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APPROXIMATING COMMON FIXED POINTS OF NEARLY ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

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Abstract. We use an iteration scheme to approximate common fixed points of nearly asymptotically nonexpansive mappings. We generalize corresponding theorems of [1] to the case of two nearly asymptotically nonexpansive mappings and those of [9] not only to a larger class of mappings but also with better rate of convergence.

Key words: *iteration scheme, nearly asymptotically nonexpansive mapping, rate of convergence, common fixed point, weak and strong convergence*

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1 Introduction

Throughout this paper, $\mathbf{N}$ denotes the set of all positive integers. Let E be a real Banach space and C a nonempty subset of E. A mapping $T : C \to C$ is called asymptotically nonexpansive if for a sequence $\{k_n\} \subset [1,\infty)$ with $\lim\limits_{n\to\infty} k_n = 1$, we have

$$\|T^n x - T^n y\| \le k_n \|x - y\|$$

for all $x, y \in C$ and $n \in \mathbf{N}$. T is called uniformly L-Lipschitzian if for some $L > 0$, $\|T^n x - T^n y\| \le L\|x - y\|$ for all $x, y \in C$ and $n \in \mathbf{N}$. Also, T is called a contraction if for some $0 < k < 1$, $\|Tx - Ty\| \le k\|x - y\|$ for all $x, y \in C$.

Fix a sequence $\{a_n\} \subset [0,\infty)$ with $\lim\limits_{n\to\infty} a_n = 0$, then according to Agarwal et al[1], T is said to be nearly asymptotically nonexpansive if $k_n \ge 1$ for all $n \in \mathbf{N}$ with $\lim\limits_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \le k_n (\|x - y\| + a_n)$$

for all $x, y \in C$. T will be nearly uniformly L-Lipschitzian if $k_n \leq L$ for all $n \in \mathbf{N}$.

Note that every asymptotically nonexpansive mapping is nearly asymptotically nonexpansive and every nearly asymptotically nonexpansive mapping is nearly uniformly L-Lipschitzian.

We know that Picard and Mann iteration processes for a mapping $T : C \to C$ are defined as:

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = Tx_n\,,\ n \in \mathbf{N} \end{cases} \tag{1.1}$$

and

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n,\ \ n \in \mathbf{N} \end{cases} \tag{1.2}$$

respectively, where $\{\alpha_n\}$ is in $(0,1)$.

Recently, Agarwal et al.[1] introduced the following iteration scheme:

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = (1-\alpha_n)T^n x_n + \alpha_n T^n y_n, \\ y_n = (1-\beta_n)\,x_n + \beta_n T^n x_n,\ \ n \in \mathbf{N}, \end{cases} \tag{1.3}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are in $(0,1)$. They showed that this scheme converges at a rate same as that of Picard iteration.

On the other hand, we state without error terms the iteration scheme studied by Yao and Chen [9] for common fixed points of two mappings:

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = \alpha_n x_n + \beta_n T^n x_n + \gamma_n S^n x_n,\ n \in \mathbf{N}, \end{cases} \tag{1.4}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are in $[0,1]$ and $\alpha_n + \beta_n + \gamma_n = 1$. They did not show the rate of convergence of this scheme.

We introduce the following iteration scheme to compute the common fixed points of two mappings.

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = (1-\alpha_n)T^n x_n + \alpha_n S^n y_n, \\ y_n = (1-\beta_n)\,x_n + \beta_n T^n x_n, n \in \mathbf{N}, \end{cases} \tag{1.5}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are in $(0,1)$.

It is to be noted that (1.5) reduces to

• (1.3) when $S = T$.

• (1.2) when $T = I$.

Moreover, when $T = I$, (1.4) reduces to Mann iteration scheme.

Having noted that both (1.5) and (1.4) reduce to Mann iteration scheme, we will show that (1.5) is better than (1.4). Actually, we will see that the rate of convergence of (1.5) is the same as that of Picard iteration while that of (1.4) is the same as Mann iteration thus establishing that our iteration scheme (1.5) converges faster than (1.4). We will then use it to prove that a common fixed point exists for nearly asymptotically nonexpansive self mappings. In this way, we will generalize corresponding theorems of [1] to the case of two nearly asymptotically nonexpansive mappings and those of [9] not only for a larger class of mappings but also with better rate of convergence.

Let $S = \{x \in E : \|x\| = 1\}$ and let E^* be the dual of E, that is, the space of all continuous linear functionals f on E. The space E has : (i) Gâteaux differentiable norm if

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each x and y in S; (ii) Fréchet differentiable norm (see e.g. [8]) if for each x in S, the above limit exists and is attained uniformly for y in S and in this case, it is also well-known that

$$\langle h, J(x) \rangle + \frac{1}{2}\|x\|^2 \leq \frac{1}{2}\|x + h\|^2 \leq \langle h, J(x) \rangle + \frac{1}{2}\|x\|^2 + b(\|h\|) \tag{1.6}$$

for all x, h in E, where J is the Fréchet derivative of the functional $\frac{1}{2}\|.\|^2$ at $x \in E$, $\langle .,. \rangle$ is the pairing between E and E^*, and b is an increasing function defined on $[0, \infty)$ such that $\lim_{t \downarrow 0} \frac{b(t)}{t} = 0$; (iii) *Opial property* [6] if for any sequence $\{x_n\}$ in $E, x_n \rightharpoonup x$ implies that $\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$ for all $y \in E$ with $y \neq x$ and (iv) *Kadec-Klee property* if for every sequence $\{x_n\}$ in E, $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$ together imply $x_n \to x$ as $n \to \infty$.

