Anal. Theory Appl.
Vol. 27, No. 2 (2011), 158–166
DOI10.1007/s10496-011-0158-8

SHARP MAXIMAL FUNCTION ESTIMATE AND WEIGHTED INEQUALITIES FOR MAXIMAL MULTILINEAR SINGULAR INTEGRALS

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Received Dec. 9, 2010

Abstract. For maximal multilinear Calderón-Zygmund singular integral operators, the sharp maximal function estimate and some weighted norm inequalities are obtained.

Key words: *multilinear Calderón-Zygmund operator, sharp maximal function, weighted norm inequality*

AMS (2010) subject classification: 42B20, 42B25

1 Introduction

Let T be a multilinear operator initially defined on the m-fold product of Schwartz spaces and taking values into the space of tempered distributions,

$$T:\mathcal{S}(\mathbf{R}^n)\times\cdots\times\mathcal{S}(\mathbf{R}^n)\longrightarrow\mathcal{S}'(\mathbf{R}^n).$$

We say that T is an m-linear Calderón-Zygmund operator, if for some $1\le q_j<\infty$, it extends to a bounded multilinear operator from $L^{q_1}\times\cdots\times L^{q_m}$ to L^q, where $1/q=1/q_1+\cdots+1/q_m$, and if there exists a function K, defined off the diagonal $x=y_1=\cdots=y_m$ in $(\mathbf{R}^n)^{m+1}$, for $\vec{f}=(f_1,\cdots,f_m)$, satisfying

$$T(\vec{f})(x)=\int_{(\mathbf{R}^n)^m}K(x,y_1,\cdots,y_m)f_1(y_1)\cdots f_m(y_m)\mathrm{d}\vec{y}$$

*Supported by the Natural Science Foundation of Hebei Province (08M001) and the National Natural Science Foundation of China (10771049).

for all $x \notin \bigcap_{j=1}^{m} \mathrm{supp} f_j$, where $d\vec{y} = dy_1 \cdots dy_m$ and $\vec{y} = (y_1, \cdots, y_m)$;

$$|K(y_0, y_1, \cdots, y_m)| \le \frac{A}{\left(\sum\limits_{k,j=0}^{m} |y_k - y_l|\right)^{mn}}; \tag{1}$$

and

$$|K(y_0, \cdots, y_j, \cdots, y_m) - K(y_0, \cdots, y_j', \cdots, y_m)| \le \frac{A|y_j - y_j'|^{\gamma}}{\left(\sum\limits_{k,j=0}^{m} |y_k - y_l|\right)^{mn+\gamma}}, \tag{2}$$

for some $\gamma > 0$ and all $0 \le j \le m$, whenever $|y_j - y_j'| \le \frac{1}{2} \max_{0 \le k \le m} |y_j - y_k|$.

The multilinear Calderón-Zygmund theory has been developed by Grafakos and Torres[1][2]. These articles and the references therein contain the background and applications about this subject. It was shown in [1] that if $1/r = 1/r_1 + \cdots + 1/r_m$, then an m-linear Calderón-Zygmund operator satisfies

$$T : L^{r_1} \times \cdots \times L^{r_m} \longrightarrow L^r \tag{3}$$

when $1 < r_j < \infty$ for all $j = 1, \cdots, m$; and

$$T : L^{r_1} \times \cdots \times L^{r_m} \longrightarrow L^{r,\infty}, \tag{4}$$

when $1 \le r_j < \infty$ for all $j = 1, \cdots, m$, and at least one $r_j = 1$. In particular

$$T : L^1 \times \cdots \times L^1 \longrightarrow L^{1/m,\infty}. \tag{5}$$

Given $\varepsilon > 0$, for $x \in \mathbf{R}^n$, define the truncated operator by

$$T_\varepsilon(\vec{f})(x) = \int_{|x-y_1|^2+\cdots+|x-y_m|^2>\varepsilon^2} K(x, \vec{y}) f_1(y_1) \cdots f_m(y_m) \mathrm{d}\vec{y}$$

and the associated maximal operator by

$$T^*(\vec{f})(x) = \sup_{\varepsilon>0} |T_\varepsilon(\vec{f})(x)|.$$

Grafakos and Torres in [2] proved that the maximal operator T^* satisfies the same boundedness as T in (3), (4), (5) and some weighted norm inequalities.

