tested by the Dirac-delta function to give the following:

$$\sum_{j=1}^N\lambda_j\sqrt{c^2+\|\mathbf{x}^i-\xi_j\|^2}=g(\mathbf{x}^i),\quad i=1,2,\cdots,M_1. \tag{2.2}$$

For each node located in the interior, $\mathbf{x}^i\in\Omega$, we define a circular subdomain $\Omega_{s^i}\subset\Omega$ centered at $\mathbf{x}^i$ with radius r_i. Testing is done by the step function defined as

$$v_i(\mathbf{x})=\begin{cases}\dfrac{1}{vol(\Omega_{s^i})}, & \mathbf{x}\in\Omega_{s^i},\\ 0, & \mathbf{x}\notin\Omega_{s^i},\end{cases} \tag{2.3}$$

and we can obtain a set of M_2 test equations

$$\frac{1}{vol(\Omega_{s^i})}\sum_{j=1}^{N}\Big(\int_{\partial\Omega_{s^i}}\frac{\partial\Phi_{c,j}}{\partial n}d\mathbf{s}\Big)=\frac{1}{vol(\Omega_{s^i})}\int_{\Omega_{s^i}}f d\mathbf{x},\quad i=1,2,\cdots,M_2. \tag{2.4}$$

Appropriate numerical integration scheme is necessary for the evaluation of (2.4). Putting the equations in (2.2) and (2.4) together, a matrix system is yielded for $[\lambda_1,\cdots,\lambda_N]^T$.

In most applications, researchers find the original MLPG formulation results in solvable exactly-determined systems. For example, in [14,15], the trial space is formed by the MQ kernel and the testings are performed exactly at the RBF centers. From the following illustration, we can see the importance of choosing *suitable* test functions in applying the solvability theory of the MLPG methods.

2.2 Example: unsymmetric MLPG may yield singular systems

Let us consider a simple differential equation $du/dx=f(x)$ for $x\in[0,4]$ subject to $u(0)=u_0$. Suppose we use the fourier basis to form the trial space

$$\{1,\sin\pi x,\cos\pi x,\sin 2\pi x,\cos 2\pi x,\cdots\}=\{g_1,g_2,\cdots\},$$

to give approximation in the form of $u_h=\sum_{j=1}^{N}\lambda_j g_j$. The local weak-form of equation is given as

$$\int_{\Omega_{s^i}}\frac{du}{dx}dx=\int_{\Omega_{s^i}}f(x)dx,$$

and the MLPG matrix A now contains entries in the form of

$$a_{i,j}=\int_{\Omega_{s^i}}\frac{dg_j}{dx}dx.$$

For any test node x^i that lies in the interval $(1,3)$, if we choose the subdomain Ω_{s^i} for (2.3) with radius 1, the corresponding entries $a_{i,j}$ will be zero for $j=1,2,\cdots,n$ and the MLPG matrix therefore contains zero rows.

To investigate the sufficient conditions on test functions in order to guarantee MLPG solvability, we can modify the proof for strong-form RBF collocation method, i.e., the Kansa method, see [16]. We consider $\Omega\subset R^d$ to be a *bounded* domain and $G=\{g_1,g_2,\cdots,g_n\}$ to be a linearly independent set of n continuous functions defined on Ω. We take a dense sequence X from $\mathcal{V}$ in (2.1) to test the problem. Suppose the MLPG matrix is not of full-rank. There exists $\alpha\neq 0$ such that $\sum_{j=1}^{n}\alpha_j a_j=0$, where a_j is the j-th column of A. Define another function using these weights a_j by

$$f(x)=\sum_{j=1}^{n}\alpha_j g_j(x).$$

If we test f by the same sequence, we will find that

$$\int_\Omega f(x)v_i(x)dx=0$$

for all $v_i\in X$. Due to the denseness of X in $L^1(\Omega)$ and continuity of f, we can conclude that $f(x)=0$ for all $x\in\Omega$. This implies the dependency of the set G that is a contradiction. By the axiom of choice, there must exist N suitable unit-$L^1(\Omega)$ functions in X so that the resulting MLPG matrix is nonsingular. In practice, we cannot determine these suitable test functions *a priori* . Hence, we are motivated to perform a large number of testings (e.g., $M\gg N$) and solve overdetermined MLPG systems instead of exactly-determined (with $M=N$) ones.

