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BOUNDEDNESS FOR THE COMMUTATOR OF FRACTIONAL INTEGRAL ON GENERALIZED MORREY SPACE IN NONHOMOGENEOUS SPACE

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Abstract. In this paper, we will establish the boundedness of the commutator generated by fractional integral operator and RBMO(μ) function on generalized Morrey space in the non-homogeneous space.

Key words: *fractional integral operator, commutator, generalized Morrey space, RBMO(μ)*

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1 Introduction

Suppose μ is a non-negative Radon measure on $\mathbf{R}^d$ satisfying only the following growth condition: there exist constants $C > 0$ and $n \in (0,d]$ such that for all $x \in \mathbf{R}^d$ and $r > 0$

$$\mu(B(x,r)) \leqslant Cr^n, \tag{1}$$

where

$$B(x,r) = \{y \in \mathbf{R}^d : |y-x| < r\}.$$

The Euclidean space $\mathbf{R}^d$ with a non-negative Radon measure only satisfying the growth condition is called a nonhomogeneous space.

In 2001, Tolsa developed a series of basic theory on nonhomogeneous space and introduced RBMO(μ) space. Recently, the properties of fractional integral commutator on Morrey space

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are studied in [2]. The purpose of this article is to establish the boundedness of the commutator generated by fractional integral operator and RBMO(μ) function on the generalized Morrey space in nonhomogeneous space. Before giving the main result, we introduce some necessary notations.

Let $(\mathbf{R}^d,\mu)$ be a nonhomogeneous space and Q a closed cube in $\mathbf{R}^d$ with sides parallel to the axes, we denote its sidelength by $l(Q)$. For $\alpha>0$, αQ stands for the cube with the same center as Q and having sidelength $\alpha l(Q)$. Given $\alpha>1, \beta>\alpha^n$, where n is the fixed number in growth condition , we say Q is a (α,β) doubling cube if $\mu(\alpha Q)\leqslant\beta\mu(Q)$. In the following, if α and β are not specified, by a doubling cube we mean a $(2,2^{d+1})$ doubling cube. Given two cubes $Q_1\subset Q_2$ in $\mathbf{R}^d$, we set

$$K_{Q_1,Q_2}=1+\sum_{k=1}^{N_{Q_1,Q_2}}\frac{\mu(2^kQ_1)}{l(2^kQ_1)^n},$$

where N_{Q_1,Q_2} is the first positive integer k such that $l(2^kQ_1)\geqslant l(Q_2)$.

Remark. In this paper, for $b\in L^1_{loc}(\mu)$, we denote the mean of b over the cube Q by m_Qb, that is ,

$$m_Qb=\frac{1}{\mu(Q)}\int_Q b(x)\,\mathrm{d}\mu(x).$$

Definition 1.[1] *Let $(\mathbf{R}^d,\mu)$ be a nonhomogeneous space, $b\in L^1_{loc}(\mu)$ and $\rho>1$ a fixed constant, we say that b is in* RBMO(μ)*, if there exists a constant $C>0$ such that for any cube Q,*

$$\frac{1}{\mu(\rho Q)}\int_Q\left|b(x)-m_{\widetilde{Q}}b\right|\mathrm{d}\mu(x)\leqslant C, \tag{2}$$

and for any two doubling cubes $Q_1\subset Q_2$,

$$\left|m_{Q_1}b-m_{Q_2}b\right|\leqslant CK_{Q_1,Q_2}, \tag{3}$$

where $\widetilde{Q}$ is the smallest doubling cube in the family $\{2^kQ\}_{(k\in\mathbf{N})}$. The minimal constant C in (2) *and* (3) *is the* RBMO(μ) *norm of b, and it will be denoted by $\|b\|_*$.*

Remark. The definition of RBMO(μ) does not depend on the choice of ρ, see [1].

Definition 2.[2] *Let $(\mathbf{R}^d,\mu)$ be a nonhomogeneous space, n the fixed number in the growth condition and $0<s<n$, the fractional integral operator I_s on the nonhomogeneous space $(\mathbf{R}^d,\mu)$ is defined by*

$$I_sf(x)=\int_{\mathbf{R}^d}\frac{f(y)}{|x-y|^{n-s}}\,\mathrm{d}\mu(y). \tag{4}$$

Moreover, if $b\in$ RBMO(μ)*, the commutator $[b,I_s]$ is defined by*

$$[b,I_s]f(x)=\int_{\mathbf{R}^d}\left[b(x)-b(y)\right]\frac{f(y)}{|x-y|^{n-s}}\,\mathrm{d}\mu(y). \tag{5}$$