Let δ be the modulus of uniform convexity. Recall that if E is a uniformly convex Banach space then (see e.g. [3])

$$\|tx + (1-t)y\| \leq 1 - 2t(1-t)\delta\,\|x - y\| \tag{1.7}$$

for all $t \in [0, 1]$ and for all $x, y \in E$ such that $\|x\| \leq 1, \|y\| \leq 1$.

A mapping $T : C \to E$ is demiclosed at $y \in E$ if for each sequence $\{x_n\}$ in C and each $x \in E$, $x_n \rightharpoonup x$ and $Tx_n \to y$ imply that $x \in C$ and $Tx = y$.

First we state the following lemmas to be used later on.

Lemma 1[7]. *Suppose that E is a uniformly convex Banach space and $0 < p \leq t_n \leq q < 1$ for all $n \in \mathbf{N}$. Let $\{x_n\}$ and $\{y_n\}$ be two sequences of E such that $\limsup_{n\to\infty} \|x_n\| \leq r$, $\limsup_{n\to\infty} \|y_n\| \leq r$ and $\lim_{n\to\infty} \|t_n x_n + (1-t_n)y_n\| = r$ hold for some $r \geq 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Lemma 2. *If $\{r_n\}$, $\{t_n\}$ and $\{s_n\}$ are sequences of nonnegative real numbers such that $r_{n+1} \le (1+t_n)r_n + s_n$, $\sum_{n=1}^{\infty} t_n < \infty$ and $\sum_{n=1}^{\infty} s_n < \infty$, then $\lim_{n\to\infty} r_n$ exists.*

Lemma 3[1]. *Let E be a uniformly convex Banach space satisfying Opial's condition and let C be a nonempty closed convex subset of E. Let T be a uniformly continuous nearly asymptotically nonexpansive mapping of C into itself. Then $I-T$ is demiclosed with respect to zero.*

Lemma 4[5]. Let *E be a reflexive Banach space such that E^* has the Kadec-Klee property. Let $\{x_n\}$ be a bounded sequence in E and $x^*, y^* \in W = \omega_w(x_n)$(weak limit set of $\{x_n\}$). Suppose $\lim_{n\to\infty} \|tx_n + (1-t)x^* - y^*\|$ exists for all $t \in [0,1]$. Then $x^* = y^*$.*

2 Convergence Theorems

Follwing the method of Agarwal et al. [1], first we calculate the rate of convergence of both (1.4) and (1.5). Recall that if $x_n \to q$, $y_n \to q$, then we say that $\{x_n\}$ is better than $\{y_n\}$ if $\|x_n - q\| \le \|y_n - q\|$ for all n. See [2].

Proposition 1. *Let C be a nonempty closed convex subset of a normed space E. Let S and T be two self contractions of C. If $\{x_n\}$ defined by both (1.4) and (1.5) converge to a common fixed point p of S and T, then $\{x_n\}$ in (1.4) converges at a rate same as that of Mann while $\{x_n\}$ in (1.5) converges at a rate same as that of Picard.*

Proof. Let p be a common fixed point of S and T. For Picard iteration scheme,

$$\|x_{n+1} - p\| \;=\; \|Tx_n - p\| \le k\|x_n - p\|.$$

For Mann iteration scheme,

$$\begin{aligned}\|x_{n+1} - p\| &= \|(1-\alpha_n)(x_n - p) + \alpha_n(Tx_n - p)\| \le (1-\alpha_n)\|x_n - p\| + \alpha_n k\|x_n - p\| \\ &= (1-(1-k)\alpha_n)\|x_n - p\| \le \|x_n - p\|.\end{aligned}$$

For the scheme (1.4) studied by Yao and Chen,

$$\begin{aligned}\|x_{n+1} - p\| &= \|\alpha_n(x_n - p) + \beta_n(Tx_n - p) + \gamma_n(Sx_n - p)\| \\ &\le (\alpha_n + \beta_n k + \gamma_n k)\|x_n - p\| = (\alpha_n + (1-\alpha_n)k)\|x_n - p\| \\ &= (\alpha_n(1-k) + k)\|x_n - p\| \le \|x_n - p\|,\end{aligned}$$

because $k \le (\alpha_n(1-k) + k) \le 1$ for all $k \in (0,1)$.

Finally, for our iteration scheme (1.5),

$$
\begin{aligned}
\|x_{n+1}-p\| &= \|(1-\alpha_n)(Tx_n-p)+\alpha_n(Sy_n-p)\| \\
&\leq (1-\alpha_n)k\,\|x_n-p\|+\alpha_n k\,\|y_n-p\| \\
&= k[(1-\alpha_n)\,\|x_n-p\|+\alpha_n(\|(1-\beta_n)(x_n-p)+\beta_n(Tx_n-p)\|)] \\
&\leq k[(1-\alpha_n+\alpha_n(1-\beta_n)+\alpha_n\beta_n k]\,\|x_n-p\| \\
&= k[(1-(1-k)\alpha_n\beta_n]\,\|x_n-p\| \\
&\leq k\,\|x_n-p\|.
\end{aligned}
$$

Clearly, (1.4) converges at the rate equal to Mann iteration while (1.5) at that equal to Picard.

Hence our scheme has a better rate of convergence.

Our next theorem is the key for our later results. From here onwards, F denotes the set of common fixed points of the mappings T and S.