3 Convergent RBF-MLPG scheme

For the convergence analysis of the overdetermined RBF-MLPG method, we focus on the symmetric positive definite kernels $\Phi:\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}$. If we take the reproducing kernel Hilbert space associated with Φ (native space) as the trial space $\mathcal{U}=\mathcal{N}_\Phi$, the standard h^β-type error bound for the RBF approximation [17,18] can be used; here, h is the fill distance of nodes and β is the smoothness of the kernel Φ. Also to our interest, the exponential error bounds for the Gaussian and the MQ kernels, see [19], will become handful too.

Let us consider second order elliptic differential equations of the form

$$Lu=a^{ij}(\mathbf{x})D_{ij}u+b^i(\mathbf{x})D_iu+c(\mathbf{x})u,\quad a^{ij}=a^{ji},\tag{3.1}$$

where $\mathbf{x}=(x_1,x_2,\cdots,x_n)\in\Omega\subset R^n$, $n\geq 2$. Suppose that $\lambda>0$ (so that L is elliptic) is the smallest eigenvalue of the coefficient matrix $[a^{ij}]$.

Theorem 3.1 (see [20]). *Let $Lu=f$ in a bounded domain Ω, where L is elliptic and $u\in C^0(\overline{\Omega})\cap C^2(\Omega)$. Suppose that Ω lies between two parallel planes a distant d apart. Put $\beta=\sup\{|b|/\lambda\}$, $C=e^{(\beta+1)d}-1$ and $c^+=\max(c,0)$. Suppose that*

$$C_1=1-C\sup_\Omega\frac{c^+}{\lambda}>0,$$

then

$$\sup_\Omega|u|\leq\frac{1}{C_1}\Big(\sup_{\partial\Omega}|u|+C\sup_\Omega\frac{|f|}{\lambda}\Big).\tag{3.2}$$

If we further assume that L is strictly elliptic, then there exists a constant $\lambda_0>0$ so that $\lambda\geq\lambda_0$. Putting

$$C_2=\max\Big\{\frac{1}{C_1},\frac{C}{\lambda_0}\Big\},$$

we have

$$\sup_\Omega|u|\leq C_2\Big(\sup_{\partial\Omega}|u|+\sup_\Omega|f|\Big).\tag{3.3}$$

To make use of (3.2) and (3.3) in the setting of the MLPG with test function (2.4), we define a norm on $C(\overline{\Omega})$ by

$$\|f\|_{\mathcal{V}} := \sup_{v\in\mathcal{V}} \left| \int_{\Omega} v(x)f(x)dx \right|, \tag{3.4}$$

where $\mathcal{V}$ is defined as in (2.1). Since the test space $\mathcal{V}$ contains the dirac-delta functions, we must have $\|f\|_{\mathcal{V}} \geq \|f\|_{L^\infty(\Omega)}$ (which is true even if the test space $\mathcal{V}$ does not contain the dirac-delta functions). In fact, using the mean value theorem and continuity of f, it is easy to show that the norms $\|f\|_{\mathcal{V}}$ and $\|f\|_{L^\infty(\Omega)}$ are indeed equivalent.

Using (3.4), we define a (continuous) PDE residual norm or Λ-norm on $\mathcal{U}$ as

$$\|u\|_{\Lambda} := \max(\|Lu\|_{\mathcal{V}}, \|u\|_{L^\infty(\partial\Omega)}). \tag{3.5}$$

For the sake of numerical computations, the test space $\mathcal{V}$ must be discretized by a finite dimensional test space $\mathcal{V}_M$. Such discretized space must be dense enough in the sense that

$$\|\cdot\|_{\Lambda} \leq C_0 \|\cdot\|_{\Lambda_M},$$

where $\|\cdot\|_{\Lambda_m}$ is defined similarly by $\|u\|_{\Lambda_M} := \max(\|Lu\|_{\mathcal{V}_M}, \|u\|_{L^\infty(\partial\Omega)})$. In order words, the discrete set of test functions, in the form of (2.4) is fine enough to capture the $L^\infty(\Omega)$-norm of any $C(\overline{\Omega})$ function. We are now ready to formally define our RBF-MLPG approximation to the exact solution u^* to (3.1) as

$$U_{M,\epsilon(N)} := \arg\min_{w\in\mathcal{U}_N} \|w - u^*\|_{\Lambda_M}.$$

We use the notation $\epsilon(N)$ to emphasize the fact that the approximation power of $\mathcal{U}_N$ depends on the nodes distribution of Ξ_N (i.e., fill-distance of these points) instead of N alone. In this setting, let C be a generic constant, we can show

$$\begin{aligned} \|u^* - U_{M,\epsilon(N)}\|_{L^\infty(\Omega)} &\leq C\|u^* - U_{M,\epsilon(N)}\|_{\Lambda} \leq C\|u^* - U_{M,\epsilon(N)}\|_{\Lambda_M} \\ &\leq C\|u^* - s_{u^*,\epsilon}\|_{\Lambda_M} \leq C\|u^* - s_{u^*,\epsilon}\|_{\Lambda} \leq C\epsilon\|u^*\|_{\mathcal{U}}. \end{aligned}$$

