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ON SOME GENERALIZED DIFFERENCE PARANORMED SEQUENCE SPACES ASSOCIATED WITH MULTIPLIER SEQUENCE DEFINED BY MODULUS FUNCTION

Binod Chandra Tripathy

(*Institute of Advanced Study in Science and Technology, India*)

Prabhat Chandra

(*Patna University, India)*

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Abstract. In this article we introduce the paranormed sequence spaces (f,Λ,Δ_m,p), $c_0(f,\Lambda,\Delta_m,p)$ and $\ell_\infty(f,\Lambda,\Delta_m,p)$, associated with the multiplier sequence $\Lambda=(\lambda_k)$, defined by a modulus function f. We study their different properties like solidness, symmetricity, completeness etc. and prove some inclusion results.

Key words: *paranorm, solid space, symmetric space, difference sequence, modulus function, multiplier seuence*

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1 Introduction

Throughout the article w, c, c_0, ℓ_∞ denote the spaces of *all*, *convergent*, *null* and *bounded* sequences, respectively. The zero sequence is denoted by $\theta=(0,\ 0,\ 0,\cdots)$. The scope for the studies on sequence spaces was extended on introducing the notion of an associated multiplier sequence. S. Goes and G. Goes in [3] defined the differentiated sequence space dE and the integrated sequence space $\int E$ for a given sequence space E, by using the multiplier sequence (k^{-1}) and (k), respectively. P.K. Kamthan in [4] used $(k!)$ as the multiplier sequence for studying some sequence spaces. We shall use a general multiplier sequence $\Lambda=(\lambda_k)$ for our study.

The notion of difference sequence was introduced by H. Kizmaz in [5] as follows:

$$Z(\Delta) = \{(x_k) \in w : (\Delta x_k) \in \mathbf{Z}\},$$

for $Z = c, c_0$ and ℓ_∞, where $\Delta x_k = x_k - x_{k+1}$, for all $k \in N$.

It was further generalized in [12] as follows:

$$Z(\Delta_m) = \{(x_k) \in w : (\Delta_m x_k) \in \mathbf{Z}\},$$

for $Z = c, c_0$ and ℓ_∞, where $\Delta_m x_k = x_k - x_{k+m}$, for all $k \in \mathbf{N}$.

Throughout the article $p = (p_k)$ is a sequence of strictly positive real numbers. The notion of paranormed sequences was studied by [10] at the initial stage. It was further investigated by [6], [7], [11], [13] and many others.

The notion of modulus function was introduced by Nakano in [8]. It was further investigated with applications to sequence spaces by [1], [9] and many others.

Remark 1.1. It is well known that $\ell_\infty(p) = \ell_\infty$, $c(p) = c$ and $c_0(p) = c_0$ if and only if $0 < h = \inf p_k \leq H = \sup p_k < \infty$, (one may refer to [6] and [7]).

2 Definitions and Preliminaries

Definition 2.1. A modulus f is a mapping from $[0,\infty)$ into $[0,\infty)$ such that

(i) $f(x) = 0$ if and only if $x = 0$;

(ii) $f(x+y) \leq f(x) + f(y)$;

(iii) f is increasing;

(iv) f is continuous from the right at 0.

Hence f is continuous everywhere in $[0,\infty)$.

Definition 2.2. A sequence space E is said to be solid (or normal) if $(\alpha_k x_k) \in E$, whenever $(x_k) \in E$ and for all sequences (α_k) of scalars with $|\alpha_k| \leq 1$, for all $k \in \mathbf{N}$.

Definition 2.3. A sequence space E is said to be monotone if it contains the canonical preimages of all its step spaces.

Remark 2.1. From the above definitions it is clear that " A sequence space E is solid implies that E is monotone".

Definition 2.4. A sequence space E is said to be symmetric if $(x_{\pi(n)}) \in E$, whenever $(x_n) \in E$, where π is a permutation of N.

Definition 2.5. A sequence space E is said to be convergence free if $(y_n) \in E$, whenever $(x_n) \in E$ and $x_n = 0$ implies $y_n = 0$.

For (a_k) and (b_k) two sequences of complex terms and $p=(p_k)\in\ell_\infty$, we have the following known inequality:

$$|a_k+b_k|^{p_k}\le D\{|a_k|^{p_k}+|b_k|^{p_k}\},$$

where $H=\sup p_k$ and $D=\max\{1,2^{H-1}\}$.

