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INTEGRABILITY AND L^1-CONVERGENCE OF DOUBLE COSINE TRIGONOMETRIC SERIES

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Abstract. We study here L^1-convergence of new modified double cosine trigonometric sum and obtain a new necessary and sufficient condition for L^1-convergence of double cosine trigonometric series. Also, the results obtained by Moricz[1],[2] are particular cases of ours.

Key words: *L^1-convergence, conjugate Dirichlet kernel*

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1 Introduction

We consider the double cosine series

$$f(x,y)=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\lambda_j\lambda_k\ \ a_{jk}\cos jx\cos ky \tag{1.1}$$

on the positive quadrant $T^2=[0,\pi]\times[0,\pi]$ of the two dimensional torus, where $\lambda_0=\dfrac{1}{2}$ and $\lambda_j=1$ for $j=1,2,3,\cdots$ and $\{a_{jk}\}$ is a double sequence of real numbers.

We denote by

$$S_{mn}(x,y)=\sum_{j=0}^{m}\sum_{k=0}^{n}\lambda_j\lambda_k\ \ a_{jk}\cos jx\cos ky,\qquad m,n\geq 0$$

the rectangular partial sum of the series (1.1) and $f(x,y)=\lim\limits_{m+n\to\infty}S_{mn}(x,y)$.

We remind the reader the following classes of coefficient sequences due to [1].

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Definition 1.1[1]. We say that $\{a_{jk}\}$ belongs to the class BV_2 if

$$a_{jk} \to 0 \qquad \text{as} \qquad j+k \to \infty, \tag{1.2}$$

and

$$\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} |\triangle_{11} a_{jk}| < \infty, \tag{1.3}$$

where

$$\triangle_{11} a_{j,k} = a_{j,k} - a_{j+1,k} - a_{j,k+1} + a_{j+1,k+1}.$$

The condition (1.2) implies that $\{a_{jk}\}$ is a null sequence while (1.3) implies that $\{a_{jk}\}$ is a sequence of bounded variation.

Defintion 1.2[1] A null sequence $\{a_{jk}\}$ belongs to the class $\mathcal{C}_2$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $0 \le m \le M$ and $0 \le n \le N$, we have

$$C(m,M;n,N;\delta) := \int\int_{D_\delta} \left| \sum_{j=m}^{M}\sum_{k=n}^{N} D_j(x)D_k(y)\triangle_{11} a_{jk} \right| \mathrm{d}x\mathrm{d}y \le \varepsilon \tag{1.4}$$

or

$$\int\int_{D_\delta} \left| \sum_{j=m}^{\infty}\sum_{k=n}^{\infty} D_j(x)D_k(y)\triangle_{11} a_{jk} \right| \mathrm{d}x\mathrm{d}y \le \varepsilon, \qquad \forall\ m,n \ge 0,$$

where

$$D_\delta := T - (\delta,\pi] \times (\delta,\pi] = \{(x,y) : 0 \le x,y \le \pi\ \ \&\ \ \min(x,y) \le \delta\}.$$

Definition 1.3[1]. A double sequence $\{a_{jk}\}$ is said to be quasi-convex if

$$\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} (j+1)(k+1)|\triangle_{22} a_{jk}| < \infty. \tag{1.5}$$

Moricz[1] introduced the following modified double cosine trigonometric sum

$$u_{mn}(x,y) = \sum_{j=0}^{m}\sum_{k=0}^{n} \lambda_j\lambda_k \Big(\sum_{i=j}^{m}\sum_{l=k}^{n} \triangle_{11} a_{il}\Big) \cos jx \cos ky \tag{1.6}$$

and studied the L^1-convergence of double cosine trigonometric series whose coefficients belong to the class BV_2, $\mathcal{C}_2$ and the class of quasi-convex coefficients by making use of L^1-convergence of these modified double cosine trigonometric sums.

We introduce here the following new modified rectangular partial sums g_{mn} of the series (1.1)

$$g_{mn}(x,y) \quad = \quad \frac{a_{00}}{2} + \sum_{j=1}^{m}\sum_{k=1}^{n} \left\{ \sum_{r=j}^{m}\sum_{l=k}^{n} \triangle_{11}(a_{rl} \cos rx \cos ly) \right\}. \tag{1.7}$$

It will turn out that $g_{mn}(x,y)$ approximate f better than $S_{mn}(x,y)$ since they converge to $f(x,y)$ in $L^1(T)$-metric while the classical rectangular partial sums $S_{mn}(x,y)$ may not.

We note that the single cosine series analogous to the modified sums was introduced by Jatinderdeep Kaur and S.S. Bhatia[3].