Note that the numerical solution $U_{M,\epsilon(N)}$ is replaced by the interpolant $s_{u^*,\epsilon}$ of u^* in $\mathcal{U}$. This is valid due to the minimization property of $U_{M,\epsilon(N)}$ and, most importantly, this trick allows us to make use of the approximation power of the trial space $\mathcal{U}_N$. Hence, we can conclude that the RBF-MLPG procedure will have a convergence rate as fast as that in the RBF interpolation problem.

4 Numerical demonstration

In all examples of this section, we employ the MQ kernel with shape parameter $c = 1$ to form the trial space. To obtain the entries (1.3) in MLPG matrices numerically, some integration schemes are necessary. Here, all local boundary integrals (1.3) are approximated

by the q-point Gauss-Legendre quadrature rule as below:

$$\begin{aligned}\int_{\partial\Omega_{s^i}}\phi_j(\mathbf{x})d\mathbf{s}&=\int_0^{2\pi}\phi_j(x^i+r_0\cos(\theta),y^i+r_0\sin(\theta))r_0d\theta\\&=\pi r_0\int_{-1}^{1}\phi_j(x^i+r_0\cos(\pi\theta+\pi),y^i+r_0\sin(\pi\theta+\pi))d\theta\\&=\pi r_0\sum_{p=1}^{q}w_p\phi_j(x^i+r_0\cos(\pi\theta_p+\pi),y^i+r_0\sin(\pi\theta_p+\pi)),\end{aligned}$$

where w_p and θ_p are the Gauss quadrature weights and points in $[-1,1]$, respectively. Although our theories suggest that the overdetermined MLPG systems should be solved by the linear optimization (an effective solution method can be found in [21]), it is numerically shown in [22] that, for strong-form RBF collocation, the least-squares solution obtained by arbitrary precision computations has the same convergent rate as that for the linear optimization and the least-squares ones have higher accuracy. In this section, all RBF-MLPG approximations are obtained by the more efficient and practical least-squares approach.

Other factors, like the support size, distributions, or numbers of test functions in (2.3), will affect the final accuracy of the RBF-MLPG scheme. Before we demonstrate the exponential convergence suggested in Section 3, we shall perform a few sensitivity analysis to obtain a (rough) idea on suitable parameters-the first example seeks for a sufficiently large value for q used in the Gauss-Legendre quadrature rule, the support size of the test function and lastly for the appropriate numbers of test functions. All of the above is done based on a single Poisson problem in $\Omega=[-1,1]\times[-1,1]$ with Dirichlet boundary condition and the Matlab's peaks function as the exact solution

$$u^*(x,y)=3e^{-x^2-(y+1)^2}(1-x)^2-\frac{1}{3}e^{-(x+1)^2-y^2}-10e^{-x^2-y^2}\Big(-y^5-x^3+\frac{x}{5}\Big).$$

All numerical accuracies are measured by the maximum error

$$\text{Max error} = \text{Max}\big\{|u_N(z_i)-u^*(z_i)|,\ z_i\in Z\big\},$$

and/or the root mean square (RMS) error

$$RMS=\Big(\sum_{z_i\in Z}\frac{(u_N(z_i)-u^*(z_i))^2}{|Z|}\Big)^{\frac{1}{2}},$$

where u_N is the numerical solution and Z is a set of 81^2 regularly placed evaluation points. Using the obtained parameters, the exponential convergence of the RBF-MLPG scheme is demonstrated under *double precision computations* for different types of elliptic PDEs including a problem with variable coefficient.

4.1 Appropriate test functions

To study the effect of the accuracy of test equations, we consider $N=M=21^2$ nodes which are regularly spaced in with separating distance $h=0.1$ in the domain $[-1,1]\times[-1,1]$. In this example, the support size of test functions is fixed $r_i=0.05$. Table 1 lists all the numerical errors for different number of gaussian points, q, which is used to evaluate boundary integrals (1.3).

By going through each column in Table 1, it is obvious that the numerical error varies with q. When the number of gaussian points is small, the error in the numerical integrations is large and this affects the MLPG accuracy. With too many gaussian points, the cancelation error becomes dominant and starts affecting the accuracy.