Definition 3.[3],[4] *Let* $(\mathbf{R}^d, \mu)$ *be a nonhomogeneous space, n the fixed number in the growth condition and* $1 \leqslant p < \infty$. *Given a function* $\phi : (0, +\infty) \to (0, +\infty)$, *the generalized Morrey space in nonhomogeneous space is defined by*

$$\mathcal{L}^{p,\phi}(\mu) = \left\{ f \in L^p_{\rm loc}(\mu) : \|f\|_{\mathcal{L}^{p,\phi}(\mu)} < \infty \right\},$$

where

$$\|f\|_{\mathcal{L}^{p,\phi}(\mu)} = \sup_{Q=Q(x,r)} \frac{1}{\phi(r)} \left(\frac{1}{r^n} \int_Q |f(y)|^p \, d\mu(y) \right)^{\frac{1}{p}},$$

$Q(x,r)$ *stands for the cube centered at x and the sidelength r.*

Remark. Taking $\phi(r) = r^{-\frac{n}{\lambda}}$, then

$$\mathcal{L}^{p,\phi}(\mu) = \mathcal{M}^{\lambda}_p(\mu),$$

where $\mathcal{M}^{\lambda}_p(\mu)$ is the Morrey space in nonhomogeneous space, see [2].

The following lemmas will be used in the proof of the main result.

Lemma 1.[5] *Let* $(\mathbf{R}^d, \mu)$ *be a nonhomogeneous space, n the fixed number in the growth condition,* $0 < s < n$ *and* $b \in \mathrm{RBMO}(\mu)$, *the commutator* $[b, I_s]$ *is defined as* (5), $1 < p < \frac{n}{s}$, $\frac{1}{q} = \frac{1}{p} - \frac{s}{n}$, *then* $[b, I_s]$ *is bounded from* $L^p(\mu)$ *into* $L^q(\mu)$, *namely,* $\|[b, I_s]f\|_{L^q(\mu)} \leqslant C\|b\|_* \|f\|_{L^p(\mu)}$.

Lemma 2.[6] *Let* $\{Q_j\}$ *be a sequence of cubes with the same center, and* $l(Q_{j+1}) = 2l(Q_j)$, $\widetilde{Q_j}$ *the smallest doubling cube in the family* $\{2^k Q_j\}_{(k \in \mathbf{N})}$. *If* $j < k$, *then we have* $\widetilde{Q_j} \subset \widetilde{Q_k}$ *and* $K_{\widetilde{Q_j}, \widetilde{Q_k}} \leqslant C(k - j)$.

Lemma 3.[1] *Let* $\rho > 1$ *be some fixed constant and* $b \in \mathrm{RBMO}(\mu)$, $1 \leqslant p < \infty$, *then there exists a constant* $C > 0$ *such that for any cube Q*

$$\left\{ \int_Q |b(x) - m_{\widetilde{Q}} b|^p \, \mathrm{d}\mu(x) \right\}^{\frac{1}{p}} \leqslant C\|b\|_* \mu(\rho Q)^{\frac{1}{p}},$$

where $\widetilde{Q}$ *is the smallest doubling cube in the family* $\{2^k Q\}_{(k \in \mathbf{N})}$.

Remark. In the following proof , C always denotes a positive constant that is independent of the main parameters involved, but its value may differ from one occurrence to another. For any index $1 < p < \infty$, we denote by p' its conjugate index, namely, $\dfrac{1}{p} + \dfrac{1}{p'} = 1$.

2 Main Results

In this paper, we obtain the following result:

Theorem. *Let* $(\mathbf{R}^d, \mu)$ *be a nonhomogeneous space, n the fixed number in the growth condition,* $0 < s < n$, $b \in \mathrm{RBMO}(\mu)$ *and the commutator* $[b, I_s]$ *be defined as* (5), $1 < p < \frac{n}{s}$,

$\frac{1}{q}=\frac{1}{p}-\frac{s}{n}$. *Suppose the function* $\phi:(0,+\infty)\to(0,+\infty)$ *satisfies the following condition: there exists a constant* $C>0$*, such that* $\phi(2r)\leqslant C\phi(r)$ *for any* $r>0$. *The function* $\psi:(0,+\infty)\to(0,+\infty)$ *satisfies:*

(i) *there exists a constant* $C>0$*, such that* $r^s\phi(r)\leqslant C\psi(r)$ *for any* $r>0$*;*

(ii) *there exists a constant* $0<C_\psi<1$*, such that*

$$\psi(2^{k+1}r)\leqslant C_\psi^{k+1}\psi(r)$$

for any $r>0$ *and* $k\in\mathbf{N}_+$.

