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Numerical Inversion for the Initial Distribution in the Multi-Term Time-Fractional Diffusion Equation Using Final Observations

Chunlong Sun[1,2], Gongsheng Li[1,*] and Xianzheng Jia[1]

[1] *School of Science, Shandong University of Technology, Zibo, Shandong 255049, China*
[2] *School of Mathematics, Southeast University, Nanjing, Jiangsu 210096, China*

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Abstract. This article deals with numerical inversion for the initial distribution in the multi-term time-fractional diffusion equation using final observations. The inversion problem is of instability, but it is uniquely solvable based on the solution's expression for the forward problem and estimation to the multivariate Mittag-Leffler function. From view point of optimality, solving the inversion problem is transformed to minimizing a cost functional, and existence of a minimum is proved by the weakly lower semi-continuity of the functional. Furthermore, the homotopy regularization algorithm is introduced based on the minimization problem to perform numerical inversions, and the inversion solutions with noisy data give good approximations to the exact initial distribution demonstrating the efficiency of the inversion algorithm.

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1 Introduction

The fractional diffusion equations have played an important role in modeling of the anomalous diffusion phenomena and in the theory of complex systems (see e.g., [1, 3–5, 9, 31], and references therein) instead of the classical diffusion equations during the last few decades. The so-called time fractional diffusion equation arises when replacing the first-order time derivative by a fractional derivative of order α with $0<\alpha<1$. On the other hand, by the attempts to describe some real processes with the fractional diffusion equations, some researches confronted with the situation that the time-fractional derivative from the corresponding models did not remain constant and changed, say, in the

*Corresponding author.
Email: ligs@sdut.edu.cn (G. S. Li)

interval from 0 to 1, from 1 to 2 or even from 0 to 2. To manage these phenomena, several approaches were suggested. One of them introduces the fractional derivatives of the variable order, i.e., the derivatives with the order that can change with the time or/and depending on the spatial coordinates, and the other way is to employ the time-fractional diffusion equations of distributed order, in which the so-called multi-term time-fractional diffusion equation is an important particular case (see e.g., [8, 11, 12, 18, 22–24, 28, 29], and references therein). A general multi-term time fractional diffusion equation is given as

$$P({}_0^C D_t)u(x,t)=L_x(u(x,t))+F(x,t), \quad (x,t)\in\Omega_T:=\Omega\times(0,T], \tag{1.1}$$

where $\Omega\subset\mathbf{R}^d$ $(d\geq 1)$ is an open bounded domain with smooth boundary, $F(x,t)$ is a source term, and L_x is a symmetric uniformly elliptic operator given by

$$L_x(u):=\mathrm{div}(p(x)\nabla u)-q(x)u, \quad x\in\Omega, \tag{1.2}$$

in which the coefficients satisfy

$$p\in C^1(\overline{\Omega}), \quad q\in C(\overline{\Omega}), \quad 0<p(x), \quad 0\leq q(x), \quad x\in\Omega, \tag{1.3}$$

and the liner differential operator on the time is defined as

$$P({}_0^C D_t)={}_0^C D_t^{\alpha}+\sum_{j=1}^{m} r_j {}_0^C D_t^{\alpha_j}, \tag{1.4}$$

where α denotes the principal fractional order, and $\alpha_1,\alpha_2,\cdots,\alpha_m$ are the multi-term fractional orders of the derivatives, which satisfy the condition:

$$0<\alpha_m<\alpha_{m-1}<\cdots<\alpha_1<\alpha<1, \tag{1.5}$$

and $r_1,r_2,\cdots,r_m$ are positive constants; and all of the time-fractional derivatives in (1.4) are defined in the meaning of Caputo from the left-hand side, which for example for $\alpha\in(0,1)$ the fractional derivative of a function $h=h(t)$ at $t\in(0,T]$ is given as

$${}_0^C D_t^{\alpha} h(t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{h'(s)}{(t-s)^{\alpha}}ds, \tag{1.6}$$

where $\Gamma(\cdot)$ is the usual Gamma function. See, e.g., Podlubny [34] and Kilbas et al. [16] for the definition and properties of the Caputo's derivative.

From the past few years, there are quite a few researches on the forward problem of multi-term time-fractional diffusion equations like Eq. (1.1), for instance, see [27] for the maximum principle, see [22,28,29] for the uniqueness and existence of the solution, see [8, 11] for the analytic solution, and see [18, 24] for numerical solutions of finite difference method, and see [12] for the finite element solution, etc.

However, for real problems, the part of the boundary, or the initial data, or the diffusion coefficient, or the source term can not be obtained directly and we have to determine them by some additional measurements, which yields to inverse problems arising in the fractional diffusion models. There are still some researches on inverse problems in the one-term time fractional diffusion equation, see, e.g., Murio [33], Cheng et al. [6], Sakamoto and Yamamoto [35], Tuan [36], Jin and Rundell [13], Yamamoto and Zhang [40], Li et al. [19], Luchko et al. [30], Miller and Yamamoto [32], Zhang and Wei [43], and Wei et al. [38]. Recently, see Jin et al. [14] for a tutorial review on inverse problems for anomalous diffusion processes, and see Liu et al. [26] for the research on recovering a time-dependent boundary source using nonlocal measurement data, and see Zhang [42] for an inverse problem of determining a time-dependent diffusion coefficient by the monotonicity of the input-output operator and fixed point method, and see Wei et al. [37] for the research on uniqueness and stability for determining a time-dependent source term also in the one-term time-fractional diffusion equation.

To our knowledge, there are few literatures concerned with inverse problems for the multi-term time fractional differential equations. Li and Yamamoto [23] considered an inverse problem of identifying the multiple fractional orders, and they gave uniqueness of the solution to the inverse problem using Laplace transform and analytical method, and Li et al. [21] considered an inverse boundary value problem for the multi-term time fractional differential equation, and they proved the uniqueness of determining the fractional orders, the number of the fractional terms and the spatially varying coefficient simultaneously. As for the research on backward problems in the space/time fractional diffusion equations, Liu and Yamamoto [25] studied the backward problem in one-term time-fractional diffusion equation by the quasi-reversibility regularization method, Sakamoto and Yamamoto [35] investigated the backward problem in the one-term time-fractional diffusion equation utilizing the eigenfunction expansion of the solution to the forward problem and estimation on the Mittag-Leffler function, and Wei and Wang [39] considered the backward problem also for the one-term time-fractional diffusion equation with variable coefficients using a modified quasi-boundary value regularization method. The researches up to now revealed that the backward problem for the single-term time-fractional equation is of moderate ill-posedness, and the problem can be solved using proper regularization strategy. What we are concerned with is the situation for the multi-term time fractional diffusion equation which is the main purpose of this paper.

In this article, we consider with the backward problem of recovering the initial distribution using the final observations for the multi-term time fractional diffusion equation. By utilizing the properties of the multivariate Mittag-Leffler function and linearity of the solution operator, we prove that the backward problem can be uniquely solvable for any finite final time $T>0$, and it is unstable with moderate ill-posedness based on a fine estimation to the multivariate Mittag-Leffler function. From the view point of optimality, the backward problem is transformed to minimizing a cost functional, and existence of a minimum to the functional is proved by the weakly lower semi-continuity of the functional. Furthermore, the homotopy regularization algorithm is introduced based on the minimization problem, and numerical inversions are performed with noisy data. The paper is organized as follows.

