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ON SIMULTANEOUS WEAKLY-CHEBYSHEV SUBSPACES

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Abstract. In this paper, we shall introduce and characterize simultaneous quasi-Chebyshev (and weakly-Chebyshev) subspaces of normed spaces with respect to a bounded set S by using elements of the dual space.

Key words: *dual space, best simultaneous approximation, simultaneous quasi-Chebyshev subspace, simultaneous weakly-Chebyshev subspace*

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1 Introduction

Let W be a subspace of the normed space X and $x \in X$. We say that $w_0 \in W$ is a best approximation of x whenever $\|x-w_0\| = d(x,W) = \inf_{w\in W}\|x-w\|$. We denote the set of all best approximations of x by $P_W(x)$. It is known that $P_W(x)$ is a closed, bounded and convex subset for all $x \in X$. A subspace W is called pseudo-Chebyshev (proximinal) if dim $P_W(x) < \infty$ ($P_W(x) \neq \emptyset$) for all $x \in X$[19]. In 2000, Mohebi and Radjavi generalized this notion to quasi-Chebyshev subspaces[9],[10]. Then, Mohebi, Rezapour and Mazaheri generalized the notion of quasi-Chebyshev subspaces to weakly-Chebyshev subspaces[11],[13]. In 2008, Shams, Mazaheri and Vaezpour provided the notion of w-Chebyshev subspaces[17]. On the other hand, the theory of best simultaneous approximation is a generalization of best approximation theory and has been studied by many authors (for example, [1]-[8], [14]- [16], [18], [20]-[22]). But, in these work there is not any characterization about best simultaneous approximation by using the dual space of the normed space. In this paper, we shall introduce and characterize simultaneous quasi-Chebyshev (and weakly-Chebyshev) subspaces of normed spaces with respect to a bounded set S by using elements of the dual space.

Suppose that X is a normed linear space, W a subset of X and S a bounded set in X. We define

$$d(S,W) = \inf_{w \in W} \sup_{s \in S} \|s - w\|.$$

An element $w_0 \in W$ is called a best simultaneous approximation to S from W whenever $d(S,W) = \sup_{s \in S} \|s - w_0\|$. The set of all best simultaneous approximation to S from W will be denoted by $S_W(S)$. In the case $S = \{x\}$, $x \in X$, $S_W(S)$ is the set of all best approximations of x in W, $P_W(x)$. Thus, the simultaneous approximation theory is a generalization of best approximation theory in a sense.

A subset W is called S-simultaneous proximinal (or simply, simultaneous proximinal) if $S_W(S) \neq \emptyset$. Also, W is called simultaneous quasi-Chebyshev (simultaneous weakly-Chebyshev) if $S_W(S)$ is nonempty and compact (weakly compact) subset of W. Throughout this paper, we suppose that S is either a bounded or a finite set. Also, we shall use the following Lemma in the sequel which has been proved in [1] and [5].

Lemma 1.1. *Let X be a normed linear space and M a proximinal subspace of X. Then, for each non-empty bounded set S in X we have*

$$d(S,M) = \sup_{s \in S} \inf_{m \in M} \|s - m\|.$$

2 Main Results

Now, we are ready to state and prove our main results. Usually, using finite sets are interesting and give us some ideas. Therefore, in this section first we suppose that S is a finite set. First, we give some elementary results.

Lemma 2.1. *Let X be a normed space, W a subspace of X and $S = \{s_1, \cdots, s_m\}$ a subset of X with linearly independent elements. Suppose that $Y =$ span S. If $Y \cap W = \{0\}$, then there is a bounded linear functional f_0 on X such that $f_0|_W = 0$, $f_0|_S = 1$ and $\dfrac{1}{d(S,W)} \leq \|f_0\|$.*

Proof. First, consider the subspace

$$Z = \text{span } \{W,S\} = \left\{ w - \sum_{i=1}^{m} \alpha_i s_i \ \alpha_1, \alpha_2, \cdots, \alpha_m \text{ are scalars and } w \in W \right\}.$$

Now, define the linear functional f_0 on Z by $f_0(w-\sum_{i=1}^{m}\alpha_i s_i)=\sum_{i=m}^{m}\alpha_i$. Note that if

$$w-\sum_{i=1}^{m}\alpha_i s_i = w'-\sum_{i=1}^{m}\beta_i s_i,$$

then we have

$$w-w'=\sum_{i=1}^{m}\alpha_i s_i-\sum_{i=1}^{m}\beta_i s_i=\sum_{i=1}^{m}(\alpha_i-\beta_i)s_i.$$