Recently, Lerner, Ombrosi, Pérez and Trujillo-González[3] defined a new multilinear maximal function associated to the m-linear Calderón-Zygmund operator as

$$\mathcal{M}(\vec{f})(x) = \sup_{Q \ni x} \prod_{j=1}^{m} \frac{1}{|Q|} \int_Q |f_j(y_j)| \mathrm{d}y_j,$$

and developed a $A_{\vec{P}}$ weighted theory for the this multilinear maximal function and multilinear Calderón-Zygmund operators.

Motivated by the work in [4], we consider here the sharp maximal function estimate and weighted norm inequalities for the maximal operator T^*.

2 Main Results and Preliminaries

For $\delta > 0$, let M_δ be the maximal function

$$M_\delta f(x) = \sup_{x\in Q}\left(\frac{1}{|Q|}\int_Q |f(y)|^\delta \mathrm{d}y\right)^{\frac{1}{\delta}},$$

and $M_\delta^\sharp$ be the sharp maximal function of Fefferman and Stein,

$$M_\delta^\sharp f(x) = \sup_{x\in Q}\inf_{c}\left(\frac{1}{|Q|}\int_Q ||f(y)|^\delta - |c|^\delta|\mathrm{d}y\right)^{\frac{1}{\delta}}.$$

If $\delta = 1$, M_1 and $M_1^\#$ are the Hardy-Littlewood maximal function and sharp maximal function denoted by M and $M^\#$, respectively.

For m exponents $p_1,\cdots,p_m$, we will write p for the number given by $1/p = 1/p_1 + \cdots + 1/p_m$, and $\vec{P} = (p_1,\cdots,p_m)$. Let $1 \le p_1,\cdots,p_m < \infty$. Given $\vec{w} = (w_1,\cdots,w_m)$, w_j are nonnegative locally integrable functions on $\mathbf{R}^n$, $j = 1,\cdots,m$, set $v_{\vec{w}} = \prod_{j=1}^m w_j^{p/p_j}$. We say that $\vec{w}$ satisfies the $A_{\vec{P}}$ condition if

$$\sup_Q\left(\frac{1}{|Q|}\int_Q v_{\vec{w}}\right)^{1/p}\prod_{j=1}^m\left(\frac{1}{|Q|}\int_Q w_j^{1-p_j'}\right)^{1/p_i'} < \infty.$$

When $p_j = 1$, $\left(\frac{1}{|Q|}\int_Q w_j^{1-p_j'}\right)^{1/p_i'}$ is understood as $(\inf_Q w_j)^{-1}$. For $m = 1$, $A_{\vec{P}}$ is the Muckenhoupt weight class A_p. We denote $A_\infty = \cup_{p>1}A_p$.

Our main results can be stated as follows.

Theorem 2.1. *Let T be an m-linear Calderón-Zygmund operator and T^* be the corresponding maximal operator, $0 < \delta < 1/m$. Then there exists a constant $C > 0$ such that for any m-tuples $\vec{f}$ of bounded measurable functions with compact support,*

$$M_\delta^\#(T^*(\vec{f}))(x) \le C\mathcal{M}(\vec{f})(x). \tag{6}$$

Corollary 2.2. *Let T be an m-linear Calderón-Zygmund operator and T^* be the corresponding maximal operator, w be a weight in A_∞ and $p > 0$. Then there exists $C > 0$ (depending on A_∞ constant of w) so that the inequalities*

$$\|T^*(\vec{f})\|_{L^p(w)} \le C\|\mathcal{M}(\vec{f})\|_{L^p(w)} \tag{7}$$

and

$$\|T^*(\vec{f})\|_{L^{p,\infty}(w)} \le C\|\mathcal{M}(\vec{f})\|_{L^{p,\infty}(w)} \tag{8}$$

hold for all m-tuples $\vec{f}$ of bounded functions with compact support.