Definition 2.6. Let f be a modulus function, then for a given multiplier sequence $\Lambda=(\lambda_k)$, we introduce the following sequence spaces:

$c(f,\Lambda,\Delta_m,p)=\{(x_k)\in w:(f(|\lambda_k(\Delta_m x_k-L)|))^{p_k}\to 0,\ \text{as } k\to\infty,\ \text{for some } L\}$,

$c_0(f,\Lambda,\Delta_m,p)=\{(x_k)\in w:(f(|\lambda_k(\Delta_m x_k)|))^{p_k}\to 0,\ \text{as } k\to\infty\}$,

$\ell_\infty(f,\Lambda,\Delta_m,p)=\{(x_k)\in w:\sup_k(f(|\lambda_k(\Delta_m x_k-L)|))^{p_k}<\infty\}$.

When $f(x)=x$, for all $x\in[0,\infty)$, the above sequence spaces are denoted as $c(\Lambda,\Delta_m,p)$, $c_0(\Lambda,\Delta_m,p)$ and $\ell_\infty(\Lambda,\Delta_m,p)$ respectively. When $\lambda_k=1$ for all $k\in N$, the above sequence spaces are denoted as $c(f,\Delta_m,p)$, $c_0(f,\Delta_m,p)$ and $\ell_\infty(f,\Delta_m,p)$ respectively.

Taking $f(x)=x$, for all $x\in[0,\infty)$ and $\lambda_k=1$, for all $k\in N$, the above sequence spaces are denoted as $c(\Delta_m,p),c_0(\Delta_m,p)$ and $\ell_\infty(\Delta_m,p)$ respectively. Further taking $p_k=1$ for all $k\in N$, the above spaces are denoted as $c(\Delta_m),c_0(\Delta_m)$ and $\ell_\infty(\Delta_m)$ respectively (please refer to [12]). Further taking $m=1$, we get the spaces $c(\Delta),c_0(\Delta)$ and $\ell_\infty(\Delta)$ respectively, studied by [5].

Similarly taking different combinations of restrictions, we will get different paranormed sequence spaces.

The following result will be used for establishing a result of this article.

Lemma 2.1.[7] *Let $c_0(p)$ denote the set of sequences $x=(x_k)$ such that $|x_k|^{p_k}\to 0$, as $k\to\infty$. If $p_k>0$ and $q_k>0$, then $c_0(q)\subset c_0(p)$ if and only if $\lim-\inf\frac{p_k}{q_k}>0$.*

3 Main Results

In this section we prove the results involving the classes of sequences (f,Λ,Δ_m,p), $c_0(f,\Lambda,\Delta_m,p)$ and $\ell_\infty(f,\Lambda,\Delta_m,p)$.

Theorem 3.1. *The classes of sequences (f,Λ,Δ_m,p), $c_0(f,\Lambda,\Delta_m,p)$ and $\ell_\infty(f,\Lambda,\Delta_m,p)$ are linear spaces.*

Proof. We prove the theorem for the class of sequences $c_0(f,\Lambda,\Delta_m,p)$. The other cases can be proved similarly. Let $(x_k),(y_k)\in c_0(f,\Lambda,\Delta_m,p)$. Then

$$(f(|\lambda_k(\Delta_m x_k)|))^{p_k}\to 0,\qquad \text{as}\qquad k\to\infty, \tag{1}$$

and

$$(f(|\lambda_k(\Delta_m y_k)|))^{p_k}\to 0,\qquad \text{as}\qquad k\to\infty. \tag{2}$$

For $\alpha, \beta \in C$, we have

$$(f(|\lambda_k \Delta_m(\alpha x_k + \beta y_k)|))^{p_k} \leq D([\alpha]+1)f(|\lambda_k \Delta_m x_k|))^{p_k} + D([\beta]+1)f(|\lambda_k \Delta_m y_k|))^{p_k} \to 0, \text{ as } k \to \infty,$$

by (1) and (2)

Hence $(\alpha x_k + \beta y_k) \in c_0(f, \Lambda, \Delta_m, p)$.

Thus $c_0(f, \Lambda, \Delta_m, p)$ is a linear space.

Theorem 3.2. *Let $p = (p_k) \in \ell_\infty$. Then the spaces $(f, \Lambda, \Delta_m, p), c_0(f, \Lambda, \Delta_m, p)$ and $\ell_\infty(f, \Lambda, \Delta_m, p)$, are paranormed spaces paranormed by g,*

$$g(x) = \sup_k (f(|\lambda_k \Delta_m x_k|))^{\frac{p_k}{M}},$$

where $M = \max(1, \sup p_k)$.

Proof. Clearly $g(x) \geq 0, g(-x) = g(x), g(x+y) \leq g(x) + g(y)$. Next we show the continuity of the product. Let α be fixed and $g(x) \to 0$. Then it is obvious that $g(\alpha x) \to 0$.