Theorem 1. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E. Let T and S be two nearly asymptotically nonexpansive self mappings of C with a sequence $\{a_n\}$ such that $\sum_{n=1}^{\infty} a_n < \infty$ and $\sum_{n=1}^{\infty}(k_n-1) < \infty$. Let $\{x_n\}$ be defined by the iteration scheme* (1.5), *where $\{\alpha_n\},\{\beta_n\}$ are in $[\varepsilon, 1-\varepsilon]$ for all $n \in \mathbf{N}$ and for some ε in $(0,1)$. If $F \neq \varnothing$, then*

$$
\lim_{n\to\infty}\|x_n-Tx_n\| = 0 = \lim_{n\to\infty}\|x_n-Sx_n\|.
$$

Proof. Let $q \in F$. Then

$$
\begin{aligned}
\|x_{n+1}-q\| &= \|(1-\alpha_n)T^n x_n+\alpha_n S^n y_n-q\| \\
&\leq (1-\alpha_n)\,\|T^n x_n-q\|+\alpha_n\,\|S^n y_n-q\| \\
&\leq (1-\alpha_n)k_n\,(\|x_n-q\|+a_n)+\alpha_n k_n\,(\|y_n-q\|+a_n) \\
&= k_n\,[(1-\alpha_n)\,\|x_n-q\|+\alpha_n\,\|y_n-q\|+a_n] \\
&\leq k_n\left[\begin{array}{c} (1-\alpha_n)\,\|x_n-q\|+\alpha_n\,(1-\beta_n)\,\|x_n-q\| \\ +\alpha_n\beta_n\,\|T^n x_n-q\|+a_n \end{array}\right] \\
&\leq k_n\left[\begin{array}{c} (1-\alpha_n)\,\|x_n-q\|+\alpha_n\,(1-\beta_n)\,\|x_n-q\| \\ +k_n\alpha_n\beta_n\,\|x_n-q\|+k_n\alpha_n\beta_n a_n+a_n \end{array}\right] \\
&= k_n\left[\begin{array}{c} (1-\alpha_n+\alpha_n\,(1-\beta_n)+k_n\alpha_n\beta_n)\,\|x_n-q\| \\ +k_n\alpha_n\beta_n a_n+a_n \end{array}\right]
\end{aligned}
$$

$$\begin{aligned}
&\leq k_n\left[(1+(k_n-1))\|x_n-q\|+(k_n+1)a_n\right] \\
&= \left(1+\left(k_n^2-1\right)\right)\|x_n-q\|+k_n(k_n+1)a_n \\
&\leq \left(1+\left(k_n^2-1\right)\right)\|x_n-q\|+K(K+1)a_n,
\end{aligned}$$

where $K=\sup_{n\in\mathbf{N}} k_n$. Thus by Lemma 1, $\lim\limits_{n\to\infty}\|x_n-q\|$ exists. Call it c.

Now

$$\begin{aligned}
\|y_n-q\| &= \|\beta_n T^n x_n+(1-\beta_n)x_n-q\| \\
&= \|\beta_n(T^n x_n-q)+(1-\beta_n)(x_n-q)\| \\
&\leq \beta_n\|T^n x_n-q\|+(1-\beta_n)\|x_n-q\| \\
&\leq \beta_n k_n(\|x_n-q\|+a_n)+(1-\beta_n)\|x_n-q\| \\
&= (1+\beta_n(k_n-1))\|x_n-q\|+\beta_n k_n a_n
\end{aligned}$$

implies that

$$\limsup_{n\to\infty}\|y_n-q\|\leq c. \tag{2.1}$$

Also

$$\|T^n x_n-q\|\leq k_n\left(\|x_n-q\|+a_n\right)$$

for all n = 1, 2,..., so

$$\limsup_{n\to\infty}\|T^n x_n-q\|\leq c. \tag{2.2}$$

Next,

$$\|S^n y_n-q\|\leq k_n\left(\|y_n-q\|+a_n\right)$$

gives by (2.1) that

$$\limsup_{n\to\infty}\|S^n y_n-q\|\leq c.$$

Moreover, $c=\lim\limits_{n\to\infty}\|x_{n+1}-q\|=\lim\limits_{n\to\infty}\|(1-\alpha_n)(T^n x_n-q)+\alpha_n(S^n y_n-q)\|$ gives by Lemma 1,

$$\lim_{n\to\infty}\|T^n x_n-S^n y_n\|=0. \tag{2.3}$$

Now

$$\begin{aligned}
\|x_{n+1}-q\| &= \|(1-\alpha_n)T^n x_n+\alpha_n S^n y_n-q\| \\
&= \|(T^n x_n-q)+\alpha_n(S^n y_n-T^n x_n)\| \\
&\leq \|T^n x_n-q\|+\alpha_n\|T^n x_n-S^n y_n\|
\end{aligned}$$

yields that

$$c\leq\liminf_{n\to\infty}\|T^n x_n-q\|$$

so that (2.2) gives $\lim_{n\to\infty} \|T^n x_n - q\| = c$.
In turn,

$$\begin{aligned} \|T^n x_n - q\| &\le \|T^n x_n - S^n y_n\| + \|S^n y_n - q\| \\ &\le \|T^n x_n - S^n y_n\| + k_n (\|y_n - q\| + a_n) \end{aligned}$$

implies

$$c \le \liminf_{n\to\infty} \|y_n - q\|. \tag{2.4}$$

By (2.1) and (2.4), we obtain

$$\lim_{n\to\infty} \|y_n - q\| = c. \tag{2.5}$$

Moreover, $\|T^n x_n - q\| \le k_n (\|x_n - q\| + a_n)$ implies that

$$\limsup_{n\to\infty} \|T^n x_n - q\| \le c.$$

Thus $c = \lim_{n\to\infty} \|y_n - q\| = \lim_{n\to\infty} \| (1-\beta_n)(x_n - q) + \beta_n (T^n x_n - q) \|$ gives by Lemma 1 that

$$\lim_{n\to\infty} \|T^n x_n - x_n\| = 0. \tag{2.6}$$

Now

$$\|y_n - x_n\| = \beta_n \|T^n x_n - x_n\|.$$

Hence by (2.6),

$$\lim_{n\to\infty} \|y_n - x_n\| = 0. \tag{2.7}$$

Also note that

$$\begin{aligned} \|x_{n+1} - x_n\| &= \|(1-\alpha_n) T^n x_n + \alpha_n S^n y_n - x_n\| \\ &\le \|T^n x_n - x_n\| + \alpha_n \|T^n x_n - S^n y_n\| \to 0 \qquad \text{as} \quad n \to \infty, \end{aligned}$$

so that

$$\|x_{n+1} - y_n\| \le \|x_{n+1} - x_n\| + \|y_n - x_n\| \to 0 \qquad \text{as} \quad n \to \infty.$$