Since $W\cap Y=\{0\}$, $w-w'=0$ and $\sum_{i=1}^{m}(\alpha_i-\beta_i)s_i=0$. Hence, $\alpha_i=\beta_i$ for all $i=1,2,\cdots,m$. Thus,

$$\sum_{i=1}^{m}\alpha_i=\sum_{i=1}^{m}\beta_i$$

and so f_0 is a function. Also, It is clear that f_0 is linear, $f_0|_W=0$ and $f_0|_S=1$. Since f_0 is continuous on W, $\|f_0\|$ is bounded. Now, suppose that $\alpha_i\geq 0$, $\sum_{i=1}^{l}\alpha_i=1$ and $c\geq 0$ is an arbitrary constant such that $\|f_0 z\|\leq c\|z\|$ for all $z\in Z$. Then,

$$\left|\sum_{i=1}^{m}\alpha_i\right|=1\leq c\|w-\sum_{i=1}^{m}\alpha_i s_i\|=c\|\sum_{i=1}^{m}\alpha_i w-\sum_{i=1}^{m}\alpha_i s_i\|,$$

for all $w\in W$. Hence,

$$1\leq c\sum_{i=1}^{m}\alpha_i\|w-s_i\|\leq c\sum_{i=1}^{m}\alpha_i(\max_{1\leq i\leq m}\|w-s_i\|)=c\max_{1\leq i\leq m}\|w-s_i\|,$$

for all $w\in W$. Thus, $\frac{1}{\max_{1\leq i\leq m}\|w-s_i\|}\leq c$ and so

$$\sup_{w\in W}\frac{1}{\max_{1\leq i\leq m}\|w-s_i\|}=\frac{1}{\inf_{w\in W}\max_{1\leq i\leq m}\|w-s_i\|}=\frac{1}{d(S,W)}\leq c.$$

Therefore, $\frac{1}{d(S,W)}\leq\|f_0\|$.

If in Lemma 2.1 we have $\max_{1\leq i\leq m}\|s_i\|\leq\frac{1}{\|f_0\|^2 d(S,W)}$, then $\|f_0\|=\frac{1}{d(S,W)}$.

Corollary 2.2. *Let X be a normed space, W a subspace of X and $S=\{s_1,\cdots,s_m\}$ a subset of X with linearly independent elements. Suppose that $Y=\operatorname{span} S$. If $Y\cap W=\{0\}$, then there exists $f_1\in X^*$ such that $f_1|_W=0$ and $\|f_1\|\leq\frac{1}{d(S,W)}$.*

Proof. Take the functional f_0 in Lemma 2.1. If $\frac{1}{d(S,W)}=\|f_0\|$, then put for $f_1=f_0$. If $\|f_0\|d(S,W)>1$, then choose a natural number n such that

$$(\|f_0\|d(S,W))^n>d(S,W).$$

Now, put $f_1 = \dfrac{f_0}{d(S,W)^n \|f_0\|^{n+1}}$.

Corollary 2.3. *Let X be a normed space, W a subspace of X and $S = \{s_1, \cdots, s_m\}$ a subset of X with linearly independent elements. Suppose that $Y = \text{span } S$. If $Y \cap W = \{0\}$, then there exists $f_2 \in X^*$ such that $f_2|_W = 0$, $\|f_2\| \leq 1$ and $f_2(w - s_i) \leq d(S,W)$ for all $w \in W$ and $i = 1, 2, \cdots, m$.*

Proof. Take the functional f_0 in Lemma 2.1 and put $f_2 = \dfrac{f_0}{d(S,W)\|f_0\|^2}$.

Now, we consider that S is only a finite set and note that we shall use finiteness of the set S to prove of the following lemma.