In Section 2, the forward problem for the multi-term time fractional equation is discussed and some preliminary lemmas are given. In Section 3, the backward problem is set forth, and the uniquely solvability and instability of the inverse problem are proved based on the properties and estimations to the multivariate Mittag-Leffler function. In Section 4, the existence of a minimum to the cost functional is proved by using the weakly lower semi-continuity of the functional. In Section 5, the homotopy regularization algorithm is utilized to solve the backward problem and numerical inversions with noisy data are presented, and several concluding remarks are given in Section 6.

2 The forward problem

Consider the forward problem

$$\begin{cases} P({}_0^C D_t)u(x,t) - L_x(u(x,t)) = 0, & (x,t)\in\Omega_T, \\ u(x,0) = f(x), & x\in\Omega, \\ u(x,t) = 0, & (x,t)\in\partial\Omega\times(0,T], \end{cases} \tag{2.1}$$

where the operators L_x and P are given in (1.2) and (1.4), respectively, and $\partial\Omega$ is the boundary of the domain Ω. As stated in Section 1, there are a few of researches on the well-posedness of solution to the forward problem (2.1) (see Luchko [28, 29] and Li et al. [22,23]). We give some notations and lemmas to show expression and regularity of the solution to the forward problem.

Let $L^2(\Omega)$ be a usual L^2 space with the inner production $(\cdot,\cdot)$ and $H_0^1(\Omega)$, $H^2(\Omega)$ denote the Sobolev spaces. On the other hand, let $\{\lambda_n, X_n\}_{n=1}^{\infty}$ be the eigensystem of the elliptic operator $-L_x$ such that $0<\lambda_1\le\lambda_2\le\cdots\lambda_n\to\infty$ as $n\to\infty$ and $\{X_n\}\subset H^2(\Omega)\cap H_0^1(\Omega)$ forms an orthonormal basis of $L^2(\Omega)$. Then we define a fractional power space $(-L_x)^\gamma$ for $\gamma\ge 0$ by

$$(-L_x)^\gamma f := \sum_{n=1}^{\infty}\lambda_n^\gamma (f,X_n)X_n,\ \mathcal{D}((-L_x)^\gamma) = \Big\{ f\in L^2(\Omega) : \sum_{n=1}^{\infty} |\lambda_n^\gamma (f,X_n)|^2 < \infty \Big\},$$

and $\mathcal{D}((-L_x)^\gamma)$ is a Hilbert space with the norm

$$\|f\|_{\mathcal{D}((-L_x)^\gamma)} = \Big(\sum_{n=1}^{\infty} |\lambda_n^\gamma (f,X_n)|^2 \Big)^{1/2}.$$

Also we note that $\mathcal{D}((-L_x)^\gamma)\subset H^{2\gamma}(\Omega)$ for $\gamma>0$ and especially there hold $\mathcal{D}((-L_x)^0) = L^2(\Omega)$, $\mathcal{D}((-L_x)^{1/2}) = H_0^1(\Omega)$ and $\mathcal{D}((-L_x)^1) = H^2(\Omega)\cap H_0^1(\Omega)$.

For an explicit expression of the solution to the forward problem (2.1), we need to utilize the multinomial Mittag-Leffler function, which is defined as

$$E_{(\beta_1,\beta_2,\cdots,\beta_m),b}(z_1,z_2,\cdots,z_m) := \sum_{k=0}^{\infty}\sum_{k_1+\cdots+k_m=k} \frac{(k;k_1,\cdots,k_m)\Pi_{j=1}^m z_j^{k_j}}{\Gamma(b+\sum_{j=1}^m \beta_j k_j)}, \tag{2.2}$$

where $0<b<2$, $0<\beta_j<1$ and $z_j\in\mathbf{C}$ $(j=1,2,\cdots,m)$, and $(k;k_1,\cdots,k_m)$ denotes the multinomial coefficient $(k;k_1,\cdots,k_m):=\frac{k!}{k_1!\cdots k_m!}$ with $k=\sum_{j=1}^m k_j$, where k_j, $j=1,\cdots,m$, are nonnegative integers. Concerning the relation between the multinomial Mittag-Leffler functions with different parameters, we give the following lemmas.

Lemma 2.1 (see [22])**.** Let $0<b<2$, $\beta_j\in(0,1)$ $(j=1,2,\cdots,m)$, and $z_j\in\mathbf{C}$ $(j=1,2,\cdots,m)$ be fixed, there holds

$$E_{(\beta_1,\beta_2,\cdots,\beta_m),b}(z_1,z_2,\cdots,z_m)=\frac{1}{\Gamma(b)}+\sum_{j=1}^{m}z_jE_{(\beta_1,\beta_2,\cdots,\beta_m),b+\beta_j}(z_1,z_2,\cdots,z_m). \tag{2.3}$$

Lemma 2.2 (see [22])**.** Let $0<b<2$ and $0<\alpha_m<\cdots<\alpha_1<1$ be given. Assume that $\alpha_1\pi/2<\mu<\alpha_1\pi$, $\mu\le|arg(z_1)|\le\pi$ and there exists $K>0$ such that $-K\le z_j<0$ $(j=2,\cdots,m)$. Then there exists a constant $M>0$ depending only on μ,K,α_j $(j=1,2,\cdots,m)$ and b such that

$$|E_{(\alpha_1,\alpha_1-\alpha_2,\cdots,\alpha_1-\alpha_m),b}(z_1,z_2,\cdots,z_m)|\le\frac{M}{1+|z_1|}. \tag{2.4}$$

Now we give the expression and some regularities of the solution to the forward problem (2.1), also see [22,23] for detailed statements.

Lemma 2.3 (see [22,23])**.** Under the assumptions in Section 1 for Eq. (1.1), and the initial function $f(x)\in\mathcal{D}((-L_x)^\gamma)$ with some $\gamma\in[0,1]$, where we interpret $\frac{1}{1-\gamma}=\infty$ if $\gamma=1$. Then there is a unique solution $u\in C([0,T];L^2(\Omega))\cap C((0,T];H^2(\Omega)\cap H_0^1(\Omega))$ to the forward problem (2.1), which is expressed by

$$u(x,t)=\sum_{n=1}^{\infty}\Big(1-\lambda_nt^\alpha\mathbf{E}^{(n)}_{\alpha',1+\alpha}(t)\Big)(f,X_n)X_n(x), \tag{2.5}$$

where

$$\mathbf{E}^{(n)}_{\alpha',1+\alpha}(t):=E_{(\alpha,\alpha-\alpha_1,\cdots,\alpha-\alpha_m),1+\alpha}(-\lambda_nt^\alpha,-r_1t^{\alpha-\alpha_1},\cdots,-r_mt^{\alpha-\alpha_m}) \tag{2.6}$$

is a multinomial Mittag-Leffler function, λ_n is the n-th eigenvalue and $X_n(x)$ is the corresponding eigenfunction of the operator $-L_x$, and α, α_j and r_j $(j=1,\cdots,m)$ are those positive constants in (1.4).