Here we formulate the new class J_d of coefficient sequences as:

Definition 1.4. A double null sequence $\{a_{jk}\}$ of positive numbers is said to belong to the class J_d if there exists a double sequence $\{A_{jk}\}$ such that

$$A_{jk} \downarrow 0, \quad j+k \to \infty, \tag{1.8}$$

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} jkA_{jk} < \infty, \tag{1.9}$$

$$\left| \triangle_{pq}\left(\frac{a_{jk}}{jk}\right)\right| \leq \frac{A_{jk}}{jk}, \quad 1 \leq p+q \leq 2 \tag{1.10}$$

for any nonnegative integers p, q and $j, k \in \{1, 2, 3, \cdots\}$.

The aim of this paper is to give necessary and sufficient conditions for the integrability and L^1-convergence of double cosine trigonometric series by using modified double cosine trigonometric sums (1.7) under a newly defined class J_d of coefficient sequences.

2 Lemma

The proof of our result is based on the following lemmas.

Lemma 2.1[4]**.** *Let $n \geq 1$, r be a nonnegative integer and $x \in [\varepsilon, \pi]$. Then $|\tilde{D}_n^r(x)| \leq C_\varepsilon \frac{n^r}{x}$, where C_ε is a positive constant depending only on ε, $0 < \varepsilon < \pi$ and $\tilde{D}_n(x)$ is the conjugate Dirichlet kernel.*

Lemma 2.2[4]**.** $||\tilde{D}_n^r(x)||_{L^1} = O(n^r \log n)$, $r = 0, 1, 2, 3, \cdots$, *where $\tilde{D}_n^r(x)$ represents the r^{th} derivative of conjugate Dirichlet-kernel.*

3 Main Result

Our main result is the following theorem:

Theorem 3.1. *If a double sequence $\{a_{jk}\}$ belongs to the class* J$_d$*, then $||g_{mn} - f|| \to 0$ as $m+n \to \infty$.*

Here $||.||$ denotes the two-dimensional $L^1(T^2)$-norm.

Proof. First we shall show the point-wise limit f of the sum (1.7) exists in T^2 and $f \in$

$L^1(T^2)$. We have

$$\begin{aligned} g_{mn}(x,y) &= \frac{a_{00}}{2}+\sum_{j=1}^{m}\sum_{k=1}^{n}\left\{\sum_{r=j}^{m}\sum_{l=k}^{n}\triangle_{11}(a_{rl}\cos rx\cos ly)\right\} \\ &= \frac{a_{00}}{2}+\sum_{j=1}^{m}\sum_{k=1}^{n}\sum_{r=j}^{m}[a_{rk}\cos rx\cos ky-a_{r,k+1}\cos rx\cos(k+1)y \\ &\quad -a_{r+1,k}\cos(r+1)x\cos ky+a_{r+1,k+1}\cos(r+1)x\cos(k+1)y \\ &\quad +a_{r,k+1}\cos rx\cos(k+1)y-a_{r,k+2}\cos rx\cos(k+2)y \\ &\quad -a_{r+1,k+1}\cos(r+1)x\cos(k+1)y+a_{r+1,k+2}\cos(r+1)x\cos(k+2)y \\ &\quad +\cdots+a_{rn}\cos rx\cos ny-a_{r,n+1}\cos rx\cos(n+1)y \\ &\quad -a_{r+1,n}\cos(r+1)x\cos ny+a_{r+1,n+1}\cos(r+1)x\cos(n+1)y] \\ &= \frac{a_{00}}{2}+\sum_{j=1}^{m}\sum_{k=1}^{n}[a_{jk}\cos jx\cos ky-a_{j+1,k}\cos(j+1)x\cos ky \\ &\quad -a_{j,n+1}\cos jx\cos(n+1)y+a_{j+1,n+1}\cos(j+1)x\cos(n+1)y \\ &\quad +a_{j+1,k}\cos(j+1)x\cos ky-a_{j+2,k}\cos(j+2)x\cos ky \\ &\quad -a_{j+1,n+1}\cos(j+1)x\cos(n+1)y+a_{j+2,n+1}\cos(j+2)x\cos(n+1)y \\ &\quad +\cdots+a_{mn}\cos mx\cos ny-a_{m+1,k}\cos(m+1)x\cos ky \\ &\quad -a_{m,n+1}\cos mx\cos(n+1)y+a_{m+1,n+1}\cos(m+1)x\cos(n+1)y]\} \\ &= S_{mn}(x,y)-\sum_{j=1}^{m}\sum_{k=1}^{n}\left\{a_{j,n+1}\cos jx\cos(n+1)y+a_{m+1,k}\cos(m+1)x\cos y\right\} \\ &\qquad +mna_{m+1,n+1}\cos(m+1)x\cos(n+1)y. \end{aligned} \tag{3.1}$$

exists in T^2 and that f is a Fourier series i.e. $f\in L^1(T^2)$.