Furthermore, from

$$\|x_n - S^n y_n\| \le \|x_n - T^n x_n\| + \|T^n x_n - S^n y_n\| \to 0 \qquad \text{as} \quad n \to \infty,$$

we find

$$\|x_{n+1} - S^n y_n\| \le \|x_{n+1} - x_n\| + \|x_n - S^n y_n\|,$$

so that

$$\lim_{n\to\infty}\|x_{n+1}-S^n y_n\|=0. \tag{2.8}$$

We shall now make use of the fact that every nearly asymptotically nonexpansive mapping is nearly uniformly L-Lipschitzian. Then

$$\begin{aligned}
\|x_{n+1}-Tx_{n+1}\| &\leq \left\|x_{n+1}-T^{n+1}x_{n+1}\right\|+\left\|T^{n+1}x_{n+1}-T^{n+1}x_n\right\| \\
&\quad +\left\|T^{n+1}x_n-Tx_{n+1}\right\| \\
&\leq \left\|x_{n+1}-T^{n+1}x_{n+1}\right\|+L(\|x_{n+1}-x_n\|+a_n) \\
&\quad +L(\|T^n x_n-x_{n+1}\|+a_n) \\
&= \left\|x_{n+1}-T^{n+1}x_{n+1}\right\|+L(\|x_{n+1}-x_n\|+a_n) \\
&\quad +L(\alpha_n\|T^n x_n-S^n y_n\|+a_n)
\end{aligned}$$

yields

$$\lim_{n\to\infty}\|x_n-Tx_n\|=0. \tag{2.9}$$

Now

$$\begin{aligned}
\|x_n-S^n x_n\| &\leq \|x_n-x_{n+1}\|+\|x_{n+1}-S^n y_n\|+\|S^n y_n-S^n x_n\| \\
&\leq \|x_n-x_{n+1}\|+\|x_{n+1}-S^n y_n\|+L(\|y_n-x_n\|+a_n)\to 0 \qquad \text{as} \quad n\to\infty.
\end{aligned}$$

and

$$\begin{aligned}
\|x_{n+1}-Sx_{n+1}\| &\leq \left\|x_{n+1}-S^{n+1}x_{n+1}\right\|+\left\|S^{n+1}x_{n+1}-Sx_{n+1}\right\| \\
&\leq \left\|x_{n+1}-S^{n+1}x_{n+1}\right\|+L(\|S^n x_{n+1}-x_{n+1}\|+a_n) \\
&\leq \left\|x_{n+1}-S^{n+1}x_{n+1}\right\|+L\begin{pmatrix}\|S^n x_{n+1}-S^n y_n\| \\ +\|S^n y_n-x_{n+1}\|+a_n\end{pmatrix} \\
&\leq \left\|x_{n+1}-S^{n+1}x_{n+1}\right\|+L^2\|x_{n+1}-y_n\|+L\|S^n y_n-x_{n+1}\|+(L+1)a_n
\end{aligned}$$

give us

$$\lim_{n\to\infty}\|x_n-Sx_n\|=0.$$

Lemma 5. *For any $p_1, p_2\in F$, $\lim\limits_{n\to\infty}\|tx_n+(1-t)p_1-p_2\|$ exists for all $t\in[0,1]$ under the condition of Theorem* 1.

Proof. By Theorem 1 $\lim\limits_{n\to\infty}\|x_n-p\|$ exists for all $p\in F$ and therefore $\{x_n\}$ is bounded. Thus there exists a real number $r>0$ such that $\{x_n\}\subseteq D\equiv\overline{B_r(0)}\cap C$, so that D is a closed convex nonempty subset of C. Put

$$g_n(t)=\|tx_n+(1-t)p_1-p_2\|$$

for all $t \in [0,1]$. Then $\lim_{n\to\infty} g_n(0) = \|p_1 - p_2\|$ and $\lim_{n\to\infty} g_n(1) = \lim_{n\to\infty} \|x_n - p_2\|$ exist. Let $t \in (0,1)$.

Define $B_n : D \to D$ by:

$$\begin{aligned} B_n x &= (1-\alpha_n)T^n x + \alpha_n S^n A_n x \\ A_n x &= (1-\beta_n)x + \beta_n T^n x. \end{aligned}$$