Furthermore, there exists a constant $M=M(\Omega,L_x,P)>0$ such that

$$\|u(x,t)\|_{L^2(0,T;L^2(\Omega))}\le M\sqrt{T}\|f(x)\|_{L^2(\Omega)}. \tag{2.7}$$

Actually, $u\in L^{\frac{1}{1-\gamma}}(0,T;H^2(\Omega)\cap H_0^1(\Omega))$ and there holds

$$\|u(x,t)\|_{L^2(0,T;H^2(\Omega))}\le\frac{MT^{\frac{1+2(\gamma-1)\alpha}{2}}}{\sqrt{1+2(\gamma-1)\alpha}}\|f(x)\|_{\mathcal{D}((-L_x)^\gamma)} \tag{2.8}$$

for $\gamma>1-\frac{1}{2\alpha}$.

3 The backward problem

The backward problem is to determine the distribution $u(x,t_0)$ for $t_0 \in [0,T)$ with final observation $u(x,T)$ for given $T \in (0,\infty)$ based on the forward problem (2.1). We deal with the inversion problem of determining the initial function $f=f(x)$ by $u_T(x)=u(x,T)$. In the follows by $\|\cdot\|_2$ we denote the L^2 norm on the space Ω, and by $(\cdot,\cdot)$ also the L^2 inner production if there is no specification.

We denote $u(f)(x,t)$ as the unique solution to the forward problem (2.1) for each given $f \in L^2(\Omega)$, and we get the input-output solution's operator $f \to u(f)(x,t)$. Then an operator from the initial function to the overposed final measurement, which is called an observation operator, can be defined by

$$G: L^2(\Omega) \to L^2(\Omega), \quad G(f)(x) := u(f)(x,t)|_{t=T} = u_T(x), \tag{3.1}$$

and the backward problem is transformed to solve the operator equation (3.1).

3.1 Uniqueness and instability

In many real problems, the observations we can get are always with perturbations in which case we have to cope with noisy data, i.e., we need to solve the perturbed operator equation

$$G(f^\varepsilon)(x) = u_T^\varepsilon(x), \tag{3.2}$$

where u_T^ε are the noisy data. Noting (3.1) and (3.2) together with the expression (2.5), we need give a fine estimation for $1-\lambda_n t^\alpha \mathbf{E}^{(n)}_{\alpha',1+\alpha}$ for the purpose of estimating $\|f^\varepsilon - f\|_2$ in term of $\|u_T^\varepsilon - u_T\|_2$. For convenience we denote

$$Q_n(t) = 1-\lambda_n t^\alpha \mathbf{E}^{(n)}_{\alpha',1+\alpha}(t), \quad t>0, \quad n=1,2,\cdots, \tag{3.3}$$

where $\mathbf{E}^{(n)}_{\alpha',1+\alpha}(t)$ is defined by (2.6). The following proposition gives a fine estimate showing that $Q_n(t)$ approaches to zero not only for given $t>0$ as $n\to\infty$, but also for given $n\in\mathbf{N}$ as $t\to\infty$.

Proposition 3.1. For $Q_n(t)$ given by (3.3), there holds

$$|Q_n(t)| \le \sum_{j=1}^{m} \frac{M(1+r_j t^{\alpha-\alpha_j})}{1+\lambda_n t^\alpha}, \tag{3.4}$$

for $0<t<\infty$ and $n=1,2,\cdots$, where $0<\alpha_m<\cdots<\alpha_1<\alpha<1$, and r_j, $j=1,\cdots,m$ are positive constants, and $M>0$ is a positive constant.

Proof. Denote $z_1=-\lambda_n t^\alpha$, $z_2=-r_1 t^{\alpha-\alpha_1},\cdots$, $z_{m+1}=-r_m t^{\alpha-\alpha_m}$, we rewrite $Q_n(t)$ by

$$Q_n(t)=1+z_1 E_{(\alpha,\alpha-\alpha_1,\cdots,\alpha-\alpha_m),1+\alpha}(z_1,z_2,\cdots,z_{m+1}). \tag{3.5}$$

By Lemma 2.1 (where $b=1$) there holds

$$\begin{aligned}&E_{(\alpha,\alpha-\alpha_1,\cdots,\alpha-\alpha_m),1}(z_1,z_2,\cdots,z_{m+1})\\ =&\frac{1}{\Gamma(1)}+z_1 E_{(\alpha,\alpha-\alpha_1,\cdots,\alpha-\alpha_m),1+\alpha}(z_1,z_2,\cdots,z_{m+1})\\ &+\sum_{j=2}^{m+1} z_j E_{(\alpha,\alpha-\alpha_1,\cdots,\alpha-\alpha_m),1+\alpha-\alpha_{j-1}}(z_1,z_2,\cdots,z_{m+1}).\end{aligned} \tag{3.6}$$

Noting $\Gamma(1)=1$, we get by (3.5)

$$\begin{aligned}Q_n(t)=&1+z_1 E_{(\alpha,\alpha-\alpha_1,\cdots,\alpha-\alpha_m),1+\alpha}(z_1,z_2,\cdots,z_{m+1})\\ =&E_{(\alpha,\alpha-\alpha_1,\cdots,\alpha-\alpha_m),1}(z_1,z_2,\cdots,z_{m+1})\\ &+\sum_{j=2}^{m+1}(-z_j) E_{(\alpha,\alpha-\alpha_1,\cdots,\alpha-\alpha_m),1+\alpha-\alpha_{j-1}}(z_1,z_2,\cdots,z_{m+1}).\end{aligned} \tag{3.7}$$

Since $r_j>0$ and $\alpha-\alpha_j>0$ ($j=1,2,\cdots,m$), using Lemma 2.2 follows that

$$|E_{(\alpha,\alpha-\alpha_1,\cdots,\alpha-\alpha_m),1}(z_1,z_2,\cdots,z_{m+1})|\le\frac{M}{1+|z_1|},$$

and

$$\Big|\sum_{j=2}^{m+1}(-z_j)E_{(\alpha,\alpha-\alpha_1,\cdots,\alpha-\alpha_m),1+\alpha-\alpha_{j-1}}(z_1,z_2,\cdots,z_{m+1})\Big|\le\sum_{j=2}^{m+1}|z_j|\frac{M}{1+|z_1|},$$

where $M>0$ is a positive constant. Thus we get

$$|Q_n(t)|\le\frac{M}{1+|z_1|}\left(1+\sum_{j=2}^{m+1}|z_j|\right). \tag{3.8}$$

Concerning with the notations of z_j ($j=1,2,\cdots,m+1$), the assertion (3.4) is valid. □

We also need the positivity and monotonicity for the multivariate Mittag-Leffler function $\mathbf{E}^{(n)}_{\alpha',1+\alpha}(t)$ given by (2.6).

Lemma 3.1 (see [2,22])**.** Let $0<\alpha_m<\cdots<\alpha_1<\alpha<1$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(t^\alpha\mathbf{E}^{(n)}_{\alpha',1+\alpha}(t)\big)=t^{\alpha-1}\mathbf{E}^{(n)}_{\alpha',\alpha}(t)>0, \tag{3.9}$$

for $t>0$, where $\mathbf{E}^{(n)}_{\alpha',1+\alpha}(t)$ is the multivariate Mittag-Leffler function given by (2.6), and then there holds $\mathbf{E}^{(n)}_{\alpha',1+\alpha}(t)>0$ for $t>0$.