Lemma 2.4. *Let X be a normed space, W a proximinal subspace of X and $S = \{s_1, \cdots, s_m\}$ a subset of $X \backslash W$. Then, $w_0 \in S_W(S)$ if and only if there exists $f_3 \in X^*$ such that $f_3|_W = 0$, $\|f_3\| \leq 1$ and $f_3(s_i) = \max_{1 \leq j \leq m} \|s_j - w_0\|$ for some $i \in \{1, 2, \cdots, m\}$.*

Proof. First suppose that there exists $f_3 \in X^*$ such that $f_3|_W = 0$, $\|f_3\| \leq 1$ and $f_3(s_i) \geq \max_{1 \leq j \leq m} \|s_j - w_0\|$ for some $i \in \{1, 2, \cdots, m\}$. Then, we have

$$\max_{1 \leq j \leq m} \|s_j - w_0\| = f_3(s_i) = f_3(s_i - w) \leq \|f_3\| \|s_i - w\| \leq \|s_i - w\| \leq \max_{1 \leq j \leq m} \|s_j - w\|,$$

for all $w \in W$. Hence, $w_0 \in S_W(S)$. Now for the converse part, suppose that $w_0 \in S_W(S)$. Since $S \subseteq X \backslash W$ and W is a proximinal subspace of X, by Lemma 1.1 we have

$$\begin{aligned} 0 &< \max_{1 \leq j \leq m} \|s_j - w_0\| = d(S,W) = \inf_{w \in W} \max_{1 \leq j \leq m} \|s_j - w\| \\ &= \max_{1 \leq j \leq m} \inf_{w \in W} \|s_j - w\| = \max_{1 \leq j \leq m} \|s_j + W\|. \end{aligned}$$

Choose i such that $\|s_i + W\| = \max_{1 \leq j \leq m} \|s_j + W\|$. Since $d(s_i, W) = \|s_i + W\| > 0$, there exists $f \in X^*$ such that $f|_W = 0$, $f(s_i) = 1$ and $\|f\| = \frac{1}{\|s_i + W\|}$. If $f_3 = \frac{f}{\|f\|}$, then $\|f_3\| = 1$, $f_3|_W = 0$ and $f_3(s_i) = \|s_i + W\| = \max_{1 \leq j \leq m} \|s_j - w_0\|$.

Theorem 2.5. *Let X be a normed space, W a proximinal subspace of X and $S = \{s_1, \cdots, s_m\}$ a subset of $X \backslash W$. Then, $w_0 \in S_W(S)$ if and only if*

$$\max_{1 \leq j \leq m} \|s_j\|_{W^\perp} = \max_{1 \leq j \leq m} \|s_j - w_0\|,$$

where $\|s_j\|_{W^\perp} = \sup\{|f(s_j)| : \|f\| \leq 1, f \in W^\perp\}$.

Proof. First suppose that $\max_{1 \leq j \leq m} \|s_j\|_{W^\perp} = \max_{1 \leq j \leq m} \|s_j - w_0\|$. Then, for each $w \in W$ we have

$$\max_{1 \leq j \leq m} \|s_j - w_0\| = \max_{1 \leq j \leq m} \|s_j\|_{W\perp} = \max_{1 \leq j \leq m} \|s_j - w\|_{W\perp} \leq \max_{1 \leq j \leq m} \|s_j - w\|.$$

Hence, $w_0 \in S_W(S)$. Now for the converse part, suppose that $w_0 \in S_W(S)$. By using Lemma 2.4, we know that there exists $f_3 \in X^*$ such that $f_3|_W = 0$, $\|f_3\| \leq 1$ and $f_3(s_i) = \max_{1 \leq j \leq m} \|s_j - w_0\|$

for some $i \in \{1,2,\cdots,m\}$. Thus, we obtain

$$\max_{1\le j\le m}\|s_j-w_0\| = f_3(s_i) \le \|s_i\|_{W^\perp} \le \max_{1\le j\le m}\|s_j\|_{W^\perp}$$
$$= \max_{1\le j\le m}\|s_j-w_0\|_{W^\perp} \le \max_{1\le j\le m}\|s_j-w_0\|.$$

Therefore, $\max_{1\le j\le m}\|s_j\|_{W^\perp} = \max_{1\le j\le m}\|s_j-w_0\|$.