Then $B_n x_n = x_{n+1}$, $B_n p = p$ for all $p \in F$. Also

$$\begin{aligned} \|A_n x - A_n y\| &= \|(1-\beta_n)x + \beta_n T^n x) - ((1-\beta_n)y + \beta_n T^n y)\| \\ &\leq \|(1-\beta_n)(x-y) + \beta_n (T^n x - T^n y)\| \\ &= (1-\beta_n)\|x-y\| + \beta_n k_n (\|x-y\| + a_n) \\ &= (1-\beta_n)\|x-y\| + \beta_n k_n \|x-y\| + \beta_n a_n k_n \\ &\leq (1-\beta_n)k_n \|x-y\| + \beta_n k_n \|x-y\| + \beta_n a_n k_n \\ &\leq k_n \|x-y\| + \beta_n a_n k_n \end{aligned}$$

and

$$\begin{aligned} \|B_n x - B_n y\| &= \|[(1-\alpha_n)T^n x + \alpha_n S^n A_n x] - [(1-\alpha_n)T^n y + \alpha_n S^n A_n y)]\| \\ &= \|[(1-\alpha_n)(T^n x - T^n y) + \alpha_n (S^n A_n x - S^n A_n y)]\| \\ &\leq (1-\alpha_n)k_n (\|x-y\| + a_n) + \alpha_n k_n (\|A_n x - A_n y\| + a_n) \\ &= (1-\alpha_n)k_n \|x-y\| + \alpha_n k_n \|A_n x - A_n y\| + k_n a_n \\ &\leq (1-\alpha_n)k_n^2 \|x-y\| + \alpha_n k_n (k_n \|x-y\| + \beta_n a_n k_n) + k_n a_n \\ &\leq \left((1-\alpha_n)k_n^2 + \alpha_n k_n^2\right)\|x-y\| + \alpha_n \beta_n a_n k_n^2 + k_n^2 a_n \\ &= k_n^2 \|x-y\| + \alpha_n \beta_n a_n k_n^2 + k_n^2 a_n = k_n^2 (\|x-y\| + b_n), \end{aligned}$$

where $b_n = \alpha_n \beta_n a_n + a_n$. Note that $b_n \to 0$ as $n \to \infty$.

Set

$$R_{n,m} = B_{n+m-1} B_{n+m-2} \ldots B_{n,}\ m \geq 1.$$

Then $R_{n,m} x_n = x_{n+m}$ and $R_{n,m} p = p$ for all $p \in F$. Also

$$\begin{aligned} \|R_{n,m} x - R_{n,m} y\| &\leq \|B_{n+m-1} B_{n+m-2} \ldots B_n x - B_{n+m-1} B_{n+m-2} \ldots B_n y\| \\ &\leq k_{n+m-1}^2 (\|B_{n+m-2} \ldots B_n x - B_{n+m-2} \ldots B_n y\| + b_{n+m-1}) \\ &\leq k_{n+m-1}^2 \|B_{n+m-2} \ldots B_n x - B_{n+m-2} \ldots B_n y\| + k_{n+m-1}^2 b_{n+m-1} \end{aligned}$$

$$
\begin{aligned}
&\le k_{n+m-1}^2 k_{n+m-2}^2 \left(\|B_{n+m-3}\dots B_n x - B_{n+m-3}\dots B_n y\| + b_{n+m-2}\right)\\
&\quad + k_{n+m-1}^2 b_{n+m-1}\\
&\le k_{n+m-1}^2 k_{n+m-2}^2 \|B_{n+m-3}\dots B_n x - B_{n+m-3}\dots B_n y\|\\
&\quad + k_{n+m-1}^2 k_{n+m-2}^2 b_{n+m-2} + k_{n+m-1}^2 b_{n+m-1}\\
&\le k_{n+m-1}^2 k_{n+m-2}^2 \|B_{n+m-3}\dots B_n x - B_{n+m-3}\dots B_n y\|\\
&\quad + k_{n+m-1}^2 k_{n+m-2}^2 (b_{n+m-2} + b_{n+m-1})\\
&\quad \vdots\\
&\le \left(\prod_{j=n}^{n+m-1} k_j^2\right)\left(\|x-y\| + \sum_{j=n}^{n+m-1} b_j\right)\\
&= K_{n,m}(\|x-y\| + \eta_{n,m}),
\end{aligned}
$$

where

$$K_{n,m} = \left(\prod_{j=n}^{n+m-1} k_j^2\right)$$

and

$$\eta_{n,m} = \sum_{j=n}^{n+m-1} b_j.$$

For the sake of simplicity, set

$$
\begin{aligned}
&\eta_{n,m} = \sum_{j=n}^{n+m-1} b_j, \quad \eta_n = \sum_{j=n}^{\infty} b_j,\\
&K_{n,m} = \left(\prod_{j=n}^{n+m-1} k_j^2\right), \ K_n = \left(\prod_{j=n}^{\infty} k_j^2\right),\\
&t_n = t x_n + (1-t) p_1,\\
&\rho_{n,m} = t\|x_n - p_1\| + \eta_{n,m},\\
&\sigma_{n,m} = (1-t)\|x_n - p_1\| + \eta_{n,m},\\
&e_{n,m} = t p_1 + (1-2t) R_{n,m} t_n - (1-t) R_{n,m} x_n,\\
&u_{n,m} = [R_{n,m} t_n - t R_{n,m} x_n - (1-t) p_1]\,\|x_n - p_1\|,\\
&v_{n,m} = [p_1 + R_{n,m} x_n - 2R_{n,m} t_n]\,\eta_{n,m},\\
&w_{n,m} = (p_1 - R_{n,m} t_n)\ /\ K_{n,m}\rho_{n,m},\\
&z_{n,m} = (R_{n,m} t_n - R_{n,m} x_n)\ /\ \ K_{n,m}\sigma_{n,m},\\
&\lambda_{n,m} = K_{n,m}\rho_{n,m}\sigma_{n,m}.
\end{aligned}
$$

Then

$$\|w_{n,m}\| = \left\| \frac{R_{n,m}t_n - p_1}{K_{n,m}(t\|x_n - p_1\| + \eta_{n,m})} \right\| \leq \frac{K_{n,m}(\|t_n - p_1\| + \eta_{n,m})}{K_{n,m}(t\|x_n - p_1\| + \eta_{n,m})} = 1$$

and similarly $\|z_{n,m}\| \leq 1$. Note that

$$\rho_{n,m} + \sigma_{n,m} = t\|x_n - p_1\| + \eta_{n,m} + (1-t)\|x_n - p_1\| + \eta_{n,m} = \|x_n - p_1\| + 2\eta_{n,m}.$$