Based on the above discussions we have

Proposition 3.2. The function $Q_n(t)$ defined by (3.3) is strictly decreasing on $t>0$ for $n=1,2,\cdots$, and there holds $0<Q_n(t)<1$.

Proof. By Lemma 3.1, we get

$$\frac{d}{dt}(Q_n(t))=-\lambda_n t^{\alpha-1}\mathbf{E}^{(n)}_{\alpha',\alpha}(t)<0, \tag{3.10}$$

for $t>0$ and $n\in\mathbf{N}$, which shows that $Q_n(t)$ is strictly decreasing on $t>0$. Noting that $Q_n(0)=\lim_{t\to 0}Q_n(t)=1$, and $\lim_{t\to\infty}Q_n(t)=0$ by Proposition 3.2, we arrive at the assertion of this proposition. $\square$

Now we give the unique solvability of the backward problem.

Theorem 3.1. Let the additional observation time be $T<\infty$, then the backward problem is uniquely solvable, and there is

$$f(x)=\sum_{n=1}^{\infty}\frac{(u_T,X_n)}{Q_n(T)}X_n(x). \tag{3.11}$$

Proof. Let $f_n=(f,X_n)_2$, $n=1,2,\cdots$. By (2.5) and the definition of $G(f)$ we have

$$G(f)(x)=\sum_{n=1}^{\infty}f_nQ_n(T)X_n(x)=u_T(x). \tag{3.12}$$

By the orthonormality of $\{X_n\}$ there holds

$$f_nQ_n(T)=(u_T,X_n), \quad n=1,2,\cdots. \tag{3.13}$$

Since $0<Q_n(T)<1$ by Proposition 3.2 we get

$$f_n=\frac{(u_T,X_n)}{Q_n(T)} \tag{3.14}$$

and then the expression (3.11) is valid.

Furthermore, by the linearity of the mapping $u_T\to f$, the solution is unique for given $T>0$. Actually in the case of $u_T(x)=0$, $x\in\Omega$, we have $(f,X_n)=f_n=0$ for $n=1,2,\cdots$, by (3.14). Henceforth there holds $f(x)=0$, a.e. $x\in\Omega$ by the complete orthonormality of X_n in $L^2(\Omega)$. The proof is over. $\square$

Now we give the instability of the backward problem.

Theorem 3.2. The backward problem for the multi-term time fractional diffusion equation is unstable and ill-posed, and the degree of ill-posedness is $\mathcal{O}(n^{-\frac{2}{d}})$, where $d\geq 1$ is the dimension of the space domain.

Proof. Let $f_n^\varepsilon = (f^\varepsilon, X_n)$, $n = 1,2,\cdots$. Noting (3.2) there is

$$G(f^\varepsilon)(x) = \sum_{n=1}^{\infty} f_n^\varepsilon Q_n(T) X_n(x) = u_T^\varepsilon(x). \tag{3.15}$$

Together with (3.12), we have

$$\sum_{n=1}^{\infty} (f_n^\varepsilon - f_n) Q_n(T) X_n(x) = u_T^\varepsilon - u_T. \tag{3.16}$$

By the orthonormality of $\{X_n\}$ there holds

$$(f_n^\varepsilon - f_n) Q_n(T) = (u_T^\varepsilon - u_T, X_n), \quad n = 1,2,\cdots. \tag{3.17}$$

Let

$$u_T^\varepsilon - u_T = \frac{1}{\sqrt{\lambda_n}} X_n(x),$$

there is

$$\|u_T^\varepsilon - u_T\|_2 = \frac{1}{\sqrt{\lambda_n}} \to 0$$

as $n \to \infty$. On the other hand, using the estimate (3.4), we get

$$\begin{aligned} |f_n^\varepsilon - f_n| &= \frac{|(u_T^\varepsilon - u_T, X_n)_2|}{|Q_n(T)|} = \frac{1}{\sqrt{\lambda_n}|Q_n(T)|} \geq \frac{1}{M\sqrt{\lambda_n}\sum\limits_{j=1}^{m} \frac{1+r_j T^{\alpha-\alpha_j}}{1+\lambda_n T^\alpha}} \\ &\geq \frac{\sqrt{\lambda_n}}{M\sum\limits_{j=1}^{m}\left(\frac{1}{T^\alpha} + \frac{r_j}{T^{\alpha_j}}\right)} = \mathcal{O}(\sqrt{\lambda_n}) \to \infty, \quad (n \to \infty), \end{aligned} \tag{3.18}$$

which shows that the backward problem is unstable and ill-posed.

By (3.11), it can be seen that the ill-posedness of the backward problem lies in the term of $Q_n(T)$ as $n \to \infty$. By Proposition 3.1 and (3.4) there holds

$$Q_n(T) \sim \sum_{j=1}^{m} \frac{M(1+r_j T^{\alpha-\alpha_j})}{1+\lambda_n T^\alpha} \sim \mathcal{O}(\lambda_n^{-1}), \quad (n \to \infty). \tag{3.19}$$

By ordinary estimation for eigenvalues of mathematical equations, there holds (see, e.g., [7])

$$C_0 n^{\frac{2}{d}} \leq \lambda_n \leq C_1 n^{\frac{2}{d}}, \tag{3.20}$$

where C_0, C_1 are positive constants, and $d \geq 1$ is the dimension of the space domain. Combining (3.19) with (3.20), we have $Q_n(T) \sim \mathcal{O}(n^{-\frac{2}{d}})$ as $n \to \infty$. The proof of the theorem is completed. □

4 Existence of minimum to the cost functional

From the view point of optimality, it is meaningful to seek for an approximate solution which is a minimizer of a cost functional of the unknown between the additional measurements and the outputs. For a positive constant $E>0$, we denote

$$S_E=\{f\in L^2(\Omega):\|f\|_2\le E\}$$

as the admissible set of the unknowns. Then by (3.1) we define an error functional for $f\in S_E$ by

$$J(f)=\|G(f)(x)-u_T(x)\|_2=\left(\int_\Omega [G(f)(x)-u_T(x)]^2dx\right)^{1/2}. \tag{4.1}$$

It is noted that the input-output solution's operator $f\to G(f)$ is linear due to the linear equation and the linear initial-boundary-value conditions, and then the functional $J(f)$ is convex. Actually, for any $\mu\in(0,1)$ and $f_1,f_2\in S_E$, utilizing the linearity of $G(f)$, there holds

$$\begin{aligned}J(\mu f_1+(1-\mu)f_2)&=\|G(\mu f_1+(1-\mu)f_2)-u_T(x)\|_2\\&=\|\mu G(f_1)+(1-\mu)G(f_2)-u_T(x)\|_2\\&=\|\mu[G(f_1)-u_T(x)]+(1-\mu)[G(f_2)-u_T(x)]\|_2\\&\le\mu\|G(f_1)-u_T(x)\|_2+(1-\mu)\|G(f_2)-u_T(x)\|_2\\&=\mu J(f_1)+(1-\mu)J(f_2).\end{aligned} \tag{4.2}$$

Furthermore, by Lemma 2.3 and the solution's expression, we get the Lipschitz continuity of the operator G.

