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LIOUVILLE PROPERTY FOR A CLASS OF QUASI-HARMONIC SPHERE

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Abstract. In this paper we obtain a Liouville type result for a class of quasi-harmonic spheres with rotational symmetry.

Key words: *Liouville property, quasi-harmonic sphere, rotational symmetry*

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1 Introduction

In [1] Lin and Wang introduced the concept of quasi-harmonic sphere in their study of the heat flow of harmonic maps, and asked whether one can show the existence of such quasi-harmonic spheres. Fan[2] provided the first examples of quasi-harmonic spheres for $N = S^n (3 \leq n \leq 6)$, and Gastel[3] gave more examples with $N = S^n$, for all $n \geq 3$. In a recent paper [4] Ding and Zhao consider the problem on the continuity of quasi-harmonic sphere at ∞, and they show that the non-constant equivariant quasi-harmonic sphere must be discontinuous at infinity. In the present paper we will prove a similar Liouville property for a class of quasi-harmonic spheres with rotational symmetry.

We say u a quasi-harmonic sphere from $\mathbf{R}^n$ to a Riemannian manifold N if it satisfies the following equations

$$\triangle u - \frac{1}{2} x \cdot \nabla u + A(u)(du, du) = 0. \tag{1.1}$$

Note that u is also a harmonic map from $(\mathbf{R}^n, g)$ to N where $g = e^{-\frac{|x|^2}{2(n-2)}} ds_0^2$ and ds_0^2 is the standard Euclidean metric.

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By Nash embedding theorem we can assume N is a Riemannian submanifold of the Euclidean space $\mathbf{R}^k$. We say u is rotational symmetry if it can be represented as

$$u(r,\theta)=(h(r),f(r,h(r))\omega(\theta)), \tag{1.2}$$

where $\omega: S^{n-1}\to S^{m-1}$ is a harmonic map and m is the dimension of N. For simplicity we denote $f(r,h(r))$ by $F(r)$ below.

Our aim in this paper is to prove the following Liouville theorem.

Theorem 1. *If u is rotational symmetry and continuous at the point ∞,* i. e.

$$\lim_{|x|\to\infty} u(x)=y\in N,$$

then u must be a constant map.

2 Proof of the Main Theorem

To prove the theorem we need a simple lemma.

Lemma 1. *Let u be any quasi harmonic sphere from $\mathbf{R}^n$ to N. Then the following equality holds*

$$r^2\frac{\partial}{\partial r}|u_r|^2+r(2(n-1)-r^2)|u_r|^2=\frac{\partial}{\partial r}|u_\theta|^2. \tag{2.3}$$

Proof. As $A(u)(du,du)$ is a norm vector on $\mathbf{N}$, we have

$$<\triangle u,u_r>=\frac{r}{2}|u_r|^2.$$

Using the polar coordinate and the fact $<u_r,u_\theta>=0$ we can obtain

$$\begin{aligned}\frac{r}{2}|u_r|^2 &= <\triangle u,u_r>\\ &= <u_{rr}+\frac{n-1}{r}u_r+\frac{\triangle_\theta u}{r^2},u_r>\\ &= \frac{1}{2}\frac{\partial}{\partial r}|u_r|^2+\frac{n-1}{r}|u_r|^2-\frac{1}{2r^2}\frac{\partial}{\partial r}|u_\theta|^2,\end{aligned}$$

which implies (2.3).

Now we begin to prove Theorem 1.

The assumption u is rotational symmetry and continuous at ∞ means that in (1.2) there must be

$$\lim_{r\to\infty}F(r)=0.$$

Noting that $\omega: S^{n-1}\to S^{m-1}$ is harmonic, there exists a constant λ such that

$$|\nabla_\theta\omega|=\lambda. \tag{2.4}$$

By Lemma 1 we can get

$$\begin{aligned}\frac{\partial}{\partial r}(r^{2n-2}e^{-\frac{r^2}{2}}|u_r|^2) &= r^{2n-2}e^{-\frac{r^2}{2}}(\frac{\partial}{\partial r}|u_r|^2+(\frac{2n-2}{r}-r)|u_r|^2)\\ &= r^{2n-4}e^{-\frac{r^2}{2}}\frac{\partial}{\partial r}|u_\theta|^2.\end{aligned}$$