Then $[b,I_s]$ *is bounded from* $\mathcal{L}^{p,\phi}(\mu)$ *into* $\mathcal{L}^{q,\psi}(\mu)$.

Remark. Taking $\phi(r)=r^{-\frac{n}{\lambda}},\psi(r)=r^{-\frac{n}{\delta}}$ (in this case, ϕ and ψ satisfy the conditions of the theorem), thus we get $[b,I_s]$ is bounded from $\mathcal{M}_p^\lambda(\mu)$ into $\mathcal{M}_q^\delta(\mu)$, and this is just the case considered in [2].

Proof of Theorem. Fix $f\in\mathcal{L}^{p,\phi}(\mu)$, for any $x_0\in\mathbf{R}^d$ and $r>0$, let $Q=Q(x_0,r)$, $Q_k=2^kQ$, $E_k=Q_{k+1}\backslash Q_k$, $k=1,2,3,\cdots$.We decompose $f(x)$ as

$$f(x)=f\chi_{2Q}+\sum_{k=1}^{\infty}f\chi_{E_k}\triangleq f_0+\sum_{k=1}^{\infty}f_k,$$

then we have

$$\begin{aligned}\frac{1}{\psi(r)}\left(\frac{1}{r^n}\int_Q\big|[b,I_s]f(x)\big|^q\mathrm{d}\mu(x)\right)^{\frac{1}{q}}&\leqslant\frac{1}{\psi(r)}\left(\frac{1}{r^n}\int_Q\big|[b,I_s]f_0(x)\big|^q\mathrm{d}\mu(x)\right)^{\frac{1}{q}}\\&+\sum_{k=1}^{\infty}\frac{1}{\psi(r)}\left(\frac{1}{r^n}\int_Q\big|[b,I_s]f_k(x)\big|^q\mathrm{d}\mu(x)\right)^{\frac{1}{q}}\triangleq D_1+D_2.\end{aligned}$$

For D_1, by Lemma 1 we get

$$\begin{aligned}D_1&\leqslant C\frac{1}{\psi(r)}\left(\frac{1}{r^n}\right)^{\frac{1}{q}}\|b\|_*\left(\int_{\mathbf{R}^d}|f_0(x)|^p\mathrm{d}\mu(x)\right)^{\frac{1}{p}}\\&=C\frac{1}{\psi(r)}\left(\frac{1}{r^n}\right)^{\frac{1}{q}}\|b\|_*\left(\int_{2Q}|f(x)|^p\mathrm{d}\mu(x)\right)^{\frac{1}{p}}\\&=C\|b\|_*\frac{1}{\psi(r)}r^{-\frac{n}{q}}\phi(2r)(2r)^{\frac{n}{p}}\frac{1}{\phi(2r)}\left(\frac{1}{(2r)^n}\int_{2Q}|f(x)|^p\mathrm{d}\mu(x)\right)^{\frac{1}{p}}\\&\leqslant C\|b\|_*\frac{1}{\psi(r)}r^{\frac{n}{p}-\frac{n}{q}}\phi(r)\|f\|_{\mathcal{L}^{p,\phi}(\mu)}\\&=C\|b\|_*\frac{1}{\psi(r)}r^s\phi(r)\|f\|_{\mathcal{L}^{p,\phi}(\mu)}\\&\leqslant C\|b\|_*\frac{1}{\psi(r)}\psi(r)\|f\|_{\mathcal{L}^{p,\phi}(\mu)}=C\|b\|_*\|f\|_{\mathcal{L}^{p,\phi}(\mu)}.\end{aligned}$$