Lemma 4.1. Based on the forward problem (2.1), suppose that $f_i(x)\in L^2(\Omega)$ $(i=1,2)$, and $G(f_i)(x)=u_T^i(x)$, $i=1,2$ are the corresponding outputs, then there exists a constant $M=M(\Omega,P,L_x)>0$ such that

$$\|G(f_1)-G(f_2)\|_2\le M\|f_1-f_2\|_2. \tag{4.3}$$

Proof. Suppose that u_1, u_2 are two solutions corresponding to the initial-value function f_1 and f_2 respectively. Let $U=u_1-u_2$, and $U=U(x,t)$ satisfies the following initial boundary value problem by the linearity

$$\begin{cases}P({}_0^CD_t)U-L_xU=0, & (x,t)\in\Omega_T,\\ U(x,0)=f_1-f_2, & x\in\Omega,\\ U(x,t)=0, & (x,t)\in\partial\Omega_T.\end{cases} \tag{4.4}$$

By (2.5) in Lemma 2.3, we have

$$U(x,t)=\sum_{n=1}^{\infty}\left(1-\lambda_n t^\alpha \mathbf{E}^{(n)}_{\alpha',1+\alpha}(t)\right)(f_1-f_2,X_n)X_n(x). \tag{4.5}$$

Thanks to the property of the multinomial Mittag-Leffler function in Lemma 2.2, there exists a positive $M>0$ such that

$$|\mathbf{E}^{(n)}_{\alpha',1+\alpha}(t)|\le \frac{M}{1+\lambda_n t^{\alpha}},$$

and there holds

$$|1-\lambda_n t^{\alpha}\mathbf{E}^{(n)}_{\alpha',1+\alpha}(t)|\le 1+\lambda_n t^{\alpha}\frac{M}{1+\lambda_n t^{\alpha}}\le 1+M. \tag{4.6}$$

Thus we get by the orthonormality of $X_n(x)$ in $L^2(\Omega)$

$$\|U(\cdot,t)\|_2\le M\|f_1-f_2\|_2, \quad 0<t\le T, \tag{4.7}$$

and taking $t=T$ follows that the assertion of this proposition is valid. □

Therefore, the functional $J(f)$ is weakly lower semi-continuous. Actually, we can prove the following assertion.

Proposition 4.1. The functional $\hat{J}(f):=\|G(f)-u_T\|_2^2$ is weakly lower semi-continuous on $f\in L^2(\Omega)$.

Proof. Suppose that f_n is weakly convergent to f_0 in $L^2(\Omega)$ as $n\to\infty$. Then we have $(f_n,f_0)\to(f_0,f_0)$ as $n\to\infty$, here $(\cdot,\cdot)$ denotes the normal L^2 inner product. Furthermore, by the continuity of the operator $f\to G(f)$ by Lemma 5 we have

$$(G(f_n)-u_T,G(f_0)-u_T)\to(G(f_0)-u_T,G(f_0)-u_T), \quad n\to\infty. \tag{4.8}$$

On the other hand, there holds

$$\begin{aligned}0\le&((G(f_n)-u_T)-(G(f_0)-u_T),(G(f_n)-u_T)-(G(f_0)-u_T))\\ =&\|G(f_n)-u_T\|_2^2-2(G(f_n)-u_T,G(f_0)-u_T)+\|G(f_0)-u_T\|_2^2.\end{aligned}$$

Henceforth we get

$$\|G(f_n)-u_T\|_2^2\ge 2(G(f_n)-u_T,G(f_0)-u_T)-\|G(f_0)-u_T\|_2^2, \tag{4.9}$$

which implies that

$$\lim_{n\to\infty}\inf\|G(f_n)-u_T\|_2^2\ge\|G(f_0)-u_T\|_2^2. \tag{4.10}$$

This ends the proof by the definition of weakly lower semi-continuous functional. □

For the existence of a minimum to the functional $J(f)$ on S_E, we still need the following lemma.

Lemma 4.2. Suppose that X is a reflexive Banach space, $K\subset X$ is closed, convex and bounded, and the functional $F:K\to\mathbf{R}$ is weakly lower semi-continuous on K. Then there exists $x_0\in K$ such that $F(x_0)\le F(x)$ for any $x\in K$.

Proof. (see [15, 20]) Firstly we prove that the functional F is bounded below. Otherwise, there exists series $\{x_n\}\subset K$ such that

$$\lim_{n\to\infty} F(x_n)=-\infty. \tag{4.11}$$

Since K is bounded, and so $\{x_n\}$ is uniformly bounded, we deduce that there exists subseries $\{x_{n'}\}$ such that $x_{n'}\to x_0$ weakly in K. Noting that K is closed and convex, we have $x_0\in K$. Then there holds

$$F(x_0)\le\lim_{n\to\infty}\inf F(x_{n'}) \tag{4.12}$$

by the weakly lower semi-continuity of the functional F on K, which is a contradiction with (4.11).

Next, we prove the functional F takes a minimum on K.

Suppose that $m=\inf_{x\in K}F(x)$, and there exists $\{x_n\}\subset K$ such that $F(x_n)\to m$ as $n\to\infty$. Hence there exists a subseries $\{x_{n'}\}\subset\{x_n\}$ such that $x_{n'}\to x_0$ weakly in K, and $F(x_{n'})\to m$. By the weakly lower semi-continuity of F we get

$$F(x_0)\le\lim_{n'\to\infty}\inf F(x_{n'})\le\lim_{n'\to\infty}F(x_{n'})=m. \tag{4.13}$$

On the other hand, we have $F(x_0)\ge m$ by the definition of m. Thus there must have $F(x_0)=m$. The proof is over. $\square$

Based on the above discussions, we deduce that the functional $J(f)$ defined by (4.1) can take a minimum on S_E since $S_E\subset L^2(\Omega)$ is a bounded, closed and convex set, and the functional is weakly lower semi-continuous on S_E.

Theorem 4.1. For the functional $J(f)$ defined by (4.1), there exists a minimal point $f^*\in S_E$ such that

$$J(f^*)=\min_{f\in S_E}J(f). \tag{4.14}$$

In the follows, we consider numerical algorithms for solving the minimization problem. Noting that the minimization problem involves in the forward problem, it is difficult to get an effective solution utilizing general optimal methods, and regularization strategy should be considered with. Combining with homotopy idea, we define a parameterized cost functional with regularizing term instead of (4.1)

$$\bar{J}(f)=(1-\mu)\|G(f)(x)-u_T(x)\|_2^2+\mu\|f\|_2^2, \tag{4.15}$$

where $\mu\in(0,1)$ is the homotopy parameter. With the same method as used in the above, we can deduce that the functional $\bar{J}(f)$ also has a minimum on S_E. The numerical algorithm and numerical inversions based on (4.15) are given in the next section.