Corollary 2.3. *Let T be an m-linear Calderón-Zygmund operator and T^* be the corresponding maximal operator, $1/p = 1/p_1 + \cdots + 1/p_m$, and $\vec{w}$ satisfy the $A_{\vec{P}}$ condition.*

(i) *If $1 < p_j < \infty$, $j = 1, \cdots, m$, then*

$$\|T^*(\vec{f})\|_{L^p(v_{\vec{w}})} \le C \prod_{j=1}^{m} \|f_j\|_{L^{p_j}(w_j)}. \tag{9}$$

(ii) *If $1 \le p_j < \infty$, $j = 1, \cdots, m$ and at least one of $p_j = 1$, then*

$$\|T^*(\vec{f})\|_{L^{p,\infty}(v_{\vec{w}})} \le C \prod_{j=1}^{m} \|f_j\|_{L^{p_j}(w_j)}.$$

Note that the weighted inequality (9) was proved in [5] using Lemma 2.9 below.

Theorem 2.4. *Let T be an m-linear Calderón-Zygmund operator and T^* be the corresponding maximal operator, $1 < p_j < \infty$, $j = 1, \cdots, m$, $1/p = 1/p_1 + \cdots + 1/p_m$. Then there is a constant $C > 0$ such that for any m-tuples weights $\vec{w}$ and any m-tuples $\vec{f}$ of bounded measurable functions with compact support,*

$$\|T(\vec{f})\|_{L^p(v_{\vec{w}})} \le C \prod_{j=1}^{m} \|f_j\|_{L^{p_j}(M^{[mp]+2} w_j)}, \tag{10}$$

and

$$\|T^*(\vec{f})\|_{L^p(v_{\vec{w}})} \le C \prod_{j=1}^{m} \|f_j\|_{L^{p_j}(M^{[mp]+2} w_j)}. \tag{11}$$

We will use the following form of the classical result of Fefferman and Stein[6]. Let $0 < p, \delta < 1$ and w be a weight in A_∞. Then, there exists $C > 0$ (depending on the A_∞ constant of w), such that

$$\int_{\mathbf{R}^n} (M_\delta f(x))^p \omega(x) \mathrm{d}x \le C \int_{\mathbf{R}^n} (M_\delta^\sharp f(x))^p \omega(x) \mathrm{d}x \tag{12}$$

for all function f for which the left hand side is finite. Similarly, if $\varphi : (0,\infty) \to (0,\infty)$ is doubling, then there exists a constant C (depending on the A_∞ constant of w and the doubling condition of φ) such that

$$\sup_{\lambda>0} \varphi(\lambda) w(\{y \in \mathbf{R}^n : M_\delta f(y) > \lambda\}) \le C \sup_{\lambda>0} \varphi(\lambda) w(\{y \in \mathbf{R}^n : M_\delta^\# f(y) > \lambda\})$$

for every function f such that the left hand side is finite.

Lemma 2.5.[7] *Let f be a nonnegative function on $\mathbf{R}^n$ such that the level set $\{x \in \mathbf{R}^n : f(x) > \lambda\}$ has finite measures for all $\lambda > 0$. Then for any weight w,*

$$\int_{\mathbf{R}^n} f(x) w(x) \mathrm{d}x \le C \int_{\mathbf{R}^n} M^\# f(x) M w(x) \mathrm{d}x.$$