For D_2, notice that if $x\in Q, y\in E_k$, then $|y-x|\sim|y-x_0|\sim 2^{k+1}r$, so we have

$$\begin{aligned}
D_2 &= \frac{1}{\psi(r)}r^{-\frac{n}{q}}\sum_{k=1}^{\infty}\left(\int_Q\left|[b,I_s]f_k(x)\right|^q\mathrm{d}\mu(x)\right)^{\frac{1}{q}}\\
&= \frac{1}{\psi(r)}r^{-\frac{n}{q}}\sum_{k=1}^{\infty}\left(\int_Q\left|\int_{\mathbf{R}^d}\left[b(x)-b(y)\right]\frac{f_k(y)}{|x-y|^{n-s}}\,d\mu(y)\right|^q\mathrm{d}\mu(x)\right)^{\frac{1}{q}}\\
&\leqslant \frac{1}{\psi(r)}r^{-\frac{n}{q}}\sum_{k=1}^{\infty}\left\{\int_Q\left(\int_{E_k}|b(x)-b(y)|\frac{|f(y)|}{|x-y|^{n-s}}\mathrm{d}\mu(y)\right)^q\mathrm{d}\mu(x)\right\}^{\frac{1}{q}}\\
&\leqslant \frac{C}{\psi(r)}r^{-\frac{n}{q}}\sum_{k=1}^{\infty}\frac{1}{\left(2^{k+1}r\right)^{n-s}}\left\{\int_Q\left(\int_{E_k}|b(x)-b(y)||f(y)|\mathrm{d}\mu(y)\right)^q\mathrm{d}\mu(x)\right\}^{\frac{1}{q}},
\end{aligned}$$

Notice that

$$\begin{aligned}
&\left\{\int_Q\left(\int_{E_k}|b(x)-b(y)||f(y)|\mathrm{d}\mu(y)\right)^q\mathrm{d}\mu(x)\right\}^{\frac{1}{q}}\\
&\leqslant\left\{\int_Q\left(\int_{E_k}|b(x)-m_{\widetilde{Q}}b||f(y)|\mathrm{d}\mu(y)+\int_{E_k}|b(y)-m_{\widetilde{Q}}b||f(y)|\mathrm{d}\mu(y)\right)^q d\mu(x)\right\}^{\frac{1}{q}}\\
&\leqslant\left\{\int_Q\left(\int_{E_k}|b(x)-m_{\widetilde{Q}}b||f(y)|\,\mathrm{d}\mu(y)\right)^q\mathrm{d}\mu(x)\right\}^{\frac{1}{q}}+\mu(Q)^{\frac{1}{q}}\int_{E_k}|b(y)-m_{\widetilde{Q}}b||f(y)|\mathrm{d}\mu(y)\\
&=\int_{E_k}|f(y)|\,\mathrm{d}\mu(y)\left(\int_Q|b(x)-m_{\widetilde{Q}}b|^q\mathrm{d}\mu(x)\right)^{\frac{1}{q}}+\mu(Q)^{\frac{1}{q}}\int_{E_k}|b(y)-m_{\widetilde{Q}}b||f(y)|\,\mathrm{d}\mu(y).
\end{aligned}$$

Thus we get

$$\begin{aligned}
D_2 \leqslant\ & \frac{C}{\psi(r)}r^{-\frac{n}{q}}\sum_{k=1}^{\infty}\frac{1}{\left(2^{k+1}r\right)^{n-s}}\int_{E_k}|f(y)|\,\mathrm{d}mu(y)\left(\int_Q|b(x)-m_{\widetilde{Q}}b|^q\mathrm{d}\mu(x)\right)^{\frac{1}{q}}\\
&+\frac{C}{\psi(r)}r^{-\frac{n}{q}}\sum_{k=1}^{\infty}\frac{1}{\left(2^{k+1}r\right)^{n-s}}\mu(Q)^{\frac{1}{q}}\int_{E_k}|b(y)-m_{\widetilde{Q}}b||f(y)|\mathrm{d}\mu(y)\\
\triangleq\ & E_1+E_2.
\end{aligned}$$