Next in Table 2, we present the suitable numbers of Gauss points q with $N=21^2$ nodes and different support sizes r. For example, from Table 1, the optimal value of q is 11, when $N=M=21^2$ and $r=0.05$. We observe that the optimal values of q should increase with the support sizes r in order to obtain the best accuracy of the RBF-MLPG scheme. Among all the exactly-determined test cases, $r=0.09$ yields the best accuracy; accuracy drops with larger support sizes.

Table 1: Number of gaussian points q and the accuracy of the RBF-MLPG scheme.

q	RMS	Max error
6	3.971506^{-3}	9.534818^{-3}
7	5.571586×10^{-4}	1.375229^{-3}
8	6.651908^{-5}	1.851960×10^{-4}
9	2.105612^{-5}	4.902434^{-5}
10	1.815665^{-5}	4.320609^{-5}
11	1.785648^{-5}	4.154598^{-5}
12	1.827903^{-5}	4.242716^{-5}
13	1.816742^{-5}	4.2997352^{-5}

Table 2: Numerical errors with different numbers of test equations and trial centers.

		$N=M=21^2$		$N=21^2$, $M=41^2$	
r	q	RMS error	Max error	RMS error	Max error
0.01	9	3.1793^{-5}	7.5100^{-5}	4.3199^{-6}	1.0742^{-5}
0.02	9	3.0009^{-5}	7.1861^{-5}	3.0628^{-6}	7.7119^{-6}
0.03	10	2.6874^{-5}	8.5981^{-5}	1.1662^{-6}	3.4846^{-6}
0.04	10	2.3065^{-5}	5.4171^{-5}	8.5172^{-7}	2.7517^{-6}
0.05	11	3.4660^{-5}	2.1272^{-4}	9.9328^{-7}	2.5706^{-6}
0.06	11	1.2849^{-5}	3.1260^{-5}	1.6961^{-6}	4.5509^{-6}
0.07	11	1.2467^{-5}	6.3504^{-5}	2.2156^{-6}	6.6261^{-6}
0.08	12	1.6302^{-6}	3.7977^{-6}	2.3728^{-6}	7.5502^{-6}
0.09	14	3.1679^{-6}	8.3291^{-6}	2.1251^{-6}	7.5384^{-6}
0.10	14	7.3311^{-6}	2.4515^{-5}	1.6999^{-6}	7.6608^{-6}

If we turn our setup to an overdetermined one as the theories suggest, using $M=41^2$ testings on $N=21^2$ nodes will yield a better accuracy in each tested r. Note that the optimal q seems to depend on r only but not on M. Moreover, with the overdetermined setting, the best accuracy occurs earlier at around $r=0.04$ to 0.05. Although more boundary integrals are needed for larger M, the cost of evaluating each integral drops, that is, $q=11$ for $r=0.05$ versus $q=14$ for $r=0.09$.

4.2 Appropriate numbers of testings

We are interested in the effect of the number of test equations on the accuracy of the MLPG solution. If a trial space with dimension N is used, there are N unknown coefficients to be determined. The solvability theory in Section 2 and the convergence theory in Section 3 both suggest that we need to have $M>N$ test equations. To estimate how large M should be, we consider test cases with $N=17^2$ and $N=21^2$ regularly spaced MQ centers. We use test functions with support $r=0.05$ and present the RMS errors obtained with different numbers $M=n^2$ of test functions in Fig. 1.

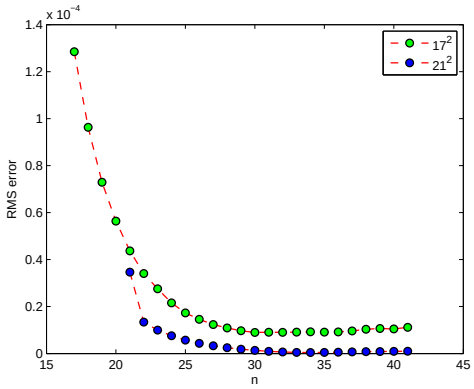

Figure 1: RMS error versus number of equations, with 21^2 and 17^2 RBFs, $r=0.05$ and $c=1$.

Each of the two curves starts with the accuracy of an exactly determined system with $M=N$. From both curves, we can see that using $M=N$ does not provide enough information to the MLPG system to make good use of the approximation power of the MQ trial spaces. When M increases, the MLPG systems become overdetermined and the extra information drives the accuracy down. Note that there is no M-convergence as the error profiles stagnate with large M. By Fig. 1, we make a very rough estimation of the suitable number of test functions; M should range between $(1.5k)^2+1$ and $(2k)^2+1$ when $N=k^2+1$.