Theorem 2.6. *Let X be a normed space, W a proximinal subspace of X, $M \subseteq W$ and $S = \{s_1,\cdots,s_m\}$ a subset of $X\backslash W$. Then, $M \subseteq S_W(S)$ if and only if there exists $f \in X^*$ and $i \in \{1,2,\cdots,m\}$ such that $f|_W = 0$, $\|f\| \le 1$ and $f(s_i) = \max_{1\le j\le m}\|s_j-m\|$ for all $m \in M$.*

Proof. First suppose that there exists $f \in X^*$ and $i \in \{1,2,\cdots,m\}$ such that $f|_W = 0$, $\|f\| \le 1$ and $f(s_i) = \max_{1\le j\le m}\|s_j-m\|$ for all $m \in M$. Then, for each $m \in M$ and $w \in W$ we have

$$\max_{1\le j\le m}\|s_j-m\| = f(s_i) = f(s_i-w) \le \|s_i-w\| \le \max_{1\le j\le m}\|s_j-w\|.$$

Thus, $m \in S_W(S)$ for all $m \in M$. Hence, $M \subseteq S_W(S)$. Now for the converse part, suppose that $M \subseteq S_W(S)$ and $m_0 \in M$. By Lemma 2.4, there exists $f \in X^*$ such that $f|_W = 0$, $\|f\| \le 1$ and $f(s_i) = \max_{1\le j\le m}\|s_j-m_0\|$ for some $i \in \{1,2,\cdots,m\}$. Since

$$M \subseteq S_W(S), \quad \max_{1\le j\le m}\|s_j-m_0\| = d(S,W) = \max_{1\le j\le m}\|s_j-m\| \quad \text{for all} \quad m \in M.$$

Therefore, $f(s_i) = \max_{1\le j\le m}\|s_j-m\|$ for all $m \in M$.

Corollary 2.7. *Let X be a normed space, W a proximinal subspace of X and $S = \{s_1,\cdots,s_m\}$ a subset of $X\backslash W$. Then, W is a simultaneous quasi-Chebyshev subspace of X if and only if there do not exist $f \in X^*$, $i \in \{1,2,\cdots,m\}$ and a sequence $\{w_n\}_{n\ge 1}$ in W without a convergent subsequence such that $\|f\| \le 1$, $f|_W = 0$ and $f(s_i) = \max_{1\le j\le m}\|s_j-w_n\|$ for all $n \ge 1$.*

Corollary 2.8. *Let X be a normed space, W a proximinal subspace of X and $S = \{s_1,\cdots,s_m\}$ a subset of $X\backslash W$. Then, W is a simultaneous weakly-Chebyshev subspace of X if and only if there do not exist $f \in X^*$, $i \in \{1,2,\cdots,m\}$ and a sequence $\{w_n\}_{n\ge 1}$ in W without a weakly convergent subsequence such that $\|f\| \le 1$, $f|_W = 0$ and $f(s_i) = \max_{1\le j\le m}\|s_j-w_n\|$ for all $n \ge 1$.*

Let X be a normed space, W a subspace of X, $S = \{s_1,\cdots,s_m\}$ a subset of $X\backslash W$ and $f \in W^\perp$. If we define

$$M_{f,S} = \left\{x \in X : \max_{1\le j\le m}\|s_j-x\| \le \max_{1\le j\le m}\|s_j\| \text{ and } f(s_i) \ge \max_{1\le j\le m}\|s_j-x\|, \text{ for some i}\right\},$$

then $M_{f,S}$ is a bounded, closed and convex subset of X.

Corollary 2.9. *Let X be a normed space, W a proximinal subspace of X, $S = \{s_1,\cdots,s_m\}$ a subset of $X\backslash W$ and $f \in W^\perp$. If $M_{f,S}$ is compact (weakly compact) for all $f \in W^\perp$ with $\|f\| \le 1$, then W is a simultaneous quasi-Chebyshev (simultaneous weakly-Chebyshev) subspace of X.*

Hereafter, we suppose that S is a bounded set. The following result is another version of Lemma 2.4.

Lemma 2.10. *Let X be a normed space, W a proximinal subspace of X and S a weakly compact subset of $X\backslash W$ with $\sup_{s\in S}\|s+W\|<\infty$. Then, $w_0\in S_W(S)$ if and only if there exists $f\in X^*$ such that $f|_W=0$, $\|f\|\leq 1$ and $f(s_0)=\sup_{s\in S}\|s-w_0\|$ for some $s_0\in S$.*

Proof. First suppose that there exists $f\in X^*$ such that $f|_W=0$, $\|f\|\leq 1$ and $f(s_0)=\sup_{s\in S}\|s-w_0\|$ for some $s_0\in S$. Then, a similar proof of Lemma 2.4 shows $w_0\in S_W(S)$. Now for the converse part, suppose $w_0\in S_W(S)$. Then, by Lemma 1.1 we have

$$d(S,W)=\inf_{w\in W}\sup_{s\in S}\|s-w\|=\sup_{s\in S}\inf_{w\in W}\|s-w\|=\sup_{s\in S}\|s+W\|.$$