Using double summation by parts and the given hypothesis, we get

$$\begin{aligned} g_{mn}(x,y) &= \frac{a_{00}}{2}+\sum_{j=1}^{m-1}\sum_{k=1}^{n-1}\triangle_{11}\left(\frac{a_{jk}}{jk}\right)\tilde{D}'_j(x)\tilde{D}'_k(y)-\sum_{j=1}^{m}\triangle_{10}\left(\frac{a_{j,n}}{jn}\right)\tilde{D}'_j(x)\tilde{D}'_n(y) \\ &\quad -\sum_{k=1}^{n}\triangle_{01}\left(\frac{a_{m,k}}{mk}\right)\tilde{D}'_m(x)\tilde{D}'_k(y)+\frac{a_{m,n}}{mn}\tilde{D}'_m(x)\tilde{D}'_n(y)-\sum_{j=1}^{m}na_{j,n+1}\cos jx\cos(n+1)y \\ &\quad -\sum_{k=1}^{n}ma_{m+1,k}\cos(m+1)x\cos y+mna_{m+1,n+1}\cos(m+1)x\cos(n+1)y \end{aligned}$$

It is known from Lemma 2.1 that

$$|\tilde{D}'_n(X)|=O(n) \qquad \text{for} \quad 0<x\leq\pi. \tag{3.2}$$

By (1.9), (1.10) we note that

$$\sum_{j=1}^{m-1}\sum_{k=1}^{n-1}\triangle_{11}\left(\frac{a_{jk}}{jk}\right)\tilde{D}'_j(x)\tilde{D}'_k(y)\leq\sum_{j=1}^{m-1}\sum_{k=1}^{n-1}\left(\frac{A_{jk}}{jk}\right)\tilde{D}'_j(x)\tilde{D}'_k(y)<\infty,$$

for all x and y such that $0<x,y\leq\pi$.

By (1.9), (1.10) and (3.2), we have

$$\sum_{j=1}^{m}\triangle_{10}\left(\frac{a_{jn}}{jn}\right)\tilde{D}'_j(x)\tilde{D}'_n(y)\leq\sum_{j=1}^{m}\sum_{k=n}^{\infty}\left(\frac{A_{jk}}{jk}\right)\tilde{D}'_j(x)\tilde{D}'_k(y)\to 0\qquad\text{as}\quad n\to\infty$$

uniformly in m, for all $0<x,y\leq\pi$.

Similarly,

$$\sum_{k=1}^{n}\triangle_{01}\left(\frac{a_{mk}}{mk}\right)\tilde{D}'_m(x)\tilde{D}'_k(y)\to 0\qquad\text{as}\quad n\to\infty$$

uniformly in n, for all $0<x,y\leq\pi$.

Since $\{a_{jk}\}$ is a double null sequence and by the use of the equation (3.2), we get

$$\frac{a_{mn}}{mn}\tilde{D}'_m(x)\tilde{D}'_n(y)\to 0\qquad\text{as}\quad m+n\to\infty$$

for all $0<x,y\leq\pi$.

Further, we know that $|\cos nx|$ is bounded in $(0,\pi]$.

Therefore, by (1.9) and (1.10) we have

$$\sum_{j=1}^{m}na_{j,n+1}\leq\sum_{j=1}^{m}\sum_{k=n+1}^{\infty}jk^2\left(\frac{A_{jk}}{jk}\right)\to 0\qquad\text{as}\quad n\to\infty.$$

This implies that

$$\sum_{j=1}^{m}na_{j,n+1}\cos jx\cos(n+1)y\to 0\qquad\text{as}\quad n\to\infty$$

uniformly in m, for all $0<x,y\leq\pi$.

Similarly,

$$\sum_{k=1}^{n}ma_{m+1,k}\cos(m+1)x\cos ky\to 0\qquad\text{as}\quad m\to\infty$$

uniformly in n, for all $0<x,y\leq\pi$.