Moreover,

$$\begin{aligned}
\|w_{n,m} - z_{n,m}\| &= \left\| \frac{p_1 - R_{n,m}t_n}{K_{n,m}\rho_{n,m}} - \frac{R_{n,m}t_n - R_{n,m}x_n}{K_{n,m}\sigma_{n,m}} \right\| \\
&= \left\| \frac{\sigma_{n,m}p_1 - \sigma_{n,m}R_{n,m}t_n - \rho_{n,m}R_{n,m}t_n + \rho_{n,m}R_{n,m}x_n}{\lambda_{n,m}} \right\| \\
&= \left\| \frac{\sigma_{n,m}p_1 - (\|x_n - p_1\| + 2\eta_{n,m})R_{n,m}t_n + \rho_{n,m}R_{n,m}x_n}{\lambda_{n,m}} \right\| \\
&= \left\| \frac{u_{n,m} - v_{n,m}}{\lambda_{n,m}} \right\|,
\end{aligned}$$

because

$$\begin{aligned}
\|u_{n,m} - v_{n,m}\| &= \left\| \begin{array}{c} \|x_n - p_1\|R_{n,m}t_n - \|x_n - p_1\|tR_{n,m}x_n - (1-t)p_1\|x_n - p_1\| \\ -p_1\eta_{n,m} - R_{n,m}x_n\eta_{n,m} + 2R_{n,m}t_n\eta_{n,m} \end{array} \right\| \\
&= \left\| \begin{array}{c} -((1-t)\|x_n - p_1\| + \eta_{n,m})p_1 + (\|x_n - p_1\| + 2\eta_{n,m})R_{n,m}t_n \\ -(t\|x_n - p_1\| + \eta_{n,m})R_{n,m}x_n \end{array} \right\|
\end{aligned}$$

and

$$\begin{aligned}
\|tw_{n,m} + (1-t)z_{n,m}\| &= \left\| \frac{t(p_1 - R_{n,m}t_n)}{K_{n,m}\rho_{n,m}} + \frac{(1-t)(R_{n,m}t_n - R_{n,m}x_n)}{K_{n,m}\sigma_{n,m}} \right\| \\
&= \left\| \frac{\sigma_{n,m}t(p_1 - R_{n,m}t_n) + \rho_{n,m}(1-t)(R_{n,m}t_n - R_{n,m}x_n)}{\lambda_{n,m}} \right\| \\
&= \frac{1}{\lambda_{n,m}} \left\| \begin{array}{c} ((1-t)\|x_n - p_1\| + \eta_{n,m})t(p_1 - R_{n,m}t_n) \\ +(t\|x_n - p_1\| + \eta_{n,m})(1-t)(R_{n,m}t_n - R_{n,m}x_n) \end{array} \right\|
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\lambda_{n,m}} \left\| \begin{array}{c} t(1-t)\,p_1\,\|x_n-p_1\| + t\,p_1\,\eta_{n,m} \\ -t(1-t)\,R_{n,m}t_n\,\|x_n-p_1\| - tR_{n,m}t_n\eta_{n,m} \\ +t(1-t)R_{n,m}t_n\,\|x_n-p_1\| + (1-t)R_{n,m}t_n\eta_{n,m} \\ -t(1-t)R_{n,m}x_n\,\|x_n-p_1\| - (1-t)R_{n,m}x_n\eta_{n,m} \end{array} \right\| \\
&= \frac{1}{\lambda_{n,m}} \left\| \begin{array}{c} t(1-t)\,\|x_n-p_1\|\,(p_1-R_{n,m}x_n) \\ +[t\,p_1+(1-2t)\,R_{n,m}t_n-(1-t)R_{n,m}x_n]\eta_{n,m} \end{array} \right\| \\
&= \frac{1}{\lambda_{n,m}} \|t(1-t)\,\|x_n-p_1\|\,(p_1-R_{n,m}x_n)+e_{n,m}\eta_{n,m}\| \\
&= \frac{1}{\lambda_{n,m}} \|t(1-t)\,\|x_n-p_1\|\,(p_1-x_{n+m})+e_{n,m}\eta_{n,m}\|\,.
\end{aligned}
$$

From (1.7), we get

$$
\begin{aligned}
2t(1-t)\,\lambda_{n,m}\delta\left(\frac{\|u_{n,m}-v_{n,m}\|}{\lambda_{n,m}}\right) &\le \lambda_{n,m} - \|t(1-t)\,\|x_n-p_1\|\,(p_1-x_{n+m})+e_{n,m}\eta_{n,m}\| \\
&\le \lambda_{n,m} - t(1-t)\,\|x_n-p_1\|\,\|x_{n+m}-p_1\| + \|e_{n,m}\|\,\eta_{n,m}.
\end{aligned}
$$

But

$$
\begin{aligned}
\lambda_{n,m} &= K_{n,m}\,(t\,\|x_n-p_1\|+\eta_{n,m})\,((1-t)\,\|x_n-p_1\|+\eta_{n,m}) \\
&\le K_n\left[t(1-t)\,\|x_n-p_1\|^2 + (\|x_n-p_1\|+\eta_n)\,\eta_n\right] \\
&\le K_n\left[t(1-t)\,\|x_n-p_1\|^2 + M_1\eta_n\right],
\end{aligned}
$$

where $M_1 = \sup(\|x_n-p_1\|+\eta_n)$.
Therefore

$$
\begin{aligned}
2\lambda_{n,m}\delta\left(\frac{\|u_{n,m}-v_{n,m}\|}{\lambda_{n,m}}\right) &\le K_n\left[\|x_n-p_1\|^2 + \frac{M_1\eta_n}{t(1-t)}\right] \\
&\quad - \|x_n-p_1\|\,\|p_1-x_{n+m}\| + \frac{\|e_{n,m}\|\,\eta_n}{t(1-t)}.
\end{aligned}
$$