5 The inversion algorithm and numerical inversion

For $f(x)\in S_E$, suppose that $\{\varphi_s(x),\ s=1,2,\cdots,\infty\}$ is a set of basis functions in $L^2(\Omega)$, and there is

$$f(x)\approx f^S(x)=\sum_{s=1}^{S}a_s\varphi_s(x), \tag{5.1}$$

where $f^S(x)$ is a S-dimensional approximate solution to $f(x)$, and $S\geq 1$ is the truncated level of $f(x)$, and a_s, $s=1,2,\cdots,S$ are the expansion coefficients. It is convenient to set a finite-dimensional space given as

$$\Phi^S=\mathrm{span}\{\varphi_1,\varphi_2,\cdots,\varphi_S\},$$

and an S-dimensional vector $\mathbf{a}=(a_1,a_2,\cdots,a_S)\in\mathbf{R}^S$. We identify an approximation $f^S\in\Phi^S$ with a vector $\mathbf{a}\in\mathbf{R}^S$ if there is no possibility of confusion, and denote

$$\|\mathbf{a}\|=\sqrt{{a_1}^2+a_2^2+\cdots+a_S^2}.$$

For any given $\mathbf{a}\in\mathbf{R}^S$, the unique solution, denoted as $u(x,t;\mathbf{a})$, to the forward problem can be worked out, and the so-called computational output by setting $t=T$ is obtained, which is given by $G(\mathbf{a})(x,T):=u(x,T;\mathbf{a})$ for $x\in\Omega$. Noting (4.15), it is transformed to solve the minimization problem for solving the inverse problem

$$\min_{\mathbf{a}\in\mathbf{R}^S}\{(1-\mu)\|u(x,T;\mathbf{a})-u_T(x)\|_2^2+\mu\|\mathbf{a}\|^2\}. \tag{5.2}$$

As for solving the nonlinear minimization problem (4.9), we employ the homotopy regularization algorithm and refer to [41] and [10] for detailed procedures, where we utilize a Sigmoid-type function depending on the number of iterations as the homotopy parameter given as:

$$\mu=\mu(j)=\frac{1}{1+\exp(\beta(j-j_0))}, \tag{5.3}$$

where j is the number of iterations of the algorithm, and j_0 is the preestimated number of iterations, $\beta>0$ is the adjust parameter. In addition, we utilize the implicit finite difference scheme (see e.g., [18]) to work out the solution of the forward problem, and choose $j_0=5$ and $\beta=0.8$ in (5.3) if there is no specification, and all computations are performed on a PC of Lenovo.

5.1 Inversion for $f(x)=\sin(x)$

We firstly consider the initial-boundary-value problem for 1D three-term time fractional diffusion equation, where we set $\Omega=(0,\pi)$ and $T=1$

$$\begin{cases} \dfrac{\partial^\alpha u}{\partial t^\alpha}+r_1\dfrac{\partial^{\alpha_1} u}{\partial t^{\alpha_1}}+r_2\dfrac{\partial^{\alpha_2} u}{\partial t^{\alpha_2}}=D\dfrac{\partial^2 u}{\partial x^2}, & (x,t)\in\Omega\times(0,1), \\ u(x,0)=f(x), & x\in\overline{\Omega}, \\ u(0,t)=0, \quad u(\pi,t)=0, & 0<t\le 1. \end{cases} \tag{5.4}$$

Let the fractional orders $\alpha=0.9$, $\alpha_1=0.7$ and $\alpha_2=0.5$, and the diffusion coefficient $D=0.8$, and the coefficients $r_1=1$ and $r_2=0.2$. Assume that

$$f(x)=\sin(x), \quad x\in\overline{\Omega}, \tag{5.5}$$

by which we obtain the solution of the forward problem and then we get the additional data at $T=1$, and further we utilize the homotopy regularization algorithm to reconstruct it in suitable approximate spaces. We here choose polynomial space as the approximate space of $f(x)$, and the exact solution to the backward problem in the approximate space can be represented by

$$\mathbf{a}=\Big(0,1,0,-\frac{1}{3!},0,\frac{1}{5!},0,-\frac{1}{7!},\cdots\Big). \tag{5.6}$$

The inversion results in $\Phi^S=\text{span}\{1,x,\cdots,x^{S-1}\}$ are listed in Table 1, where S is the dimension of the approximate space, and $\mathbf{a}^{inv}$ is the inversion solution, Err denotes the relative error in the solutions, and j is the number of iterations. Moreover, the exact and the inversion initial functions in Φ^S for $S=4,6,8,10$ are plotted in Figs. 1(a), (b), (c) and (d) respectively.

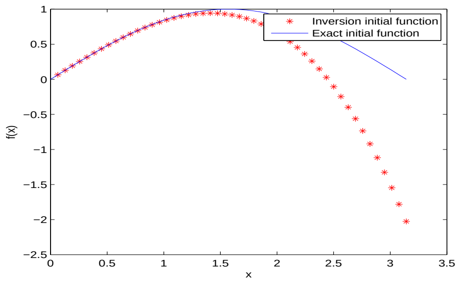
Table 1: The inversion results with dimensions of the approximate space in Subsection 5.1.

S	$\mathbf{a}^{inv}$	Err	j
4	(1.151e-11, 1.000, 2.230e-11, -0.1667)	9.424e-1	20
6	(2.614e-11, 1.000, 2.606e-10, -0.1667, 6.777e-11, 8.333e-3)	2.089e-1	29
8	(3.936e-11, 1.000, 1.178e-9, -0.1667, 1.386e-9, 8.333e-3, 1.333e-10, -1.984e-4)	2.698e-2	35
10	(-1.359e-12, 1.000, -1.141e-10, -0.1667, -4.145e-10, 8.333e-3, -1.826e-10, -1.984e-4, -9.612e-12, 2.756e-6)	2.294e-3	43

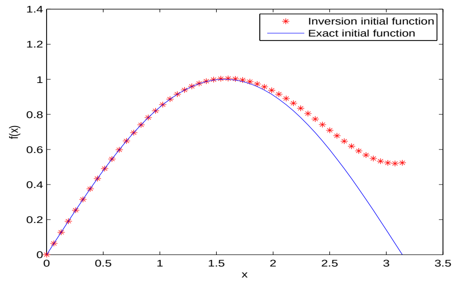
From Table 1 and Fig. 1, it can be seen that the relative errors in the solutions become small when S goes to large, and the inversion solutions give good approximations to the exact initial function when $S\ge 10$, which shows that the inversion algorithm is convergent in the meaning of finite-dimensional approximation.

5.2 Inversion for $f(x)=\sin(3x)$

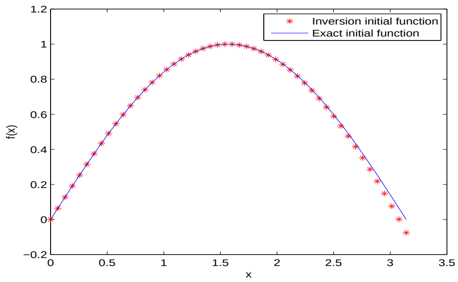
In this subsection, we consider the backward problem for the two-term time fractional diffusion equation but with a bit complicated initial function. Let $\Omega=(0,2\pi)$ and $T=1$,



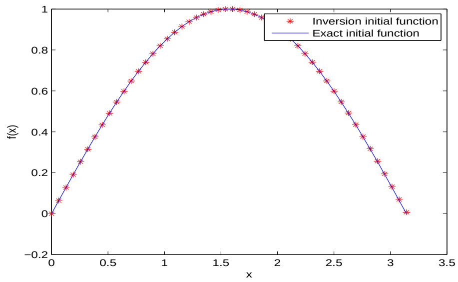
(a) $S=4$



(b) $S=6$



(c) $S=8$



(d) $S=10$

Figure 1: The exact and inversion initial functions in Φ^S in Subsection 5.1.

