Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **6**, No. 2, pp. 247-260

DOI: 10.4208/aamm.12-m12132
April 2014

Two-Dimensional Legendre Wavelets for Solving Time-Fractional Telegraph Equation

M. H. Heydari[1], M. R. Hooshmandasl[1] and F. Mohammadi[2,*]

[1] *Faculty of Mathematics, Yazd University, Yazd, Iran*
[2] *Department of Mathematics, Faculty of Seiences, Hormozgan University, Bandarabbas, Iran*

Received 10 October 2012; Accepted (in revised version) 4 September 2013

Available online 7 April 2014

Abstract. In this paper, we develop an accurate and efficient Legendre wavelets method for numerical solution of the well known time-fractional telegraph equation. In the proposed method we have employed both of the operational matrices of fractional integration and differentiation to get numerical solution of the time-telegraph equation. The power of this manageable method is confirmed. Moreover the use of Legendre wavelet is found to be accurate, simple and fast.

AMS subject classifications: 65T60, 26A33

Key words: Telegraph equation, Legendre wavelets, fractional calculus, Caputo derivative.

1 Introduction

Fractional ordinary and partial differential equations, as generalizations of classical integer order differential equations, are increasingly used to model problems in fluid flow, mechanics, viscoelasticity, biology, physics, engineering and other applications (for example see [1–3]). Fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes [4–9]. Fractional differentiation and integration operators are also used for extensions of the diffusion and wave operators [10]. The solutions of fractional differential equations are much involved, because in general, there exists no method that yields an exact solution for fractional differential equations, and only approximate solutions can be derived using linearization or perturbation methods.

*Corresponding author.
Email: heydari@stu.yazd.ac.ir (M. H. Heydari), hooshmandasl@yazd.ac.ir (M. R. Hooshmandasl), f.mohammadi62@hotmail.com (F. Mohammadi)

Wavelet methods have been applied for solving partial differential equations (PDEs) from the beginning of 1990s [11]. In the last two decades this method of solution for such problems has attracted great attention and numerous papers about this topic have been published. Due to this fact we must confine somewhat our analysis; in the following only PDEs of mathematical physics and of elastostatics are considered. From the first field of investigation [12–17] can be cited, and for elasticity problems we refer to [18–24]. In these papers different wavelet families have been applied. In most cases the wavelet coefficients have been calculated by the Galerkin or collocation method, for which we have to evaluate integrals of some combinations of the wavelet functions (also called connection coefficients).

We consider the time-fractional telegraph equation of order α $(1<\alpha\leq 2)$ as:

$$\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}}+\frac{\partial^{\alpha-1}u(x,t)}{\partial t^{\alpha-1}}+u(x,t)=\frac{\partial^{2}u(x,t)}{\partial x^{2}}+f(x,t), \quad a\leq x\leq b, \quad t\geq 0, \tag{1.1}$$

where $\partial^{\beta}/\partial t^{\beta}$ denotes Caputo fractional derivative of order β, that will be described in the next section. This equation is commonly used in the study of wave propagation of electric signals in a cable transmission line and also in wave phenomena. This equation has been also used in modeling the reaction-diffusion processes in various branches of engineering sciences and biological sciences by many researchers (see [25] and references therein).

The fractional telegraph equation has recently been considered by many authors. Cascaval et al. [26] have discussed the time-fractional telegraph equations, and have investigated its wellposedness and asymptotic behavior by using the Riemann-Liouville approach. Orsingher and Beghin [27] discussed the time-fractional telegraph equation and telegraph processes with Brownian time, showing that some processes are governed by time-fractional telegraph equations. Chen et al. [28] also discussed and derived the solution of the time-fractional telegraph equation with three kinds of nonhomogeneous boundary conditions, by the method of separations of variables. Orsingher and Zhao [29] considered the space-fractional telegraph equations, obtaining the Fourier transform of its fundamental solution and presenting a symmetric process with discontinuous trajectories, whose transition function satisfies the space-fractional telegraph equation. Momani [30] discussed exact and approximate solutions of the space- and time-fractional telegraph differential equations by means of the so-called Adomian decomposition method.

The aim of the present work is to develop Legendre wavelets method with both of the operational matrices of integration and differentiation for solving the time-fractional telegraph equation, which is fast and mathematically simple and guarantees the necessary accuracy for a relatively small number of grid points. The outline of this article is as follows: In Section 2 we describe properties of Legendre wavelets. In Section 3 the proposed method is used to approximate the solution of the problem. In Section 4 some numerical examples are solved by applying the method of this article. Finally a conclusion is drawn in Section 5.

2 Basic definitions

In this section, we briefly give some necessary definitions about fractional calculus and wavelets theory which will be used in this paper.

2.1 Fractional calculus

We give some basic definitions and properties of the fractional calculus theory which will be used further in this paper.

Definition 2.1. A real function $f(t)$, $t>0$, is said to be in the space C_μ, $\mu\in\mathbb{R}$ if there exists a real number $p\ (>\mu)$ and a function $f_1(t)\in C[0,\infty)$ such that $f(t)=t^p f_1(t)$, and it is said to be in the space C_μ^n, $n\in\mathbb{N}$ if $f^{(n)}\in C_\mu$.