Also, by (1.9) and (1.10), we have

$$\begin{aligned}mna_{m+1,n+1}&\leq\sum_{j=m+1}^{\infty}\sum_{k=n+1}^{\infty}j^2k^2\triangle_{11}\left(\frac{a_{jk}}{jk}\right)\\&\leq\sum_{j=m+1}^{\infty}\sum_{k=n+1}^{\infty}j^2k^2\frac{A_{jk}}{jk}\to 0\qquad\text{as}\quad m+n\to\infty.\end{aligned}\tag{3.3}$$

Consequently, we get $\lim\limits_{m+n\to\infty} g_{mn} = f(x,y)$ exists in $L^1(T^2)$.

Next, we consider

$$\begin{aligned}
||f-g_{mn}|| \le & \int_0^\pi\int_0^\pi \left|\sum_{j=m+1}^{\infty}\sum_{k=n+1}^{\infty}\triangle_{11}\left(\frac{a_{jk}}{jk}\right)\tilde{D}'_j(x)\tilde{D}'_k(y)\right| \mathrm{d}x\mathrm{d}y \\
& + \int_0^\pi\int_0^\pi \left|\sum_{j=1}^{m}\triangle_{10}\left(\frac{a_{jn}}{jn}\right)\tilde{D}'_j(x)\tilde{D}'_n(y)\right| \mathrm{d}x\mathrm{d}y \\
& + \int_0^\pi\int_0^\pi \left|\sum_{k=1}^{n}\triangle_{01}\left(\frac{a_{mk}}{mk}\right)\tilde{D}'_m(x)\tilde{D}'_k(y)\right| \mathrm{d}x\mathrm{d}y \\
& + \int_0^\pi\int_0^\pi \left|\frac{a_{mn}}{mn}\tilde{D}'_m(x)\tilde{D}'_n(y)\right| \mathrm{d}x\mathrm{d}y \\
& + \int_0^\pi\int_0^\pi \left|\sum_{j=1}^{m} n a_{j,n+1}\cos jx\cos(n+1)y\right| dxdy \\
& + \int_0^\pi\int_0^\pi \left|\sum_{k=1}^{n} m a_{m+1,k}\cos(m+1)x\cos y\right| \mathrm{d}x\mathrm{d}y \\
& + mn|a_{m+1,n+1}|\int_0^\pi\int_0^\pi |\cos(m+1)x\cos(n+1)y|\mathrm{d}x\mathrm{d}y \\
\le & \int_0^\pi\int_0^\pi \left|\sum_{j=m+1}^{\infty}\sum_{k=n+1}^{\infty}\left(\frac{A_{jk}}{jk}\right)\tilde{D}'_j(x)\tilde{D}'_k(y)\right| \mathrm{d}x\mathrm{d}y \\
& + \int_0^\pi\int_0^\pi \left|\sum_{j=1}^{m}\left(\frac{A_{jn}}{jn}\right)\tilde{D}'_j(x)\tilde{D}'_n(y)\right| \mathrm{d}x\mathrm{d}y \\
& + \int_0^\pi\int_0^\pi \left|\sum_{k=1}^{n}\left(\frac{A_{mk}}{mk}\right)\tilde{D}'_m(x)\tilde{D}'_k(y)\right| \mathrm{d}x\mathrm{d}y \\
& + \int_0^\pi\int_0^\pi \left|\frac{a_{mn}}{mn}\tilde{D}'_m(x)\tilde{D}'_n(y)\right| \mathrm{d}x\mathrm{d}y \\
& + \int_0^\pi\int_0^\pi \left|\sum_{j=1}^{m}\sum_{k=n+1}^{\infty} jk^2\left(\frac{A_{jk}}{jk}\right)\cos jx\cos(n+1)y\right| \mathrm{d}x\mathrm{d}y \\
& + \int_0^\pi\int_0^\pi \left|\sum_{k=1}^{n}\sum_{j=m+1}^{\infty} kj^2\left(\frac{A_{jk}}{jk}\right)\cos(m+1)x\cos y\right| \mathrm{d}x\mathrm{d}y \\
& + mn|a_{m+1,n+1}|\int_0^\pi\int_0^\pi |\cos(m+1)x\cos(n+1)y|\mathrm{d}x\mathrm{d}y
\end{aligned} \tag{3.4}$$

We note that from Lemma 2.2, $||\dfrac{\tilde{D}'_n(x)}{n^2}|| = O(1)$.

Further, by (1.9) and (1.10), we get

$$\int_0^\pi\int_0^\pi \left|\sum_{j=m+1}^{\infty}\sum_{k=n+1}^{\infty} jk\left(\frac{A_{jk}}{j^2k^2}\right)\tilde{D}'_j(x)\tilde{D}'_k(y)\right| \mathrm{d}x\mathrm{d}y \to 0 \qquad \text{as} \quad m+n\to\infty.$$

Thus by using the equation (3.3) and the given hypothesis all the terms on the right hand side of the inequality (3.4) tend to zero as $m+n\to\infty$. Hence, the conclusion of Theorem 3.1 holds.