Let $\lambda = \sup\{\lambda_n K_n : n \in \mathbf{N}\}$. Since E is uniformly convex, $\delta(s)/s$ is nondecreasing. Therefore

$$
\begin{aligned}
2\lambda\,\delta\left(\frac{\|u_{n,m}-v_{n,m}\|}{\lambda}\right) &\le K_n\left[\|x_n-p_1\|^2 + \frac{M_1\eta_n}{t(1-t)}\right] \\
&\quad - \|x_n-p_1\|\,\|p_1-x_{n+m}\| + \frac{\|e_{n,m}\|\,\eta_n}{t(1-t)}.
\end{aligned}
$$

Moreover, $\delta(0)=0$, $\lim\limits_{n\to\infty}\eta_n = 0$, $\lim\limits_{n\to\infty}K_n = 1$ and δ is continuous, therefore

$$
\lim_{m,n\to\infty}\|u_{n,m}-v_{n,m}\| = 0.
$$

By the triangle inequality,

$$\|u_{n,m}\| \leq \|u_{n,m} - v_{n,m}\| + \|v_{n,m}\| = \|u_{n,m} - v_{n,m}\| + M_2\eta_{n,m}$$

for some $M_2 > 0$. This gives

$$\lim_{m,n\to\infty} \|u_{n,m}\| = 0.$$

Since

$$\lim_{n\to\infty} \|x_n - p_1\| > 0,$$

we have

$$\lim_{m,n\to\infty} \|R_{n,m}t_n - tR_{n,m}x_n - (1-t)p_1\| = 0.$$

Finally, from

$$\begin{aligned} g_{n+m}(t) &= \|tx_{n+m} + (1-t)p_1 - p_2\| \\ &\leq \|R_{n,m}t_n - p_2\| + \|R_{n,m}t_n - tR_{n,m}x_n - (1-t)p_1\| \\ &\leq K_{n,m}(\|t_n - p_2\| + \eta_{n,m}) + \|R_{n,m}t_n - tR_{n,m}x_n - (1-t)p_1\| \\ &\leq K_n(\|t_n - p_2\| + \eta_n) + \|R_{n,m}t_n - tR_{n,m}x_n - (1-t)p_1\| \end{aligned}$$

we get

$$\begin{aligned} \limsup_{m\to\infty} g_{n+m}(t) &\leq \liminf_{n\to\infty} K_n(\|t_n - p_2\| + \eta_n) + \limsup_{m\to\infty} \|R_{n,m}t_n - tR_{n,m}x_n - (1-t)p_1\| \\ &= \liminf_{n\to\infty} g_n(t). \end{aligned}$$

Thus

$$\limsup_{n\to\infty} g_n(t) \leq \liminf_{n\to\infty} g_n(t)$$

so that $\lim_{n\to\infty} \|tx_n + (1-t)p_1 - p_2\|$ exists for all $t \in [0,1]$.

Lemma 6. *Assume that the condition of Theorem 1 is satisfied. Then, for any $p_1, p_2 \in F$, $\lim_{n\to\infty} \langle x_n, J(p_1 - p_2)\rangle$ exists; in particular, $\langle p - q, J(p_1 - p_2)\rangle = 0$ for all $p, q \in \omega_w(x_n)$.*

Proof. Take $x = p_1 - p_2$ with $p_1 \neq p_2$ and $h = t(x_n - p_1)$ in the inequality (1.7) to get:

$$\begin{aligned} \frac{1}{2}\|p_1 - p_2\|^2 + t\langle x_n - p_1, J(p_1 - p_2)\rangle &\leq \frac{1}{2}\|tx_n + (1-t)p_1 - p_2\|^2 \\ &\leq \frac{1}{2}\|p_1 - p_2\|^2 + t\langle x_n - p_1, J(p_1 - p_2)\rangle + b(t\|x_n - p_1\|). \end{aligned}$$

As $\sup_{n\geq 1} \|x_n - p_1\| \leq M'$ for some $M' > 0$, it follows that

$$\begin{aligned} \frac{1}{2}\|p_1 - p_2\|^2 + t\limsup_{n\to\infty} \langle x_n - p_1, J(p_1 - p_2)\rangle &\leq \frac{1}{2}\lim_{n\to\infty} \|tx_n + (1-t)p_1 - p_2\|^2 \\ &\leq \frac{1}{2}\|p_1 - p_2\|^2 + b(tM') + t\liminf_{n\to\infty} \langle x_n - p_1, J(p_1 - p_2)\rangle. \end{aligned}$$

That is,

$$\limsup_{n\to\infty} \langle x_n - p_1, J(p_1 - p_2)\rangle \le \liminf_{n\to\infty} \langle x_n - p_1, J(p_1 - p_2)\rangle + \frac{b(tM')}{tM'} M'.$$

If $t \to 0$, then $\lim\limits_{n\to\infty} \langle x_n - p_1, J(p_1 - p_2)\rangle$ exists for all $p_1, p_2 \in F$; in particular, we have $\langle p - q, J(p_1 - p_2)\rangle = 0$ for all $p, q \in \omega_w(x_n)$.

We now give our weak convergence theorem.