Lemma 2.6.[3] (1) *If* $1 < p_j < \infty$, $j = 1, \cdots, m$, $1/p = 1/p_1 + \cdots + 1/p_m$. *Then the inequality*

$$\|\mathcal{M}(\vec{f})\|_{L^p(v_{\vec{w}})} \leq C \prod_{j=1}^{m} \|f_j\|_{L^{p_j}(w_j)} \tag{13}$$

holds for every $\vec{f}$ *if and only if* $\vec{w}$ *satisfies* $A_{\vec{P}}$ *condition.*

(2) *If* $1 \leq p_j < \infty$, $j = 1, \cdots, m$, *then*

$$\|\mathcal{M}(\vec{f})\|_{L^{p,\infty}(v_{\vec{w}})} \leq C \prod_{j=1}^{m} \|f_j\|_{L^{p_j}(w_j)} \tag{14}$$

holds for every $\vec{f}$ *if and only if* $\vec{w}$ *satisfies* $A_{\vec{P}}$ *condition.*

Lemma 2.7.[3] *Let* $\vec{w} = (w_1, \cdots, w_m)$ *and* $1 \leq p_1, \cdots, p_m < \infty$. *Then* $\vec{w} \in A_{\vec{P}}$ *if and only if* $w_j^{1-p_j'} \in A_{mp_j'}$, $j = 1, \cdots, m$ *and* $v_{\vec{w}} \in A_{mp}$.

Lemma 2.8.[3] (1) *If* $1 < p_j < \infty$, $j = 1, \cdots, m$, $1/p = 1/p_1 + \cdots + 1/p_m$. *Then there is a constant* $C > 0$ *such that for any m-tuples weights* $\vec{w}$ *and any m-tuples* $\vec{f}$ *of measurable functions,*

$$\|\mathcal{M}(\vec{f})\|_{L^p(v_{\vec{w}})} \leq C \prod_{j=1}^{m} \|f_j\|_{L^{p_j}(Mw_j)}, \tag{15}$$

(2) *If* $1 \leq p_j < \infty$, $j = 1, \cdots, m$ *and at least one of* $p_j = 1$, *then*

$$\|\mathcal{M}(\vec{f})\|_{L^{p,\infty}(v_{\vec{w}})} \leq C \prod_{j=1}^{m} \|f_j\|_{L^{p_j}(Mw_j)}. \tag{16}$$

Lemma 2.9.[5] *Let* T *be an m-linear Calderón-Zygmund operator. Then, for all* $\eta > 0$, *there exists a constant* $C_\eta = C_\eta(n,m) < \infty$ *such that for all* $\vec{f}$ *in any product of* $L^{q_j}(\mathbf{R}^n)$ *spaces, with* $1 \leq q_j < \infty$, *the following inequality holds for all* x *in* $\mathbf{R}^n$,

$$T^*(\vec{f})(x) \leq C_\eta(\mathcal{M}(\vec{f})(x) + (M(|T(\vec{f})|^\eta)(x))^{1/\eta}). \tag{17}$$

We will employ the following Kolmogorov inequality. Let $0 < p < q < \infty$, then there is a constant $C > 0$ such that for any measurable function f

$$\|f\|_{L^p(Q, \frac{dx}{|Q|})} \leq C\|f\|_{L^{q,\infty}(Q, \frac{dx}{|Q|})}. \tag{18}$$

3 The Proof of Theorems

For a given $x \in \mathbf{R}^n$ and $\varepsilon > 0$, we denote by $S_\varepsilon(x) = \{\vec{y} : \max_{1\leq j\leq m} |x - y_j| \leq \varepsilon\}$ and $U_\varepsilon(x) = \{\vec{y} : |x - y_1|^2 + \cdots + |x - y_m|^2 > \varepsilon^2\}$.