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the forward problem is given by

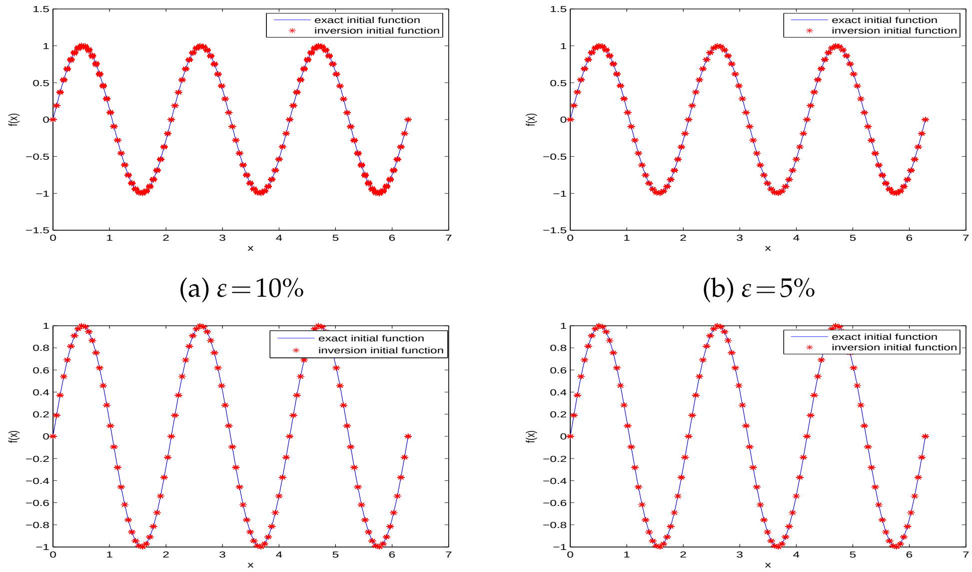
$$\begin{cases} \dfrac{\partial^\alpha u}{\partial t^\alpha}+\dfrac{1}{2}\dfrac{\partial^{\alpha_1} u}{\partial t^{\alpha_1}}=\dfrac{1}{2}\dfrac{\partial^2 u}{\partial x^2}, & (x,t)\in\Omega\times(0,1),\\ u(x,0)=\sin(3x), & x\in\overline{\Omega},\\ u(0,t)=0,\quad u(2\pi,t)=0, & 0<t\le 1. \end{cases} \tag{5.7}$$

We choose $\Phi^4=\text{span}\{1,\sin(x),\sin(2x),\sin(3x)\}$ as the approximate space for the initial function, and perform the inversion using random noisy data. Suppose that the additional information is polluted which is described by

$$u_T^\varepsilon(x)=u_T(x)(1+\varepsilon\zeta), \tag{5.8}$$

where $\varepsilon>0$ is noise level, and ζ is a random noise ranged in $[-1,1]$.

Without loss of generality, we take $\alpha=0.7$ and $\alpha_1=0.3$ as example, and repeat inversion processes ten times for each given noise level and take the average of gained inversion results. The average results are listed in Table 2, where $\overline{\mathbf{a}}^{inv}$ is the average inversion solution of the ten-time inversions, $\overline{Err}$ is the average relative error in the solutions and $\bar{j}$ denotes the average number of the iterations. Moreover, the ten-time inversion results



(a) $\varepsilon=10\%$ (b) $\varepsilon=5\%$

(c) $\varepsilon=1\%$ (d) $\varepsilon=0.1\%$

Figure 2: The exact and 10-time inversion solutions with noisy data in Subsection 5.2.

Table 2: The average inversion results with noisy data in Subsection 5.2.

ε	$\overline{\mathbf{a}}^{inv}$	$\overline{Err}$	$\overline{j}$
10%	(-2.09335e-4, 1.70645e-4, 1.35274e-3, 0.996728)	3.557e-3	12
5%	(-1.04668e-4, 8.53227e-5, 6.76372e-4, 0.998364)	1.778e-3	12
1%	(-2.09335e-5, 1.70645e-5, 1.35274e-4, 0.999673)	3.557e-4	12
0.1%	(-2.09335e-6, 1.70645e-6, 1.35274e-5, 0.999967)	3.557e-5	12

and the exact initial function with different noise levels are plotted in Figs. 2(a), (b), (c) and (d) respectively.

From the above two examples it can be seen that the approximate space plays an important role in the performance of the inversion algorithm. The inversion results in Subsection 5.2 are much better than those in Subsection 5.1 although the initial function in Ex.2 is more complicated than that in Subsection 5.1. Since the reconstructed initial functions are trigonometric functions, it is better to choose trigonometric functions space as the approximate space than to utilize polynomials space as observed in the above.

5.3 Inversion for piecewise function

In this subsection, we consider the backward problem for the three-term time fractional diffusion equation but with piecewise initial function. Let $\Omega=(0,1)$ and $T=1$, the forward

Table 3: The average inversion results with noisy data in Subsection 5.3.

ε	$\overline{\mathbf{a}}^{inv}$	$\overline{Err}$	$\overline{j}$
1%	(1.0070, 0.9911, 1.2577, 0.2723, 1.9531, -0.9400)	7.9262e-3	30
0.1%	(1.0007, 0.9991, 1.3858, 0.0272, 1.9953, -0.9940)	7.9262e-4	30
0.01%	(1.0001, 0.9999, 1.3986, 0.0027, 1.9995, -0.9994)	7.9262e-5	30
0.001%	(1.0000, 1.0000, 1.3999, 0.0002, 2.0000, -0.9999)	7.9262e-6	30

problem is given by

$$\begin{cases} \dfrac{\partial^\alpha u}{\partial t^\alpha}+r_1\dfrac{\partial^{\alpha_1} u}{\partial t^{\alpha_1}}+r_2\dfrac{\partial^{\alpha_2} u}{\partial t^{\alpha_2}}=D\dfrac{\partial^2 u}{\partial x^2}, & (x,t)\in\Omega\times(0,1),\\ u(x,0)=f(x), & x\in\overline{\Omega},\\ u(0,t)=0, u(1,t)=0, & 0<t\le 1. \end{cases} \tag{5.9}$$

Set the fractional orders $\alpha=0.8$, $\alpha_1=0.6$ and $\alpha_2=0.4$, and the diffusion coefficient $D=0.5$, and the coefficients $r_1=0.8$ and $r_2=0.5$. Let the exact initial function be

$$f(x)=\begin{cases} 1+x, & 0\le x<0.4,\\ 1.4, & 0.4\le x\le 0.6,\\ 2-x, & 0.6<x\le 1. \end{cases} \tag{5.10}$$

We utilize the homotopy regularization algorithm to reconstructed it in the piecewise linear polynomial space. The exact solution of the backward problem in the linear polynomial space is represented by $\mathbf{a}=(1,1,1.4,0,2,-1)$. The average computational results of ten-time continuous inversions are listed in Tables 3, where ε, $\overline{\mathbf{a}}^{inv}$, $\overline{Err}$ and $\overline{j}$ denote the same meanings as in the above. Moreover, the exact and ten-time average inversion initial functions with with different noise levels are plotted in Figs. 3(a), (b), (c) and (d) respectively.