But, note that the function defined by $\phi(s)=\|s+W\|$ is continuous. Since S is a weakly compact subset of X, there exists $s_0\in S$ such that

$$\phi(s_0)=\|s_0+W\|=\sup_{s\in S}\|s+W\|=d(S,W).$$

Since $S\subseteq X\backslash W$, $d(s_0,W)=\|s_0+W\|>0$ and so there exists $g\in X^*$ such that $g|_W=0$, $g(s_0)=1$ and $\|g\|=\frac{1}{d(s_0,W)}$. If $f=\frac{g}{\|g\|}$, then $\|f\|=1$, $f|_W=0$ and $f(s_0)=\sup_{s\in S}\|s-w_0\|$.

The proof of the following result is similar to that of Theorem 2.5.

Proposition 2.11. *Let X be a normed space, W a proximinal subspace of X and S a weakly compact subset of $X\backslash W$ with $\sup_{s\in S}\|s+W\|<\infty$. Then, $w_0\in S_W(S)$ if and only if*

$$\sup_{s\in S}\|s\|_{W^\perp}=\sup_{s\in S}\|s-w_0\|,$$

where $\|s\|_{W^\perp}=\sup\{|f(s)|:\|f\|\leq 1, f\in W^\perp\}$.

Theorem 2.12. *Let X be a normed space, W a proximinal subspace of X, $M\subseteq W$ and S a weakly compact subset of $X\backslash W$ with $\sup_{s\in S}\|s+W\|<\infty$. Then, $M\subseteq S_W(S)$ if and only if there exist $f\in X^*$ and $s_0\in S$ such that $f|_W=0$, $\|f\|\leq 1$ and $f(s_0)=\sup_{s\in S}\|s-m\|$ for all $m\in M$.*

Proof. First suppose that there exist $f\in X^*$ and $s_0\in S$ such that $f|_W=0$, $\|f\|\leq 1$ and $f(s_0)=\sup_{s\in S}\|s-m\|$ for all $m\in M$. Then, for each $m\in M$ and $w\in W$ we have

$$\sup_{s\in S}\|s-m\|=f(s_0)=f(s_0-w)\leq\|s_0-w\|\leq\sup_{s\in S}\|s-w\|.$$

Thus, $m\in S_W(S)$ for all $m\in M$. Hence, $M\subseteq S_W(S)$. Now for the converse part, suppose that $M\subseteq S_W(S)$ and $m_0\in M$. By Lemma 2.4, there exists $f\in X^*$ such that $f|_W=0$, $\|f\|\leq 1$ and $f(s_0)=\sup_{s\in S}\|s-m_0\|$ for some $s_0\in S$. Since $M\subseteq S_W(S)$, $\sup_{s\in S}\|s-m_0\|=d(S,W)=\sup_{s\in S}\|s-m\|$ for all $m\in M$. Therefore, $f(s_0)=\sup_{s\in S}\|s-m\|$ for all $m\in M$.

Corollary 2.13. *Let X be a normed space, W a proximinal subspace of X and S a weakly compact subset of $X \backslash W$ with $\sup_{s\in S}\|s+W\|<\infty$. Then, W is a simultaneous quasi-Chebyshev subspace of X if and only if there do not exist $f \in X^*$, $s_0 \in S$ and a sequence $\{w_n\}_{n\geq 1}$ in W without a convergent subsequence such that $\|f\| \leq 1$, $f|_W = 0$ and $f(s_0) = \sup_{s\in S}\|s-w_n\|$ for all $n \geq 1$.*

Corollary 2.14. *Let X be a normed space, W a proximinal subspace of X and S a weakly compact subset of $X \backslash W$ with $\sup_{s\in S}\|s+W\|<\infty$. Then, W is a simultaneous quasi-Chebyshev subspace of X if and only if there do not exist $f \in X^*$, $s_0 \in S$ and a sequence $\{w_n\}_{n\geq 1}$ in W without a weakly convergent subsequence such that $\|f\| \leq 1$, $f|_W = 0$ and*

$$f(s_0) = \sup_{s\in S}\|s-w_n\| \quad \text{for all} \quad n \geq 1.$$

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