Proof of Theorem 2.1. We will use the ideas from [3],[4]. Given a point x and a cube Q containing x. As in the proof of Theorem 3.2 in [3], it suffices to prove that for $0<\delta<1/m$, there exists a constant $C>0$ such that

$$\left(\frac{1}{|Q|}\int_Q |T^*(\vec{f})(z)-c_Q|^\delta \mathrm{d}z\right)^{1/\delta}\le C\mathcal{M}(\vec{f})(x)$$

for some constant c_Q depending on Q and $T^*(\vec{f})$.

Let $f_j=f_j^0+f_j^\infty$, where $f_j^0=f_j\chi_{Q^*}$, $j=1,\cdots,m$ and $Q^*=3Q$. Then

$$\begin{aligned}\prod_{j=1}^m f_j(y_j) &= \prod_{j=1}^m (f_j^0(y_j)+f_j^\infty(y_j))=\sum_{\beta_1,\cdots,\beta_m\in\{0,\infty\}} f_1^{\beta_1}(y_1)\cdots f_m^{\beta_m}(y_m)\\ &= \prod_{j=1}^m f_j^0+\sum{}' f_1^{\beta_1}(y_1)\cdots f_m^{\beta_m}(y_m),\end{aligned}$$

where each term of Σ' contains at least one $\beta_j\neq 0$. Then

$$T^*(\vec{f})(z)\le T^*(\vec{f}^0)(z)+\sum{}' T^*(f_1^{\beta_1},\cdots,f_m^{\beta_m})(z), \tag{19}$$

Applying Kolmogorov's inequality (18), we have

$$\begin{aligned}\left(\frac{1}{|Q|}\int_Q |T^*(\vec{f}^0)(z)|^\delta \mathrm{d}z\right)^{\frac{1}{\delta}} &\le C\|T^*(\vec{f}^0)\|_{L^{1/m,\infty}(Q,\frac{\mathrm{d}x}{|Q|})}\\ &\le C\prod_{i=1}^m \frac{1}{|3Q|}\int_{3Q}|f_i(z)|\mathrm{d}z\\ &\le C\mathcal{M}(\vec{f})(x),\end{aligned}$$

since $T^*:L^1\times\cdots\times L^1\longrightarrow L^{1/m,\infty}$.

In order to study the other term, we set

$$c_Q=\sum{}'(T^*(f_1^{\beta_1},\cdots,f_m^{\beta_m}))_Q,$$

where $(f)_Q=\dfrac{1}{|Q|}\displaystyle\int_Q f(z)\mathrm{d}z$. It suffices to show that for any $z\in Q$, we have an estimate of the form

$$\sum{}'|T^*(f_1^{\beta_1},\cdots,f_m^{\beta_m})(z)-T^*(f_1^{\beta_1},\cdots,f_m^{\beta_m})(x)|\le C\mathcal{M}(\vec{f})(x). \tag{20}$$

Consider first the case when $\beta_1=\cdots=\beta_m=\infty$. For any $z\in Q$ we obtain

$$\begin{aligned}&|T^*(f_1^\infty,\cdots,f_m^\infty)(z)-T^*(f_1^\infty,\cdots,f_m^\infty)(x)|\\ \le\ &\sup_{\varepsilon>0}|T_\varepsilon(f_1^\infty,\cdots,f_m^\infty)(z)-T_\varepsilon(f_1^\infty,\cdots,f_m^\infty)(x)|\\ \le\ &\int_{(\mathbf{R}^n\setminus 3Q)^m}|K(z,\vec{y})-K(x,\vec{y})|\prod_{i=1}^m|f_i(y_i)|\mathrm{d}\vec{y}+2\sup_{\varepsilon>0}\int_D|K(x,\vec{y})|\prod_{i=1}^m|f_i^\infty(y_i)|\mathrm{d}\vec{y}\\ =\ &I+II,\end{aligned}$$

where $D=\{\vec{y}:|x-y_1|^2+\cdots+|x-y_m|^2>\varepsilon^2,|z-y_1|^2+\cdots+|z-y_m|^2\leq\varepsilon^2\}$. For I, using the regularity condition (2), we have