Definition 2.2. The Riemann-Liouville fractional integration operator of order $\alpha\geq 0$ of a function $f\in C_\mu$, $\mu\geq -1$, is defined as:

$$(I^\alpha f)(t)=\begin{cases}\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-\tau)^{\alpha-1}f(\tau)d\tau, & \alpha>0,\\ f(t), & \alpha=0.\end{cases} \tag{2.1}$$

It has the following properties:

$$(i)\ \ I^\alpha I^\beta=I^{\alpha+\beta}, \qquad (ii)\ \ I^\alpha I^\beta=I^\beta I^\alpha, \tag{2.2a}$$

$$(iii)\ \ (I^\alpha I^\beta f)(t)=(I^\beta I^\alpha f)(t), \qquad (iv)\ \ I^\alpha t^\vartheta=\frac{\Gamma(\vartheta+1)}{\Gamma(\alpha+\vartheta+1)}t^{\alpha+\vartheta}, \tag{2.2b}$$

where α, $\beta\geq 0$, $t>0$ and $\vartheta>-1$.

Riemann-Liouville fractional derivative of order $\alpha>0$ is defined as:

$$D^\alpha f(t)=\Big(\frac{d}{dt}\Big)^n I^{n-\alpha}f(t), \quad (n-1<\alpha\leq n), \tag{2.3}$$

where n is an integer and $f\in C_1^n$.

The Riemann-Liouville derivatives have certain disadvantages when trying to model real-world phenomena with fractional differential equations. Therefore, we shall now introduce a modified fractional differential operator D_*^α proposed by Caputo [7]:

Definition 2.3. The fractional derivative of order $\alpha>0$ in the Caputo sense is defined as:

$$D_*^\alpha f(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-\alpha-1}f^{(n)}(\tau)d\tau, \quad (n-1<\alpha\leq n), \tag{2.4}$$

where n is an integer, $t>0$, and $f\in C_1^n$.

Caputos integral operator has a useful property:

$$I^{\alpha} D_{*}^{\alpha} f(t) = f(t) - \sum_{k=0}^{n-1} f^{(k)}(0^{+}) \frac{t^k}{k!}, \quad (n-1<\alpha\leq n), \tag{2.5}$$

where n is an integer, $t>0$, and $f\in C_1^n$.

For more details on the mathematical properties of fractional derivatives and integrals see [7].

2.2 Wavelets

Wavelets constitute a family of functions constructed from dilations and translations of a single function called the mother wavelet $\psi(t)$. When the dilation parameter a and the translation parameter b vary continuously we have the following family of continuous wavelets as [31]:

$$\psi_{a,b}(t) = |a|^{-\frac{1}{2}} \psi\Big(\frac{t-b}{a}\Big), \quad a,b\in\mathbb{R}, \quad a\neq 0. \tag{2.6}$$

If we restrict the parameters a and b to discrete values as $a=a_0^{-k}$, $b=nb_0a_0^{-k}$, $a_0>1$, $b_0>0$, and for n and k positive integers, we obtain the following family of discrete wavelets:

$$\psi_{k,n}(t) = |a_0|^{\frac{k}{2}} \psi(a_0^k t - nb_0), \tag{2.7}$$

and $\{\psi_{k,n}(t)|k,\ n\in\mathbb{Z}\}$ forms a wavelet basis for $L^2(\mathbb{R})$. In particular, when $a_0=2$ and $b_0=1$, $\psi_{k,n}(t)$ forms an orthonormal basis, that is $(\psi_{k,n}(t),\psi_{l,m}(t))=\delta_{kl}\delta_{nm}$.

2.2.1 Legendre wavelets

Legendre wavelets $\psi_{n,m}(t)=\psi(k,\hat{n},m,t)$ have four arguments; $k\in\mathbb{N}$, $n=1,2,\cdots,2^{k-1}$, and $\hat{n}=2n-1$, moreover m is the degree of the Legendre polynomials and t is the normalized time, and are defined on the interval $[0,1)$ as [31]:

$$\psi_{n,m}(t) = \begin{cases} 2^{\frac{k}{2}} \sqrt{m+\dfrac{1}{2}}\, p_m(2^k t - \hat{n}), & \dfrac{\hat{n}-1}{2^k} \leq t < \dfrac{\hat{n}+1}{2^k}, \\ 0, & \text{otherwise}, \end{cases} \tag{2.8}$$

where $m=0,1,\cdots,M-1$, and M is a fixed positive integer. The coefficient $\sqrt{m+1/2}$ in (2.8) is for orthonormality, the dilation parameter is $a=2^{-k}$ and the translation parameter is $b=\hat{n}2^{-k}$. Here, $P_m(t)$ are the well-known Legendre polynomials of degree m which are orthogonal with respect to the weight function $w(t)=1$ on the interval $[-1,1]$, and satisfy the following recursive formula:

$$\begin{aligned} &p_0(t)=1, \quad p_1(t)=t, \\ &p_{m+1}(t) = \Big(\frac{2m+1}{m+1}\Big) t p_m(t) - \Big(\frac{m}{m+1}\Big) p_{m-1}(t), \quad m=1,2,3,\cdots. \end{aligned}$$

2.3 Function approximation

An arbitrary function $f(t)\in L^2(R)$ defined over $[0,1)$ may be expanded into Legendre wavelets basis as

$$f(t)=\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}c_{nm}\psi_{nm}(t), \tag{2.9}$$

where coefficients $c_{nm}=(f(t),\psi_{nm}(t))$, in which $(,)$ denotes the inner product.