We draw the following corollaries from Theorem 3.1.

Corollary 3.2. *Under the condition of Theorem* 3.1*, the sum f of the series* (1.1) *is the integrable and* (1.1) *is Fourier series of f.*

Proof. It follows from Theorem 3.1 that $f\in L^1(T^2)$. Furthermore, it is known that the convergence in L^1-norm (the so-called strong convergence) implies that in weak convergence.

Now, consider

$$\begin{aligned} g_{mn}(x,y) &= \frac{a_{00}}{2}+\sum_{j=1}^{m}\sum_{k=1}^{n}\left\{\sum_{r=j}^{m}\sum_{l=k}^{n}\triangle_{11}(a_{rl}\cos rx\cos ly)\right\} \\ &= S_{mn}(x,y)-\sum_{j=1}^{m}\sum_{k=1}^{n}\left\{a_{j,n+1}\cos jx\cos(n+1)y+a_{m+1,k}\cos(m+1)x\cos y\right\} \\ &\quad +mna_{m+1,n+1}\cos(m+1)x\cos(n+1)y \end{aligned}$$

for fixed $r,l\geq 1$, we get

$$\begin{aligned} &\frac{4}{\pi^2}\int_0^\pi\int_0^\pi f(x,y)\cos rx\cos ly\,\mathrm{d}x\mathrm{d}y \\ &\qquad = \lim_{m+n\to\infty}\frac{4}{\pi^2}\int_0^\pi\int_0^\pi u_{mn}(x,y)\cos rx\cos ly\,\mathrm{d}x\mathrm{d}y \\ &\qquad = a_{rl}-\lim_{m+n\to\infty}\left\{\sum_{j=1}^{m}na_{j,n+1}+\sum_{k=1}^{n}ma_{m+1,k}+mna_{m+1,n+1}\right\} \\ &\qquad = a_{rl} \end{aligned}$$

Since the limit of each term in the brace is zero (as already shown in the proof of Theorem 3.1). This proves that (1.1) is the Fourier series of f.

Corollary 3.3. *If a double sequence $\{a_{jk}\}$ belongs to the class J_d, then $||S_{mn}-f||\to 0$ as $m+n\to\infty$.*

Proof. Consider

$$\begin{aligned} ||f-S_{mn}|| &= ||f-g_{mn}+g_{mn}-S_{mn}||\leq ||f-g_{mn}||+||g_{mn}-S_{mn}|| \\ &\leq ||f-g_{mn}||+\int_0^\pi\int_0^\pi\left|\frac{a_{mn}}{mn}\tilde{D}'_m(x)\tilde{D}'_n(y)\right|\mathrm{d}x\mathrm{d}y \\ &\quad +\int_0^\pi\int_0^\pi\left|\sum_{j=1}^{m}na_{j,n+1}\cos jx\cos(n+1)y\right|\mathrm{d}x\mathrm{d}y \\ &\quad +\int_0^\pi\int_0^\pi\left|\sum_{k=1}^{n}ma_{m+1,k}\cos(m+1)x\cos y\right|\mathrm{d}x\mathrm{d}y \\ &\quad +mn|a_{m+1,n+1}|\int_0^\pi\int_0^\pi|\cos(m+1)x\cos(n+1)y|\mathrm{d}x\mathrm{d}y \end{aligned}$$

Using Theorem 3.1 the conclusion of the corollary 3.3 follows.

Remark 3.4. (a) We note that Theorem 3.1, Corollaries 3.2 and 3.3 can be considered as analogous results of Jatinderdeep Kaur and S.S. Bhatia [3] from one dimensional to two dimensional case.

(b) By making use of (1.10), we note that

$$|\triangle_{11}a_{jk}| \leq |\triangle_{10}a_{jk}| + |\triangle_{10}a_{j,k+1}| \leq A_{jk} + A_{j,k+1}. \tag{3.5}$$

It follows from (3.5) and the condition (1.9) that if $\{a_{jk}\}$ belongs to class J_d, then $\{a_{jk}\} \in BV_2 \bigcap \mathcal{C}_2$. Thus, Theorem 1.1, Corollaries 1.1 and 1.2 of [1] are particular cases of ours.

(c) Further, by setting $A_{jk} = |\triangle_{22}a_{jk}|$, it is not hard to verify that the class J_d contains all quasi-convex null sequences. Therefore, Corollary 3 of [2] holds in the case $\{a_{jk}\}$ belonging to the class J_d.

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