$$\begin{aligned}
I &\leq C\int_{(\mathbf{R}^n\backslash 3Q)^m}\frac{|x-z|^\gamma}{(|z-y_1|+\cdots+|z-y_m|)^{mn+\gamma}}\prod_{i=1}^m|f_i(y_i)|\mathrm{d}\vec{y}\\
&\leq C\sum_{k=1}^\infty\int_{(3^{k+1}Q)^m\backslash(3^kQ)^m}\frac{|x-z|^\gamma}{(|z-y_1|+\cdots+|z-y_m|)^{mn+\gamma}}\prod_{i=1}^m|f_i(y_i)|\mathrm{d}\vec{y}\\
&\leq C\sum_{k=1}^\infty\frac{|Q|^{\gamma/n}}{(3^k|Q|^{1/n})^{mn+\gamma}}\int_{(3^{k+1}Q)^m}\prod_{i=1}^m|f_i(y_i)|\mathrm{d}\vec{y}\\
&\leq C\sum_{k=1}^\infty\frac{1}{3^{k\gamma}}\prod_{i=1}^m\frac{1}{3^{k+1}|Q|}\int_{3^{k+1}Q}|f_i(y_i)|\mathrm{d}y_i\leq C\mathcal{M}(\vec{f})(x).
\end{aligned}$$

For II, using the size condition (1), we have

$$\begin{aligned}
II &\leq 2\sup_{\varepsilon>0}\int_{\varepsilon^2<|x-y_1|^2+\cdots+|x-y_m|^2<4\varepsilon^2}|K(x,\vec{y})|\prod_{i=1}^m|f_i(y_i)|\mathrm{d}\vec{y}\\
&\leq C\mathcal{M}(\vec{f})(x).
\end{aligned}$$

What remains to be considered are the terms in (20) such that $\beta_{j_1}=\cdots=\beta_{j_l}=0$ for some $\{j_1,\cdots,j_l\}\subset\{1,\cdots,m\}$ and $1\leq l<m$. Then

$$\begin{aligned}
&|T^*(f_1^{\beta_1},\cdots,f_m^{\beta_m})(z)-T^*(f_1^{\beta_1},\cdots,f_m^{\beta_m})(x)|\\
\leq\ &\sup_{\varepsilon>0}|T_\varepsilon(f_1^{\beta_1},\cdots,f_m^{\beta_m})(z)-T_\varepsilon(f_1^{\beta_1},\cdots,f_m^{\beta_m})(x)|\\
\leq\ &\prod_{j\in\{j_1,\cdots,j_l\}}\int_{3Q}|f_j(y_j)|\int_{(\mathbf{R}^n\backslash 3Q)^{m-l}}\frac{|x-z|^\gamma\prod_{j\notin\{j_1,\cdots,j_m\}}|f_j(y_j)|\mathrm{d}\vec{y}}{(|z-y_1|+\cdots+|z-y_m|)^{mn+\gamma}}\\
+\ &2\sup_{\varepsilon>0}\int_D|K(x,\vec{y})|\prod_{j\in\{j_1,\cdots,j_l\}}|f_i^0(y_i)|\prod_{j\notin\{j_1,\cdots,j_l\}}|f_i^\infty(y_i)|\mathrm{d}\vec{y}\\
=\ &III+IV,
\end{aligned}$$

Similar to the proof for I, we have

$$\begin{aligned}
III &\leq \prod_{j\in\{j_1,\cdots,j_l\}}\int_{3Q}|f_j(y_j)|\mathrm{d}y_j\sum_{k=1}^\infty\frac{|Q|^{\gamma/n}}{(3^k|Q|^{1/n})^{mn+\gamma}}\int_{(3^{k+1}Q)^{m-l}}\prod_{j\notin\{j_1,\cdots,j_m\}}|f_j(y_j)|\mathrm{d}y_j\\
&\leq C\sum_{k=1}^\infty\frac{|Q|^{\gamma/n}}{(3^k|Q|^{1/n})^{mn+\gamma}}\int_{(3^{k+1}Q)^m}\prod_{j\notin\{j_1,\cdots,j_m\}}|f_j(y_j)|\mathrm{d}y_j\leq C\mathcal{M}(\vec{f})(x).
\end{aligned}$$