If the infinite series in (2.9) is truncated, then (2.9) can be written as

$$f(t)\approx\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}c_{nm}\psi_{nm}(t)=C^T\Psi(t), \tag{2.10}$$

where C and $\Psi(t)$ are $\hat{m}\times 1$ $(\hat{m}=2^{k-1}M)$ matrices, given by

$$\begin{aligned} C&=\left[c_{10},c_{11},\cdots,c_{1M-1},c_{20},\cdots,c_{2M-1},\cdots,c_{2^{k-1}0},\cdots,c_{2^{k-1}M-1}\right]^T,\\ \Psi(t)&=\left[\psi_{10}(t),\psi_{11}(t),\cdots,\psi_{1M-1}(t),\cdots,\psi_{2^{k-1}0}(t),\cdots,\psi_{2^{k-1}M-1}(t)\right]^T. \end{aligned}$$

For simplicity, we write (2.10) as

$$f(t)\approx\sum_{i=1}^{\hat{m}}c_i\psi_i(t)=C^T\Psi(t), \tag{2.11}$$

where $c_i=c_{nm}$, $\psi_i=\psi_{nm}$. The index i, is determined by the relation $i=M(n-1)+m+1$. Therefor we have

$$C=[c_1,c_2,\cdots,c_{\hat{m}}]^T,\quad \Psi(t)=[\psi_1(t),\psi_2(t),\cdots,\psi_{\hat{m}}(t)]^T. \tag{2.12}$$

Similarly, an arbitrary function of two variables $u(x,t)\in L^2(\mathbb{R}\times\mathbb{R})$ defined over $[0,1)\times[0,1)$, may be expanded into Legendre wavelets basis as

$$u(x,t)\approx\sum_{i=1}^{\hat{m}}\sum_{j=1}^{\hat{m}}u_{ij}\psi_i(x)\psi_j(t)=\Psi^T(x)U\Psi(t), \tag{2.13}$$

where $U=[u_{ij}]$ and $u_{ij}=(\psi_i(x),(u(x,t),\psi_j(t)))$.

Taking the collocation points

$$t_i=\frac{(2i-1)}{2\hat{m}},\quad i=1,2,\cdots,\hat{m}, \tag{2.14}$$

we define the $\hat{m}\times\hat{m}$ wavelet matrix Φ as

$$\Phi=\Big[\Psi\Big(\frac{1}{2\hat{m}}\Big),\Psi\Big(\frac{3}{2\hat{m}}\Big),\cdots,\Psi\Big(\frac{2\hat{m}-1}{2\hat{m}}\Big)\Big]. \tag{2.15}$$

Indeed Φ has a diagonal form [32].

2.4 The fractional operational matrices of Legendre wavelets

The fractional integration of order α of the vector $\Psi(\cdot)$ defined in (2.12) can be expressed as

$$(I^{\alpha}\Psi)(\cdot) \approx P^{\alpha}\Psi(\cdot), \tag{2.16}$$

where P^{α} is the $\hat{m}\times\hat{m}$ approximated operational matrix of fractional integration of order α for Legendre wavelets. It is shown that the matrix P^{α} can be expressed as [31]:

$$P^{\alpha} = \Phi P_B^{\alpha} \Phi^{-1}, \tag{2.17}$$

where P_B^{α} is the operational matrix of fractional integration of order α of the Block Pulse functions (BPF) [33]. In [32] it is shown that P^{α} has an upper trigonometric form. Also, From (2.11) and (2.17), it is concluded that for a function $f(t)\in C_{\mu}$, $\mu\geq -1$, we have:

$$(I^{\alpha} f)(t) \approx C^T \Phi P_B^{\alpha} B_{\hat{m}}(t).$$

The fractional differentiation of order α of the matrix $\Psi(\cdot)$ in Caputo sense can be approximated as:

$$(D_*^{\alpha}\Psi)(\cdot) \approx Q^{\alpha}\Psi(\cdot), \tag{2.18}$$

where Q^{α} is the $\hat{m}\times\hat{m}$ approximated operational matrix of fractional differentiation of order α. It can be simply shown that the matrix Q^{α} is the inverse of P^{α} that can be expressed as:

$$Q^{\alpha} = \Phi P_B^{-\alpha} \Phi^{-1}, \tag{2.19}$$

where $P_B^{-\alpha}$ is the operational matrix of fractional differentiation of order α of the BPF [33].

3 Description of numerical method

In this section, we will use the fractional operational matrices of Legendre wavelets for solving the time-fractional telegraph equation (1.1). Let us consider the time-fractional telegraph equation (1.1) as:

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} + \frac{\partial^{\alpha-1} u}{\partial t^{\alpha-1}} + u = \frac{\partial^2 u}{\partial x^2} + f(x,t), \quad (x,t)\in[0,1)\times[0,1), \tag{3.1}$$

with the Dirichlet boundary conditions:

$$u(x,0) = h_0(x), \qquad u(0,t) = g_0(t), \tag{3.2a}$$
$$u(x,1) = h_1(x), \qquad u(1,t) = g_1(t), \tag{3.2b}$$

where $h_i(x)$ and $g_i(t)$ are two times continuously differentiable functions on $[0,1]$.