Next let $\alpha \to 0$ and x be fixed. Since f is continuous, we have $f(|\alpha||\lambda_k \Delta_m x_k|) \to 0$ as $\alpha \to 0$. Thus we have

$$\sup_k [f(|\alpha \lambda_k \Delta_m x_k|)]^{\frac{p_k}{M}} \to 0, \quad \text{as} \quad \alpha \to 0.$$

Hence $g(\alpha x) \to 0$, as $\alpha \to 0$.

Therefore g is a paranorm.

Proposition 3.3. *$c_0(f, \Lambda, \Delta_m, p) \subset c(f, \Lambda, \Delta_m, p) \subset \ell_\infty(f, \Lambda, \Delta_m, p)$ and the inclusions are proper.*

Proof. The proof is a routine verification and suitable examples can be constructed to show that the inclusions are proper.

Theorem 3.4. *The spaces $(f, \Lambda, \Delta_m, p), c_0(f, \Lambda, \Delta_m, p)$ and $\ell_\infty(f, \Lambda, \Delta_m, p)$, are neither solid nor monotone in general, but the spaces $c_0(f, \Lambda, p)$* and *$\ell_\infty(f, \Lambda, p)$ are solid and as such are monotone.*

Proof. Let (x_k) be a given sequence and (α_k) be a sequence of scalars such that $|\alpha_k| \leq 1$, for all $k \in N$. Then we have

$$(f(|\lambda_k \alpha_k x_k)|)^{p_k} \leq (f(|\lambda_k x_k)|)^{p_k}, \quad \text{for all} \quad k \in \mathbf{N}.$$

The solidness of $c_0(f, \Lambda, p)$ and $\ell_\infty(f, \Lambda, p)$ follows from this inequality. The monotonicity follows by Remark 2.1.

The first part of the proof follows from the following examples.

Example 3.1. Let $f(x) = x$, for all $x \in [0, \infty), m = 1, \lambda_k = 1$, for all $k \in \mathbf{N}$. Let $p_k = 1$ for k odd and $p_k = 2$ for k even. Then define (x_k) by $x_k = k$, for all $k \in \mathbf{N}$, belongs to $c(\Delta, p)$ and

$\ell_\infty(\Delta, p)$. For E a sequence space, consider its step space E_J defined by $(y_k) \in E_J$ implies $y_k = 0$ for k odd and $y_k = x_k$ for k even. Then (y_k) neither belongs to $(c(\Delta, p))_J$ nor to $(\ell_\infty(\Delta, p))_J$. Hence the spaces are not monotone. Hence are not solid by Remark 2.1.

Example 3.2. Let $f(x) = x$, for all $x \in [0, \infty), m = 1, \lambda_k = 2 + k^{-1}$, for all $k \in N$. Let $p_k = 2$ for k odd and $p_k = 3$ for k even. Consider the sequence (x_k) defined by $x_k = 1$ for all $k \in \mathbf{N}$. Then $(x_k) \in c_0(\Lambda, \Delta, p)$. Now consider the step spaces as defined in Example 3.1. Then $(y_k) \notin c_0(\Lambda, \Delta, p)$. Hence $c_0(\Lambda, \Delta, p)$ is not monotone, as such $c_0(\Lambda, \Delta, p)$ is not solid by Remark 2.1.

Theorem 3.5. *The spaces* $(f, \Lambda, \Delta_m, p), c_0(f, \Lambda, \Delta_m, p)$ *and* $\ell_\infty(f, \Lambda, \Delta_m, p)$, *are not convergence free*.

Proof. The result follows from the following example.

Example 3.3. Let $f(x) = x$, for all $x \in [0, \infty), m = 1, \lambda_k = 1$, for all $k \in N$. Let $p_k = 1$ for k odd and $p_k = 2$ for k even. Consider the sequence (x_k) defined by $x_k = k^{-1}$, for all $k \in \mathbf{N}$, Then $(x_k) \in Z(\Delta, p)$ for $Z = c, c_o, \ell_\infty$. Consider the sequence (y_k) defined by $y_k = k^2$, for all $k \in \mathbf{N}$. Then (y_k) neither belongs to $c(\Delta, p)$ nor to $c_0(\Delta, p)$ nor to $\ell_\infty(\Delta, p)$. Hence the spaces are not convergence free.

The proof of the following results follows from the Lemma 2.1.

Proposition 3.6. *Let* (p_k) *and* (q_k), *be two sequences of real numbers. Then* $c_0(f, \Lambda, \Delta_m, p) \subset c_0(f, \Lambda, \Delta_m, q)$ *if and only if* $\liminf \frac{p_k}{q_k} > 0$.

The following result is a consequence of the above result.