Note that $\lim\limits_{r\to\infty} r^{2n-2}e^{-\frac{r^2}{2}}|u_r|^2=0$, we have

$$|u_r|^2=-r^{2-2n}e^{\frac{r^2}{2}}\int_r^\infty s^{2n-4}e^{-\frac{s^2}{2}}\frac{\partial}{\partial s}|u_\theta|^2\mathrm{d}s. \tag{2.5}$$

From (1.2) and (2.4) it is easy to check that

$$|u_r|^2=(h')^2+(F')^2; |u_\theta|^2=\lambda^2F^2. \tag{2.6}$$

Using (2.5) and (2.6) we can obtain that for any $r>\sqrt{2n-4}$, there holds

$$\begin{aligned}(F'(r))^2 &\le |u_r|^2\\ &= -r^{2-2n}e^{\frac{r^2}{2}}\int_r^\infty s^{2n-4}e^{-\frac{s^2}{2}}\frac{\partial}{\partial s}|u_\theta|^2\mathrm{d}s\\ &= -\lambda^2r^{2-2n}e^{\frac{r^2}{2}}\int_r^\infty s^{2n-4}e^{-\frac{s^2}{2}}(F^2)'(s)\mathrm{d}s\\ &= \lambda^2r^{2-2n}e^{\frac{r^2}{2}}\left(r^{2n-4}e^{-\frac{r^2}{2}}F^2(r)+\int_r^\infty\left(\frac{2n-4}{s}-s\right)s^{2n-4}e^{-\frac{s^2}{2}}F^2(s)\mathrm{d}s\right)\\ &\le \lambda^2r^{2-2n}e^{\frac{r^2}{2}}r^{2n-4}e^{-\frac{r^2}{2}}F^2(r)\\ &= \lambda^2r^{-2}F^2(r).\end{aligned} \tag{2.7}$$

Then we get

$$|F'(r)|\le\lambda\frac{F(r)}{r}. \tag{2.8}$$

Now for any $\sqrt{2n-4}<r<s$, it can be derived from (2.8) that

$$\frac{F(s)}{F(r)}=e^{\int_r^s\frac{F'(t)}{F(t)}\mathrm{d}t}\le e^{\int_r^s\frac{|F'(t)|}{F(t)}\mathrm{d}t}\le e^{\lambda\int_r^s\frac{1}{t}\mathrm{d}t}=\left(\frac{s}{r}\right)^\lambda. \tag{2.9}$$

In the proof of (2.7) we have obtained

$$(F'(r))^2\le-\lambda^2r^{2-2n}e^{\frac{r^2}{2}}\int_r^\infty s^{2n-4}e^{-\frac{s^2}{2}}(F^2)'(s)\mathrm{d}s.$$

By using (2.8) and (2.9) we obtain

$$\begin{aligned}(F'(r))^2 &\le -2\lambda^2r^{2-2n}e^{\frac{r^2}{2}}\int_r^\infty s^{2n-4}e^{-\frac{s^2}{2}}F(s)F'(s)\mathrm{d}s\\ &\le 2\lambda^3r^{2-2n}e^{\frac{r^2}{2}}\int_r^\infty s^{2n-5}e^{-\frac{s^2}{2}}F^2(s)\mathrm{d}s\\ &\le 2\lambda^3r^{2-2n}e^{\frac{r^2}{2}}\int_r^\infty s^{2n-5}e^{-\frac{s^2}{2}}\left(\frac{s}{r}\right)^{2\lambda}F^2(r)\mathrm{d}s\\ &\le C_\lambda r^{2-2n}e^{\frac{r^2}{2}}r^{2n-6+2\lambda}e^{-\frac{r^2}{2}}r^{-2\lambda}F^2(r)\\ &\le C_\lambda r^{-4}F^2(r).\end{aligned} \tag{2.10}$$

This inequality implies that there exists a positive constant $c(=\sqrt{C_\lambda})$ such that for any r big enough,

$$F'(r)+c\frac{F(r)}{r^2}\geq 0$$

which is equivalent to

$$(e^{-\frac{c}{r}}F(r))'\geq 0. \tag{2.11}$$

The fact $\lim\limits_{r\to\infty}F(r)=0$ and (2.11) imply that $F\equiv 0$, so we complete the proof of Theorem 1.

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