We suppose

$$\frac{\partial^{\alpha+2} u}{\partial t^{\alpha}\partial x^2} = \Psi(x)^T U \Psi(t), \tag{3.3}$$

where $U=[u_{i,j}]_{\hat{m}\times\hat{m}}$ is an unknown matrix which should be found and $\Psi(\cdot)$ is the vector that was defined in (2.12).

By fractional integrating of order α with respect to t of (3.3), we have:

$$\frac{\partial^2 u}{\partial x^2}=\Psi(x)^T U P^{\alpha}\Psi(t)+\frac{\partial^2 u}{\partial x^2}\Big|_{t=0}+t\frac{\partial}{\partial t}\Big(\frac{\partial^2 u}{\partial x^2}\Big)\Big|_{t=0}, \tag{3.4}$$

and by putting $t=1$ in (3.4) and considering (3.2) we have:

$$\frac{\partial}{\partial t}\Big(\frac{\partial^2 u}{\partial x^2}\Big)\Big|_{t=0}=h_1''(x)-h_0''(x)-\Psi(x)^T U P^{\alpha}\Psi(1). \tag{3.5}$$

Now by substituting (3.5) into (3.4) we obtain:

$$\frac{\partial^2 u}{\partial x^2}=\Psi(x)^T U P^{\alpha}\Psi(t)-t\Psi(x)^T U P^{\alpha}\Psi(1)+(1-t)h_0''(x)+th_1''(x). \tag{3.6}$$

Also by integrating two times with respect to x of (3.3), and considering (3.2) we obtain:

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}}=\Psi(x)^T(P^2)^T U\Psi(t)-x\Psi(1)^T(P^2)^T U\Psi(t)+(1-x)\frac{\partial^{\alpha} g_0}{\partial t^{\alpha}}+x\frac{\partial^{\alpha} g_1}{\partial t^{\alpha}}. \tag{3.7}$$

Now by integrating (3.6) two times with respect to x and considering (3.2) we have:

$$\begin{aligned}u(x,t)=&\Psi(x)^T(P^2)^T U P^{\alpha}\Psi(t)-t\Psi(x)^T(P^2)^T U P^{\alpha}\Psi(1)-x\Psi(1)^T(P^2)^T U P^{\alpha}\Psi(t)\\&+tx\Psi(1)^T(P^2)^T U P^{\alpha}\Psi(1)+H(x,t),\end{aligned} \tag{3.8}$$

where $H(x,t)$ is a known function of x and t.

Now by fractional differentiation of order $(\alpha-1)$ of (3.8) with respect to t, and considering operational matrix of fractional order differentiation we get:

$$\begin{aligned}\frac{\partial^{\alpha-1} u}{\partial t^{\alpha-1}}=&\Psi(x)^T(P^2)^T U P\Psi(t)-\frac{t^{2-\alpha}}{\Gamma(3-\alpha)}\Psi(x)^T(P^2)^T U P^{\alpha}\Psi(1)-x\Psi(1)^T(P^2)^T U P\Psi(t)\\&+\frac{xt^{2-\alpha}}{\Gamma(3-\alpha)}\Psi(1)^T(P^2)^T U P^{\alpha}\Psi(1)+\frac{\partial^{\alpha-1}H(x,t)}{\partial t^{\alpha-1}}.\end{aligned} \tag{3.9}$$

By replacing (3.6)-(3.9) into (3.1) and taking collocation points

$$x_i,t_i=\frac{2i-1}{2\hat{m}},\quad i=1,2,\cdots,\hat{m},$$

we obtain the

$$\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}}+\frac{\partial^{\alpha-1} u(x,t)}{\partial t^{\alpha-1}}+u(x,t)-\frac{\partial^2 u(x,t)}{\partial x^2}-f(x,t)\Big|_{(x,t)=(x_i,t_j)}=0,\quad i,j=1,2,\cdots,\hat{m}. \tag{3.10}$$

By solving this system and determining U, we get the numerical solution of this problem by substituting U into (3.8).

4 Numerical examples

In this section we demonstrate the efficiency of the proposed method for numerical solution of telegraph equation in the form (3.1) with the Dirichlet boundary conditions (3.2). To show the efficiency of the present method, we report the root mean square error L_2 and maximum error L_∞ in case $\alpha=2$ as:

$$L_2=\sqrt{\frac{1}{\hat{m}}\sum_{i=1}^{\hat{m}}|u(x_i,t_i)-\tilde{u}(x_i,t_i)|^2},\quad L_\infty=\max_{1\le i\le \hat{m}}|u(x_i,t_i)-\tilde{u}(x_i,t_i)|.$$