Corollary 3.7. *Let* (p_k) *and* (q_k), *be two sequences of real numbers. Then* $c_0(f, \Lambda, \Delta_m p) = c_0(f, \Lambda, \Delta_m, q)$ *if and only if* $\liminf \frac{p_k}{q_k} > 0$ *and* $\liminf \frac{q_k}{p_k} > 0$.

The proof of the following results is routine verification.

Proposition 3.8. (i) *Let* $0 < p_k < q_k < \infty$ *for each* $k \in N$, *then* $c_0(f, \Lambda, \Delta_m, q) \subset c_0(f, \Lambda, \Delta_m, p)$.

(ii) *Let* $0 < \inf p_k < p_k < 1$ *for each* $k \in \mathbf{N}$, *then* $c_0(f, \Lambda, \Delta_m) \subset c_0(f, \Lambda, \Delta_m, p)$.

(iii) *Let* $1 < p_k < \sup p_k < \infty$ *for each* $k \in \mathbf{N}$, *then* $c_0(f, \Lambda, \Delta_m, p) \subset c_0(f, \Lambda, \Delta_m)$.

Proposition 3.9. *The following are equivalent*:

(i) $h > 0$ *and* $H < \infty$.

(ii) $c_0(f, \Lambda, \Delta_m, p) \subset c_0(f, \Lambda, \Delta_m)$.

(iii) $c(f, \Lambda, \Delta_m, p) \subset c(f, \Lambda, \Delta_m)$.

(iv) $\ell_\infty(f, \Lambda, \Delta_m, p) \subset \ell_\infty(f, \Lambda, \Delta_m)$.

Theorem 3.10. *The spaces* $(f, \Lambda, \Delta_m, p), c_0(f, \Lambda, \Delta_m, p)$ *and* $\ell_\infty(f, \Lambda, \Delta_m, p)$, *are not symmetric in general*.

Proof. The result follows from the following examples.

Example 3.4. Let f be any modulus function, $m = 0$ and $\lambda_k = k$ for all $k \in \mathbf{N}$. Let $p_k = 1$ for k odd and $p_k = 4$ for k even. Consider the sequence (x_k) defined by $x_k = k^{-2}$, for all $k \in \mathbf{N}$. Then (x_k) belongs to $c(f,\Lambda,p)$ as well as $c_0(f,\Lambda,p)$. Consider its rearrangement (y_k) defined as follows:

$(y_n) = (x_1,\, x_3,\, x_4,\, x_2,\, x_6,\, x_7,\, x_8,\, \ldots,\, x_{24},\, x_5,\, x_{26},\, x_{27},\, \ldots,\, x_{624},\, x_{25},\, x_{626},\, \ldots)$.

Then (y_n) neither belongs to $c(f,\Lambda,p)$ nor to $c_0(f,\Lambda,p)$. Hence the spaces $c(f,\Lambda,\Delta_m,p)$, $c_0(f,\Lambda,\Delta_m,p)$ and $\ell_\infty(f,\Lambda,\Delta_m,p)$, are not symmetric in general.

Example 3.5. Let f be any modulus function, $m = 0$ and $\lambda_k = k$, for all $k \in \mathbf{N}$. Let $p_k = 1 + k^{-1}$ for all $k \in \mathbf{N}$. Consider the sequence (x_k) defined by $x_k = k^{-1}$, for all $k \in N$. Then (x_k) belongs to $\ell_\infty(f,\Lambda,p)$. Consider its rearrangement (y_k) as defined in Example 3.4. Then $(y_n) \notin \ell_\infty(f,\Lambda,p)$. Hence the space $\ell_\infty(f,\Lambda,p)$ is not symmetric in general.

Remark 3.1. We have $Z(f,\Lambda,\Delta_m,p) = (f,\Delta_m,p)$, for $Z = c, c_0, \ell_\infty$ if and only if $(\lambda_k) \in \ell_\infty$.

The following result is a consequence of Remark 1.1 and Remark 2.1.

Proposition 3.11. *The spaces $c_0(f,\Lambda,\Delta,p)$ and $Z(f,\Lambda,p)$, for $Z = c, c_0, \ell_\infty$ are solid if and only if*

(i) $(\lambda_k) \in \ell_\infty$.

(ii) $h > 0$ *and* $H < \infty$.

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Binod Chandra Tripathy
Mathematical Sciences Division
Institute of Advanced Study in Science and Technology
Paschim Baragaon; Garchuk; Guwahati-781035
INDIA

E-mail: tripathybc@yahoo.com; tripathybc@rediffmail.com

Prabhat Chandra
Department of Mathematics
Patna University
PATNA - 800005
INDIA

E-mail : prabhatchandra_patna@yahoo.com