Theorem 2. *Let E be a uniformly convex Banach space and let C, T, S and $\{x_n\}$ be taken as in Theorem* 1. *Assume that (a) E satisfies Opial's condition or (b)E has a Fréchet differentiable norm or (c)dual E^* of E satisfies Kadec-Klee property. If $F \neq \phi$ then $\{x_n\}$ converges weakly to a point of F.*

Proof. Let $p \in F$. Then $\lim\limits_{n\to\infty} \|x_n - p\|$ exists as proved in Theorem 1. We prove that $\{x_n\}$ has a unique weak subsequential limit in F. For, let u and v be weak limits of the subsequences $\{x_{n_i}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$, respectively. By Theorem 1, $\lim\limits_{n\to\infty} \|x_n - Tx_n\| = 0$ and $I - T$ is demiclosed with respect to zero by Lemma 3, therefore we obtain $Tu = u$. Similarly, $Su = u$. Again in the same fashion, we can prove that $v \in F$. Next, we prove the uniqueness. To this end, first assume (a) is true. If u and v are distinct then by Opial's condition,

$$\begin{aligned}\lim_{n\to\infty} \|x_n - u\| &= \lim_{n_i\to\infty} \|x_{n_i} - u\| < \lim_{n_i\to\infty} \|x_{n_i} - v\| \\ &= \lim_{n\to\infty} \|x_n - v\| = \lim_{n_j\to\infty} \|x_{n_j} - v\| < \lim_{n_j\to\infty} \|x_{n_j} - u\| = \lim_{n\to\infty} \|x_n - u\|.\end{aligned}$$

This is a contradiction so $u = v$. Next assume (b). By Lemma 6, $\langle p - q, J(p_1 - p_2)\rangle = 0$ for all $p, q \in \omega_w(x_n)$. Therefore

$$\|u - v\|^2 = \langle u - v, J(u - v)\rangle = 0$$

implies $u = v$. Finally, say (c) is true. Since $\lim\limits_{n\to\infty} \|tx_n + (1-t)u - v\|$ exists for all $t \in [0,1]$ by Lemma 5, therefore $u = v$ by Lemma 4. Consequently, $\{x_n\}$ converges weakly to a point of F and this completes the proof.

Two mappings $S, T : C \to C$, where C is a subset of a normed space E, are said to satisfy the condition (A') [4] if there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0,\infty)$ such that either $\|x - Sx\| \ge f(d(x,F))$ or $\|x - Tx\| \ge f(d(x,F))$ for all $x \in C$, where

$$d(x,F) = \inf\{\|x - p\| : p \in F = F(S) \cap F(T)\}.$$

Theorem 3. *Let E be a real Banach space and let $C, S, T, F, \{x_n\}$ be taken as in Theorem 1. Then $\{x_n\}$ converges to a point of F if and only if $\lim\limits_{n\to\infty} \inf_{n\to\infty} d(x_n, F) = 0$ where*

$$d(x,F) = \inf\{\|x - p\| : p \in F\}.$$

Proof. Necessity is obvious. Suppose that

$$\lim_{n\to\infty}\inf_{n\to\infty} d(x_n,F)=0.$$

As proved in Theorem 1, $\lim_{n\to\infty}\|x_n-w\|$ exists for all $w\in F$, therefore $\lim_{n\to\infty} d(x_n,F)$ exists. But by hypothesis, $\liminf_{n\to\infty} d(x_n,F)=0$, therefore we have $\lim_{n\to\infty} d(x_n,F)=0$. On the lines similar to [4], it can be proved that $\lim_{n\to\infty} d(x_n,F)=0$. This gives that $d(q,F)=0$ and $q\in F$.

Applying Theorem 3, we obtain a strong convergence of the scheme (1.5) under the condition (A') as follows.

Theorem 4. *Let E be a real Banach space and let $C,S,T,F,\{x_n\}$ be taken as in Theorem 1. Let S,T satisfy the condition (A') and $F\neq\varnothing$, then $\{x_n\}$ converges strongly to a common fixed point of S and T.*

Proof. As is proved in Theorem 1 that

$$\lim_{n\to\infty}\|x_n-Sx_n\|=0=\lim_{n\to\infty}\|x_n-Tx_n\|. \tag{2.10}$$

From the condition (A') and (2.10), we get

$$\lim_{n\to\infty} f(d(x_n,F))\le\lim_{n\to\infty}\|x_n-Tx_n\|=0$$

or

$$\lim_{n\to\infty} f(d(x_n,F))\le\lim_{n\to\infty}\|x_n-Sx_n\|=0.$$

In both cases,

$$\lim_{n\to\infty} f(d(x_n,F))=0.$$

Since $f:[0,\infty)\to[0,\infty)$ is a nondecreasing function satisfying $f(0)=0,\ f(r)>0$ for all $r\in(0,\infty)$, therefore we have

$$\lim_{n\to\infty} d(x_n,F)=0.$$

Now all the conditions of Theorem 2 are satisfied, therefore by its conclusion $\{x_n\}$ converges strongly to a point of F.

Remark 1. (1) Theorem 3 generalizes Theorems 3.9, 3.10 and 3.11 of [1] to the case of common fixed points of two mappings. In fact, choose $S=T$ to get the said results. Moreover, this theorem generalizes the corresponding results in the literature proved by using Mann iteration scheme by choosing $T=I$.

(2) Theorem 2 improves Theorem 3.1 of Yao and Chen [9] in two ways: (i) our result is true for a larger class of mappings (ii) the rate of convergence is better.

(3) Theorems of this paper can also be proved with error terms. Thus we have also generalized Theorem 3.2 of Yao and Chen[9] in the aforementioned two ways.

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