Example 4.1. Consider the time-fractional telegraph equation (3.1) with $f(x,t)=x^2+t-1$ and the boundary conditions:

$$u(x,0)=x^2,\qquad u(0,t)=t,$$
$$u(x,1)=1+x^2,\qquad u(1,t)=1+t.$$

The exact solution of this problem for $\alpha=2$ is $u(x,t)=x^2+t$. The space-time graph of the exact and numerical solutions for $\alpha=2$ and $\hat{m}=12$ $(M=3,\ k=3)$ are presented in Figs. 1 and 2. The graphs of analytical and approximate solutions for some nodes in $[0,1]\times[0,1]$ and different values of α are shown in Fig. 3. From Figs. 2 and 2 (in case $\alpha=2$), it can be seen that the numerical solutions are in a very good agreement with the exact solutions. Therefore, we hold that the numerical solutions for $\alpha=1.65$, 1.75 and 1.85 are also credible. A comparison between the exact solution (Ex.S) and numerical solution (Nu.S) for $\alpha=1.65$, 1.75 and 1.85 for some nodes in $[0,1]\times[0,1]$ are shown in Fig. 3. The root-mean-square error L_2 and maximum error L_∞ for some $(x,t)\in[0,1]$ in case $\alpha=2$, are presented in Table 1.

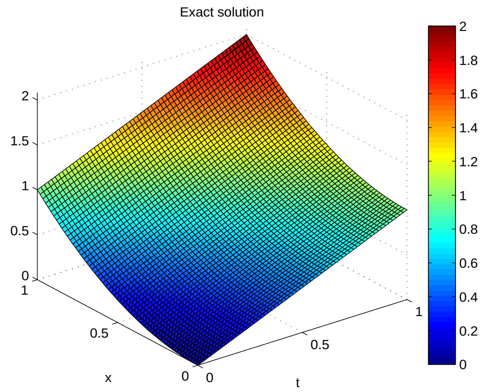

Figure 1: Exact solution of Example 4.1.

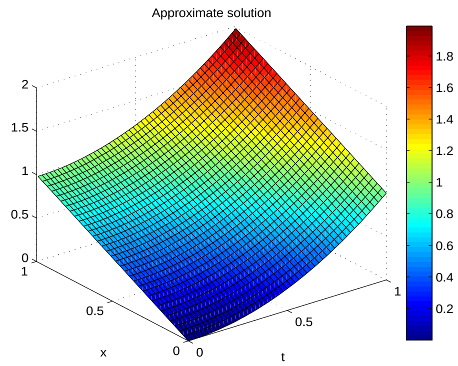

Figure 2: Approximate solution of Example 4.1.

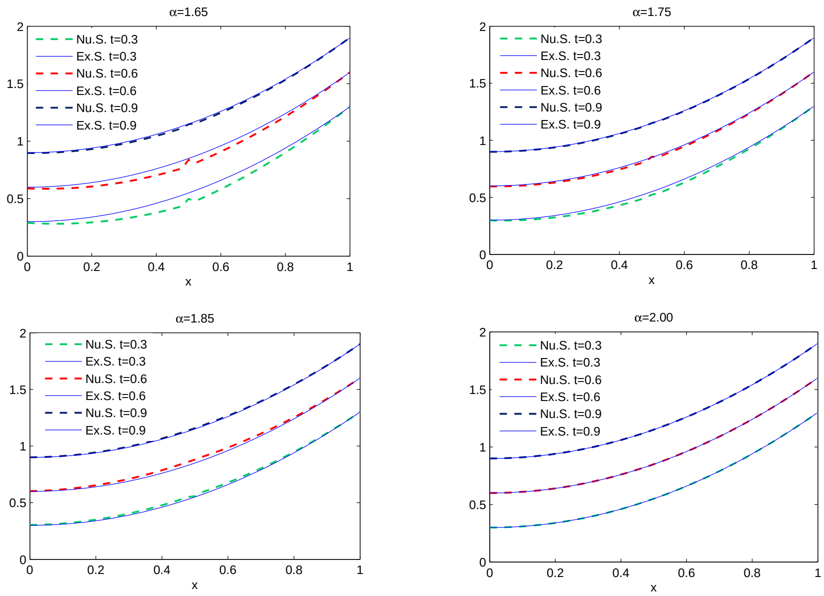

Figure 3: Comparison between numerical and exact solutions for Example 4.1.

Table 1: The L_∞ and L_2 errors for some different values of t in case $\alpha=2$.

t	$t=0.1$	$t=0.3$	$t=0.5$	$t=0.7$	$t=0.9$	$t=1.0$
L_∞	4.90×10^{-3}	1.47×10^{-3}	2.28×10^{-3}	1.17×10^{-3}	6.45×10^{-4}	0.00
L_2	8.64×10^{-4}	8.06×10^{-4}	1.38×10^{-3}	7.50×10^{-4}	8.60×10^{-5}	0.00

Example 4.2. Consider the time-fractional telegraph equation (3.1) with $f(x,t)=0$ and the boundary conditions:

$$u(x,0)=e^x, \qquad u(0,t)=e^{-t},$$
$$u(x,1)=e^{x-1}, \qquad u(1,t)=e^{1-t}.$$

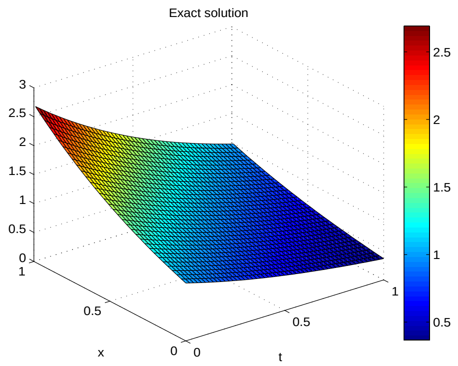

Figure 4: Exact solution of Example 4.2.