4.3 Seeing the exponential convergence of RBF-MLPG

The experience gained in the previous examples, we consider RBF-MLPG scheme with $N=k^2$ nodes and $M=(1.5k)^2+1$ testings. Accuracy will be reported in terms of the

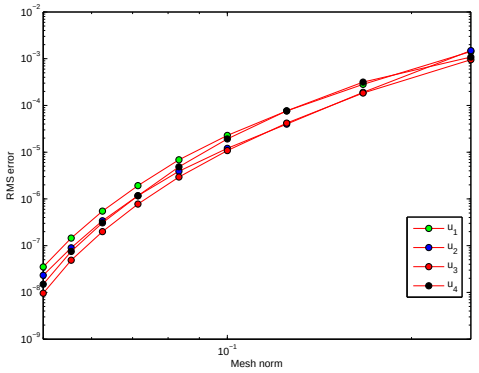

Figure 2: Convergence profiles for different Poisson problems.

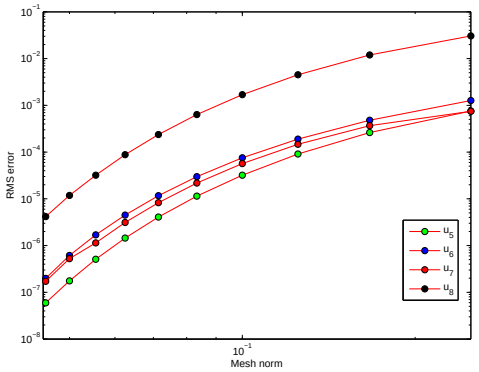

Figure 3: Convergence profiles for various kinds of elliptic PDEs.

mesh norm h that is the minimum separating distance found in the N RBF nodes. Test function (2.3) will have a fixed support size $r=0.05$ and the $q=11$-point Gauss-Legendre quadrature rule will be used to evaluate the boundary integrals. Fig. 2 shows the error profiles for four Poisson problems[†]. The right hand functions f and the Dirichlet data g in (1.1) are generated from the exact solution. Even though the RBF-MLPG method shares a lot of similarities with the strong-form Kansa method, the RBF-MLPG not only yields higher accuracy but also shows (very clear) exponential convergence behavior under double precision computations. Note that, in [22,23], the exponential convergence of the MQ-Kansa method can be observed under arbitrary precision computations.

In Fig. 3, the exponential convergence of the RBF-MLPG methods on other types of elliptic PDEs[‡] are demonstrated. Test functions u_5 and u_6 that satisfy modified Helmholtz

[†]Exact solutions are $u_1(x,y)=\left(\sqrt{((x-2)^2+(y-2)^2)}\right)^{-1}$, $u_2(x,y)=\frac{1}{2}\log((x-2)^2+(y-2)^2)$, $u_3(x,y)=\cos(x/2)\sin(y/2)$ and $u_4(x,y)=\exp(-x^2-y^2)$.

[‡]Exact solutions are $u_5(x,y)=\cos(x/2)\sin(y/2)$, $u_6(x,y)=\left(\sqrt{((x-2)^2+(y-2)^2)}\right)^{-1}$, $u_7(x,y)=\cos(x)\sin(y)$ and $u_8(x,y)=x^3-y^3$.

equations $(\Delta u - u/2 = f)$, u_7 that satisfies a Helmholtz equation $(\Delta u + 2u = f)$ and u_8 that satisfies a PDE with variable coefficients

$$(x^2+y^2+1)\Delta u + [0,y]^T \nabla u - u = f, \quad \text{in } \Omega \subset \mathbb{R}^2,$$

are used here. In terms of RMS errors, the numerical solution is not as accurate as in the Poisson cases. Nonetheless, all tests support the theoretical convergence rate of the MQ kernel.

5 Conclusions

Solvability and convergence analysis of the RBF-MLPG method are presented. Some numerical demonstrations are given to verify the proven theories: the exponential convergence rate found in multiquadrics interpolation can be carried to the RBF-MLPG approximation. For the strong-form Kansa method implemented in double precision, it usually requires some additional techniques, e.g., adaptive greedy method [24], to obtain obvious exponential convergent profiles. The RBF-MLPG, having an *obtainable* exponential convergence instead of a *theoretical* one only, is an attractive alternative towards its strong-from siblings. Lastly, note that, in [25], the convergence of the least-squares strong-form Kansa method is proven for the modified Helmholtz equations. It is very hopeful that the same conclusion also holds true for the RBF-MLPG method.

Acknowledgments

This project was supported by the CERG Grant of the Hong Kong Research Grant Council and the FRG Grant of the Hong Kong Baptist University.

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