By Lemma 3 and Hölder inequality, we have

$$\begin{aligned}
E_1 &\leqslant \frac{C}{\psi(r)}r^{-\frac{n}{q}}\sum_{k=1}^{\infty}\frac{1}{\left(2^{k+1}r\right)^{n-s}}\|b\|_*\mu(\rho Q)^{\frac{1}{q}}\int_{Q_{k+1}}|f(y)|\mathrm{d}\mu(y)\\
&\leqslant \frac{C}{\psi(r)}r^{-\frac{n}{q}}\|b\|_*\sum_{k=1}^{\infty}\frac{1}{\left(2^{k+1}r\right)^{n-s}}r^{\frac{n}{q}}\mu(Q_{k+1})^{\frac{1}{p'}}\left(\int_{Q_{k+1}}|f(y)|^p\mathrm{d}\mu(y)\right)^{\frac{1}{p}}\\
&\leqslant \frac{C}{\psi(r)}r^{-\frac{n}{q}}\|b\|_*\sum_{k=1}^{\infty}\frac{r^{\frac{n}{q}}\left(2^{k+1}r\right)^{\frac{n}{p'}}}{\left(2^{k+1}r\right)^{n-s}}\phi(2^{k+1}r)\left(2^{k+1}r\right)^{\frac{n}{p}}\frac{1}{\phi(2^{k+1}r)}\left(\frac{1}{(2^{k+1}r)^n}\int_{Q_{k+1}}|f(y)|^p\mathrm{d}\mu(y)\right)^{\frac{1}{p}}
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \frac{C}{\psi(r)}\|b\|_* \sum_{k=1}^{\infty} \frac{1}{\left(2^{k+1}r\right)^{n-s}} (2^{k+1}r)^n \phi(2^{k+1}r)\|f\|_{\mathcal{L}^{p,\phi}(\mu)} \\
&= \frac{C}{\psi(r)}\|b\|_*\|f\|_{\mathcal{L}^{p,\phi}(\mu)} \sum_{k=1}^{\infty} (2^{k+1}r)^s \phi(2^{k+1}r) \\
&\leqslant \frac{C}{\psi(r)}\|b\|_*\|f\|_{\mathcal{L}^{p,\phi}(\mu)} \sum_{k=1}^{\infty} \psi(2^{k+1}r) \\
&\leqslant \frac{C}{\psi(r)}\|b\|_*\|f\|_{\mathcal{L}^{p,\phi}(\mu)} \sum_{k=1}^{\infty} C_\psi^{k+1}\psi(r) \\
&\leqslant C\|b\|_*\|f\|_{\mathcal{L}^{p,\phi}(\mu)}.
\end{aligned}
$$

Notice that

$$
\begin{aligned}
E_2 &\leqslant \frac{C}{\psi(r)} r^{-\frac{n}{q}} \sum_{k=1}^{\infty} \frac{1}{\left(2^{k+1}r\right)^{n-s}} r^{\frac{n}{q}} \int_{Q_{k+1}} \left|b(y)-m_{\widetilde{Q}}b\right|\left|f(y)\right| \mathrm{d}\mu(y) \\
&= \frac{C}{\psi(r)} \sum_{k=1}^{\infty} \frac{1}{\left(2^{k+1}r\right)^{n-s}} \int_{Q_{k+1}} \left|b(y)-m_{\widetilde{Q}}b\right|\left|f(y)\right| \mathrm{d}\mu(y) \\
&\leqslant \quad \frac{C}{\psi(r)} \sum_{k=1}^{\infty} \frac{1}{\left(2^{k+1}r\right)^{n-s}} \int_{Q_{k+1}} \left|b(y)-m_{\widetilde{Q_{k+1}}}b\right|\left|f(y)\right| \mathrm{d}\mu(y) \\
&\quad + \frac{C}{\psi(r)} \sum_{k=1}^{\infty} \frac{1}{\left(2^{k+1}r\right)^{n-s}} \int_{Q_{k+1}} \left|m_{\widetilde{Q_{k+1}}}b-m_{\widetilde{Q}}b\right|\left|f(y)\right| \mathrm{d}\mu(y) \\
&\triangleq F_1+F_2.
\end{aligned}
$$

By Hölder inequality , Lemma 3, and Lemma 2, we get

$$
\begin{aligned}
F_1 &\leqslant \frac{C}{\psi(r)} \sum_{k=1}^{\infty} \frac{1}{\left(2^{k+1}r\right)^{n-s}} \left(\int_{Q_{k+1}} \left|b(y)-m_{\widetilde{Q_{k+1}}}b\right|^{p'} \mathrm{d}\mu(y)\right)^{\frac{1}{p'}} \left(\int_{Q_{k+1}} \left|f(y)\right|^p \mathrm{d}\mu(y)\right)^{\frac{1}{p}} \\
&\leqslant \frac{C}{\psi(r)} \sum_{k=1}^{\infty} \frac{\|b\|_* \mu(\rho Q_{k+1})^{\frac{1}{p'}}}{\left(2^{k+1}r\right)^{n-s}} \phi(2^{k+1}r)\left(2^{k+1}r\right)^{\frac{n}{p}} \\
&\qquad \frac{1}{\phi(2^{k+1}r)} \left(\frac{1}{(2^{k+1}r)^n} \int_{Q_{k+1}} \left|f(y)\right|^p \mathrm{d}\mu(y)\right)^{\frac{1}{p}} \\
&\leqslant \frac{C}{\psi(r)}\|b\|_* \sum_{k=1}^{\infty} \frac{1}{\left(2^{k+1}r\right)^{n-s}} \left(2^{k+1}r\right)^{\frac{n}{p'}} \phi(2^{k+1}r)\left(2^{k+1}r\right)^{\frac{n}{p}} \|f\|_{\mathcal{L}^{p,\phi}(\mu)} \\
&\leqslant \frac{C}{\psi(r)}\|b\|_*\|f\|_{\mathcal{L}^{p,\phi}(\mu)} \sum_{k=1}^{\infty} (2^{k+1}r)^s \phi(2^{k+1}r) \\
&\leqslant \frac{C}{\psi(r)}\|b\|_*\|f\|_{\mathcal{L}^{p,\phi}(\mu)} \sum_{k=1}^{\infty} \psi(2^{k+1}r) \\
&\leqslant \frac{C}{\psi(r)}\|b\|_*\|f\|_{\mathcal{L}^{p,\phi}(\mu)} \sum_{k=1}^{\infty} C_\psi^{k+1}\psi(r) \\
&\leqslant C\|b\|_*\|f\|_{\mathcal{L}^{p,\phi}(\mu)},
\end{aligned}
$$