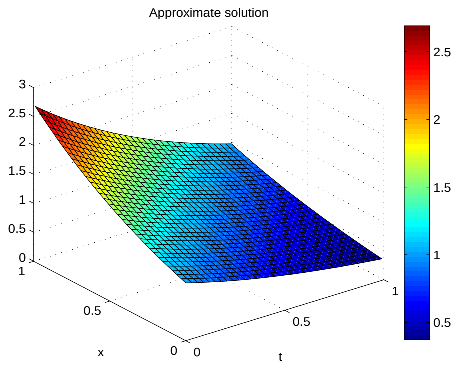

Figure 5: Approximate solution of Example 4.2.

The exact solution of this problem for $\alpha=2$ is $u(x,t)=e^{x-t}$. The space-time graph of the exact and numerical solutions for $\alpha=2$ and $\hat{m}=12$ are shown in Figs. 4 and 5. The graphs of analytical and approximate solutions for some nodes in $[0,1]\times[0,1]$ and different values of α are presented in Fig. 6. From Figs. 5 and 6 (in case $\alpha=2$), it can be seen that the numerical solutions are in a very good agreement with the exact solutions. Therefore, we hold that the numerical solutions for α=1.65, 1.75 and 1.85 are also credible. A comparison between exact and numerical solutions for $\alpha=1.65$, 1.75 and 1.85 for some nodes in $[0,1]\times[0,1]$ are shown in Fig. 6. The root-mean-square error L_2 and maximum error L_∞ for some $(x,t)\in[0,1]$ in case $\alpha=2$ are presented in Table 2.

Table 2: The L_∞ and L_2 errors for some different values of t in case $\alpha=2$.

t	$t=0.1$	$t=0.3$	$t=0.5$	$t=0.7$	$t=0.9$	$t=1.0$
L_∞	8.64×10^{-3}	2.05×10^{-3}	2.43×10^{-4}	8.54×10^{-5}	1.50×10^{-5}	0.00
L_2	5.50×10^{-3}	9.61×10^{-4}	9.43×10^{-5}	5.41×10^{-5}	6.53×10^{-6}	0.00

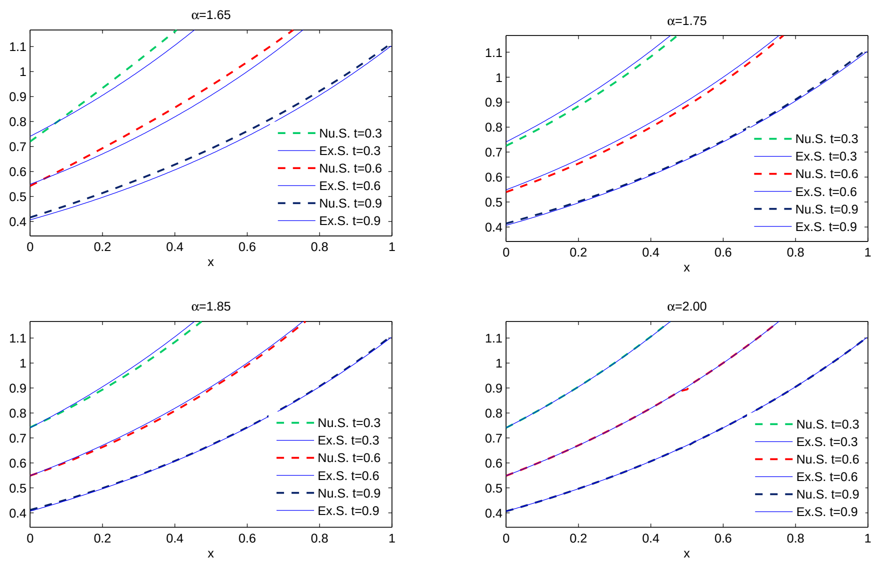

Figure 6: Comparison between numerical and exact solutions for Example 4.2.

Example 4.3. Finally consider the time-fractional telegraph equation (3.1) with $f(x,t) = 2\sin(x)e^{-t}$ and the boundary conditions:

$$\begin{aligned} &u(x,0)=\sin(x), && u(0,t)=0, \\ &u(x,1)=e^{-1}\sin(x), && u(1,t)=\sin(1)e^{-t}. \end{aligned}$$

The exact solution of this problem for $\alpha=2$ is $u(x,t)=\sin(x)e^{-t}$. The space-time graph of the exact and numerical solutions for $\alpha=2$ and $\hat{m}=12$ are presented in Figs. 7 and 8. The graphs of analytical and approximate solutions for some nodes in $[0,1]\times[0,1]$ and different values of α are shown in Fig. 9. From Figs. 8 and 9, it is obvious that the nu-

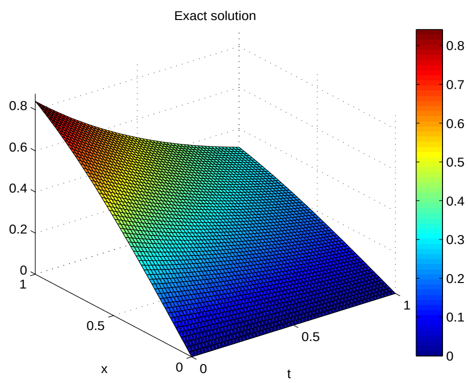

Figure 7: Exact solution of Example 4.3.