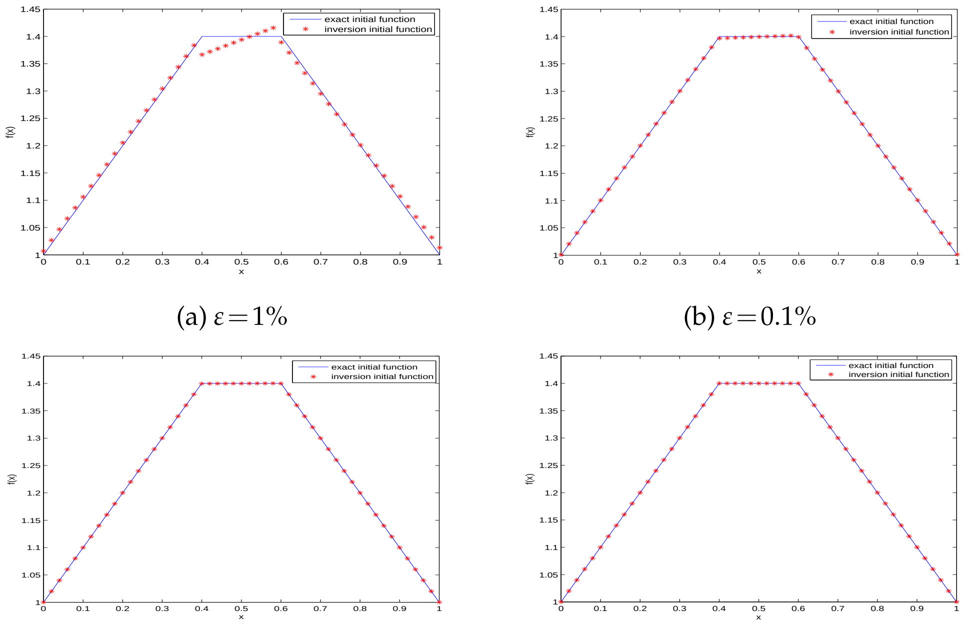
5.4 Inversion in 2D case

Let $\Omega=(0,1)\times(0,1)$, and $\Omega_T=\Omega\times(0,T)$, we consider the backward problem for the 2D two-term time-fractional diffusion equation. The forward problem is given by

$$\begin{cases} \dfrac{\partial^\alpha u}{\partial t^\alpha}+r_1\dfrac{\partial^{\alpha_1} u}{\partial t^{\alpha_1}}=D\Delta u, & (x,y,t)\in\Omega_T,\\ u(x,y,0)=f(x,y), & (x,y)\in\overline{\Omega},\\ u(x,y,t)=0, & (x,y)\in\partial\Omega,\quad t\in(0,T], \end{cases} \tag{5.11}$$

where $\partial\Omega$ is the boundary of domain Ω. We set the exact initial function be

$$f(x,y)=xy(x-1)(y-1), \tag{5.12}$$



(a) $\varepsilon=1\%$ (b) $\varepsilon=0.1\%$

(c) $\varepsilon=0.01\%$ (d) $\varepsilon=0.001\%$

Figure 3: The exact and average inversion solutions with noisy data in Subsection 5.3.

and we take $D=0.8$, $r_1=0.5$ and $T=1$, and the fractional orders $\alpha=0.8$ and $\alpha_1=0.5$ respectively, and we also choose polynomial space as the approximate space in performance of the inversion algorithm. By (5.12), the exact solution of the backward problem in $\Phi^7=\text{span}\{1,x,y,xy,xy^2,x^2y,x^2y^2\}$ can be represented by

$$\mathbf{a}=(0,0,0,1,-1,-1,1). \tag{5.13}$$

As done in the above, the average inversion results by ten-time continuous inversions with different noise levels are listed in Table 4, where ε, $\overline{\mathbf{a}}^{inv}$, $\overline{Err}$ and $\overline{j}$ denote the same meanings as in the above. The exact and the average inversion initial functions with the noise level $\varepsilon=10\%$ are plotted in Fig. 4, respectively.

From Tables 2-4, and Figs. 2-4, it can be seen that the inversion solutions with random noisy data give good approximations to the exact solutions as the noise level goes

Table 4: The average inversion results with noisy data in Subsection 5.4.

ε	$\overline{\mathbf{a}}^{inv}$	$\overline{Err}$	$\overline{j}$
10%	(-2.140e-3, 1.037e-2, 1.574e-3, 1.007, -1.021, -1.064, 1.080)	3.69e-2	33.6
5%	(-1.070e-3, 5.184e-3, 7.869e-4, 1.003, -1.011, -1.032, 1.040)	1.85e-2	33.6
1%	(-2.140e-4, 1.037e-3, 1.574e-4, 1.001, -1.002, -1.006, 1.008)	3.69e-3	33.1
0.1%	(-2.140e-5, 1.037e-4, 1.574e-5, 1.000, -1.000, -1.001, 1.001)	3.69e-4	33.0

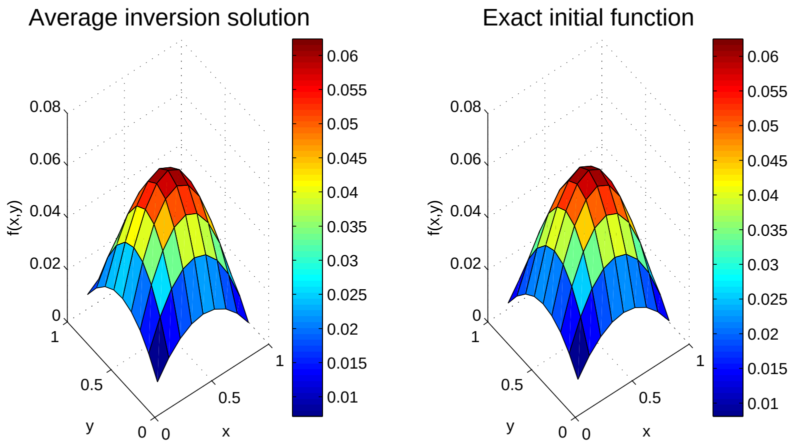


Figure 4: The exact and average inversion solutions with $\varepsilon = 10\%$ in Subsection 5.4.

to small. Although the inversion results in Subsection 5.3 with large noises are not so good as those in Subsection 5.2 and Subsection 5.4, they are still satisfactory and show a numerical stability for the backward problem.

Remark 5.1. It seems to be unbelievable that the inversions results are very good even using large noisy data especially in Subsection 5.2 and Subsection 5.4. One reason maybe lies in the mildly ill-posedness of the inverse problem by Theorem 3.2 and the degree of the ill-posedness is in the order of polynomial. Another reason is that for the unknown initial distribution in the trigonometric/polynomial form, we utilize the corresponding trigonometric/polynomial functions as the basis functions of the approximate space. This is an important aspect for implementation of the inversion algorithm. If choosing polynomials as the basis functions in Subsection 5.2, or choosing trigonometric functions as the basis functions in Subsection 5.4, we can also perform the inversions but the inversion results are not so good as given in the above.

6 Conclusions

The backward problem of determining the initial function using final observations for the multi-term time-fractional diffusion equation is investigated. Based on the analytical properties of the multivariate Mittag-Leffler function, the backward problem is proved to be uniquely solvable. On the other hand, the backward problem is unstable not only for $T \to \infty$ but also for finite $T > 0$, however, its instability is of moderate ill-posedness which is verified by the numerical inversions with noisy data. Numerical inversions for piece-wise continuous and discontinuous functions for the backward problem using the data with large noises are our task in the near future.

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