and

$$\begin{aligned} F_2 &= \frac{C}{\psi(r)}\sum_{k=1}^{\infty}\frac{|m_{\widetilde{Q_{k+1}}}b-m_{\widetilde{Q}}b|}{(2^{k+1}r)^{n-s}}\int_{Q_{k+1}}|f(y)|\mathrm{d}\mu(y) \\ &\leqslant \frac{C}{\psi(r)}\sum_{k=1}^{\infty}\frac{\|b\|_* K_{\widetilde{Q},\widetilde{Q_{k+1}}}}{(2^{k+1}r)^{n-s}}\mu(Q_{k+1})^{\frac{1}{p'}}\left(\int_{Q_{k+1}}|f(y)|^p\mathrm{d}\mu(y)\right)^{\frac{1}{p}} \\ &\leqslant \frac{C}{\psi(r)}\sum_{k=1}^{\infty}\frac{\|b\|_*(k+1)}{(2^{k+1}r)^{n-s}}(2^{k+1}r)^{\frac{n}{p'}}\phi(2^{k+1}r)(2^{k+1}r)^{\frac{n}{p}} \\ &\qquad \frac{1}{\phi(2^{k+1}r)}\left(\frac{1}{(2^{k+1}r)^n}\int_{Q_{k+1}}|f(y)|^p\mathrm{d}\mu(y)\right)^{\frac{1}{p}} \\ &\leqslant \frac{C}{\psi(r)}\sum_{k=1}^{\infty}\frac{\|b\|_*(k+1)}{(2^{k+1}r)^{n-s}}(2^{k+1}r)^n\phi(2^{k+1}r)\|f\|_{\mathcal{L}^{p,\phi}(\mu)} \\ &\leqslant \frac{C}{\psi(r)}\|b\|_*\|f\|_{\mathcal{L}^{p,\phi}(\mu)}\sum_{k=1}^{\infty}(k+1)(2^{k+1}r)^s\phi(2^{k+1}r) \\ &\leqslant \frac{C}{\psi(r)}\|b\|_*\|f\|_{\mathcal{L}^{p,\phi}(\mu)}\sum_{k=1}^{\infty}(k+1)\psi(2^{k+1}r) \\ &\leqslant \frac{C}{\psi(r)}\|b\|_*\|f\|_{\mathcal{L}^{p,\phi}(\mu)}\sum_{k=1}^{\infty}(k+1)C_{\psi}^{k+1}\psi(r) \\ &\leqslant C\|b\|_*\|f\|_{\mathcal{L}^{p,\phi}(\mu)}. \end{aligned}$$

The above estimates yield that

$$\frac{1}{\psi(r)}\left(\frac{1}{r^n}\int_Q|[b,I_s]f(x)|^q\mathrm{d}\mu(x)\right)^{\frac{1}{q}}\leqslant C\|b\|_*\|f\|_{\mathcal{L}^{p,\phi}(\mu)},$$

Thus we have

$$\|[b,I_s]f\|_{\mathcal{L}^{q,\psi}(\mu)}\leqslant C\|b\|_*\|f\|_{\mathcal{L}^{p,\phi}(\mu)}.$$

Namely, $[b,I_s]$ is bounded from $\mathcal{L}^{p,\phi}(\mu)$ into $\mathcal{L}^{q,\psi}(\mu)$.

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