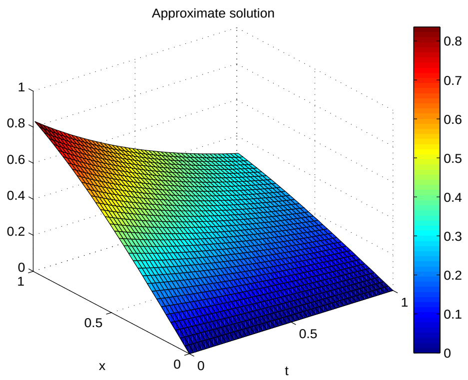

Figure 8: Approximate solution of Example 4.3.

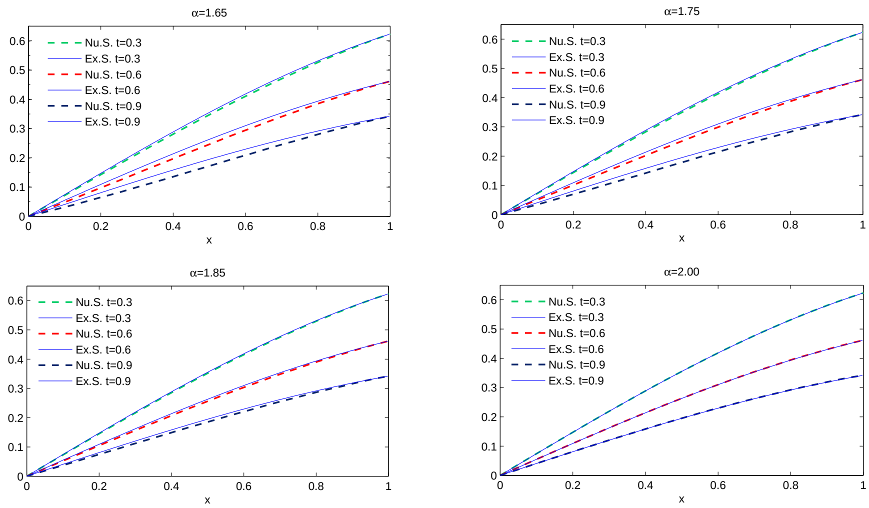

Figure 9: Comparison between numerical and exact solutions for Example 4.3.

merical solutions are in a very good agreement with the exact solutions. Therefore, we hold that the numerical solutions for $\alpha=1.65$, 1.75 and 1.85 are also credible. A comparison between exact and numerical solutions for $\alpha=1.65$, 1.75 and 1.85 for some nodes in $[0,1]\times[0,1]$ are shown in Fig. 9. The root-mean-square error L_2 and maximum error L_∞ for some $(x,t)\in[0,1]$ in case $\alpha=2$ are presented in Table 3.

Table 3: The L_∞ and L_2 errors for some different values of t in case $\alpha=2$.

t	$t=0.1$	$t=0.3$	$t=0.5$	$t=0.7$	$t=0.9$	$t=1.0$
L_∞	6.95×10^{-3}	2.53×10^{-3}	5.28×10^{-3}	8.17×10^{-4}	8.65×10^{-4}	0.00
L_2	6.34×10^{-3}	5.26×10^{-4}	6.33×10^{-4}	5.52×10^{-4}	7.61×10^{-4}	0.00

5 Conclusion

In this paper, numerical solutions of the nonhomogeneous time-fractional telegraph equation are derived by combining wavelet function with operational matrices of fractional integration and derivative. In the proposed method already a small number of grids points guarantees the necessary accuracy. The method is very convenient for solving boundary value problems, since the boundary condition are taken into account automatically. Also the basic idea described in this paper is expected to be further employed to solve other fractional partial differential equations.

References

[1] H. G. SUN, W. CHEN, C. P. LI AND Y. Q. CHEN, *Fractional differential models for anomalous diffusion*, Phys. A-Statist. Mech. Appl., 389(14) (2010), pp. 2719–2724.

[2] W. CHEN, *Time-space fabric underlying anomalous diffusion*, Chaos Solitons Fract., 28(4) (2006), pp. 923–929.

[3] H. SUN, W. CHEN, H. WEI AND Y. Q. CHEN, *A comparative study of constant-order and variable-order fractional models in characterizing memory property of systems*, Euro. Phys. J. Special Topics, 193 (2011), pp. 185–193.

[4] A. CARPINTERI AND F. MAINARDI, Fractals and Fractional Calculus in Continuum Mechanics, Wien, New York, Springer Verlag, 1997.

[5] K. S. MILLER AND B. ROSS, An Introduction to the Fractional Calculus and Fractional Differential Equations, New York, Wiley, 1993.

[6] K. B. OLDHAM AND J. SPANIER, The Fractional Calculus, New York, Academic Press, 1974.

[7] I. PODLUBNY, Fractional Differential Equations, San Diego, Academic Press, 1999.

[8] I. PODLUBNY, *Fractional-order systems and fractional-order controllers*, Institute of Experimental Physics, Kosice, Slovakia, Report UEF-03-94, Slovak Academy of Sciences, November 1994.

[9] R. GORENFLO AND F. MAINARDI, *Fractional calculus: integral and differential equations of fractional order*, in: A. Carpinteri, F. Mainardi (eds.), Fractals and Fractional Calculus in Continuum Mechanics, Springer, New York, pp. 223–276, 1997.

[10] W. R. SCHNEIDER AND W. WYESS, *Fractional diffusion and wave equations*, J. Math. Phys., 30 (1989), pp. 134–144.

[11] U. LEPIK, *Solving pdes with the aid of two-dimensional haar wavelets*, Comput. Math. Appl., 61 (2011), pp. 1873–1879.

[12] U. ANDERSON AND B. ENGQUIST, *A contribution to wavelet-based subgrid modeling*, Appl. Comput. Harmon. Model, 7 (1999), pp. 151–164.

[13] C. CATTANI, *Haar wavelets based technique in evolution problems*, Chaos Proc. Estonian Acad. Sci. Phys. Math., 1 (2004), pp. 45–53.

[14] N. COULT, *Explicit formulas for wavelet-homogenized coefficients of elliptic operators*, Appl. Comput. Harmon. Anal., 21 (2001), pp. 360–375.

[15] X. CHEN, J. XIANG, B. LI, AND Z. HE, *A study of multiscale wavelet-based elements for adaptive finite element analysis*, Adv. Eng. Software, 41 (2010), pp. 196–205.

[16] G. HARIHARAN, K. KANNAN, AND K. SHARMA, *Haar wavelet method for solving fishers equation*, Appl. Math. Comput., 211(2) (2009), pp. 284–292.

[17] P. Mrazek and J. Weickert, *From two-dimensional nonlinear diffusion to coupled haar wavelet shrinkage*, J. Vis. Commun. Image. Represent, 18 (2007), pp. 162–175.

[18] W. Fan and P. Qiao, *A 2-d continuous wavelet transform of mode shape data for damage detection of plate structures*, Int. J. Solids Structures, 46 (2003), pp. 6473–6496.

[19] J. E. Kim, G.-W. Yang, and Y. Y. Kim, *Adaptive multiscale wavelet-galerkin analysis for plane elasticity problems and its application to multiscale topology design optimation*, Int. J. Solids Structures, Comput Appl. Math., 40 (2003), pp. 6473–6496.

[20] Y. Shen and W. Li, *The natural integral equations of plane elasticity problems and its wavelet methods*, Appl. Math. Comput., 150(2) (2004), pp. 417–438.

[21] Z. Chun and Z. Zheng, *Three-dimensional analysis of functionally graded plate based on the haar wavelet method*, Acta Mech. Solida Sin., 20(2) (2007), pp. 95–102.

[22] H. F. Lam and C. T. Ng, *A probabilistic method for the detection of obstructed cracks of beam-type structures using spacial wavelet transform*, Probab. Eng. Mech., 23 (2008), pp. 239–245.

[23] J. Majak, M. Pohlak, M. Eerme, and T. Lepikult, *Weak formulation based haar wavelet method for solving differential equations*, Appl. Math. Comput., 211 (2009), pp. 488–494.

[24] L. M. S. Castro, A. Ferreira, S. Bertoluzza, R. Patra, and J. Reddy, *A wavelet collocation method for the static analysis of sandwich plates ussing a layerwise theory*, Compos. Struct., 92 (2010), pp. 1786–1792.

[25] M. Lakestani and B. N. Saray, *Numerical solution of telegraph equation using interpolating scaling functions*, Comput. Math. Appl., 60 (2010), pp. 1964–1972.

[26] R. C. Cascaval, E. C. Eckstein, C. L. Frota, and J. A. Goldstein, *Fractional telegraph equations*, Math. Anal. Appl., 276 (2002), pp. 145–159.

[27] E. Orsingher and L. Beghin, *Time-fractional telegraph equations and telegraph processes with brownian time*, Prob. Theory Relat. Fields, 128 (2004), pp. 141–160.

[28] J. Chen, F. Liu, and V. Anh, *Analytical solution for the time-fractional telegraph equation by the method of separating variables*, J. Math. Anal. Appl., 338 (2008), pp. 364–377.

[29] E. Orsingher and X. Zhao, *The space-fractional telegraph equation and the related fractional telegraph process*, Chinese Annal. Math. Ser. B, 24 (2003), pp. 45–56.

[30] S. Momani, *Analytic and approximate solutions of the space- and time-fractional telegraph equations*, Appl. Math. Comput., 170 (2005), pp. 1126–1134.

[31] R. M. u and R. A. Khan, *The legendre wavelet method for solving fractional differential equations*, Commun. Nonlinear Sci. Numer. Simul., 227(2) (2009), pp. 234–244.

[32] M. H. Heydari, M. R. Hooshmandasl, F. M. M. Ghaini, and F. Mohammadi, *Wavelet collocation method for solving multi order fractional differential equations*, J. Appl. Math., Volume 2012, Article ID 542401, 19 pages doi:10.1155/2012/542401.

[33] A. Kilicman and Z. A. Zhour, *Kronecker operational matrices for fractional calculus and some applications*, Appl. Math. Comput., 187(1) (2007), pp. 250–265.