Similar to the proof for II, using (1) we have

$$VI\leq 2\int_{S_{2r(Q)}(x)\bigcap U_{r(Q)/2}(x)}|K(x,\vec{y})|\prod_{i=1}^m|f_i(y_i)|\mathrm{d}\vec{y}\leq C\mathcal{M}(\vec{f})(x),$$

where $r(Q)$ denotes the side length of Q. This concludes the proof of the theorem.

Proof of Corollary 2.2. It is enough to prove (7) when the right-side is finite (otherwise there is nothing to prove). Using (12) and (6), we have

$$\|T^*(\vec{f})\|_{L^p(w)} \leq \|M_\delta(T^*(\vec{f}))\|_{L^p(w)} \leq C\|M_\delta^\#(T^*(\vec{f}))\|_{L^p(w)} \leq C\|\mathcal{M}(\vec{f})\|_{L^p(w)},$$

provided we can show that $\|M_\delta(T^*(\vec{f})\|_{L^p(w)}$ is finite. Since $w \in A_\infty$, w is also in A_{p_0} with $0 < \max(1, pm) < p_0 < \infty$. So with $\delta < p/p_0 < 1/m$, we have

$$\|M_\delta(T^*(\vec{f}))\|_{L^p(w)} \leq C\|M(T^*(\vec{f})^{p/p_0})\|_{L^{p_0}(w)}^{p_0/p} \leq C\|T^*(\vec{f})\|_{L^p(w)}.$$

It is enough to prove that $\|T^*(\vec{f})\|_{L^p(w)}$ is finite for each $\vec{f}$ of bounded functions with compact support for which $\|\mathcal{M}(\vec{f})\|_{L^p(w)}$ is finite.

The proof of Corollary 3.8 in [3] shows that $\|M_\delta(T(\vec{f}))\|_{L^p(w)}$ is finite, by Lemma 2.9 with $\eta = \delta$, we get $\|T^*(\vec{f})\|_{L^p(w)}$ is finite. Similar argument give the weak-type estimate (8).

Proof of Corollary 2.3. Since $v_{\vec{w}} \in A_\infty$, the corollary immediately follows from Corollary 2.2 and Lemma 2.6.

Proof of Theorem 2.4. We only need to prove (11), since the proof of (10) is the same because T satisfies the sharp function estimate (6) which is proved in [3].

For fixed $p > 1/m$, choose $q \in (0, 1/m)$ such that $[p/q] = [mp]$. Set $\nu = p/q > 1$. It follows from the duality that

$$\left(\int_{\mathbf{R}^n} (T^*(\vec{f})(x))^p v_{\vec{w}}(x)\mathrm{d}x\right)^{1/\nu} = \sup_{g \geq 0, \|g\|_{L^{\nu'}(v_{\vec{w}}^{1-\nu'})} \leq 1} \int_{\mathbf{R}^n} (T^*(\vec{f})(x))^q g(x)\mathrm{d}x.$$

By Lemma 2.5, Theorem 2.1, Lemma 2.8(1) and note that $Mv_{\vec{w}}(x) \leq \prod_{j=1}^m (Mw_j(x))^{p/p_j}$, we have

$$\begin{aligned}
\int_{\mathbf{R}^n} (T^*(\vec{f})(x))^q g(x)\mathrm{d}x &\leq C\int_{\mathbf{R}^n} (M_q^\#(T^*(\vec{f})(x))^q Mg(x)\mathrm{d}x \\
&\leq C\int_{\mathbf{R}^n} (\mathcal{M}(\vec{f})(x))^q Mg(x)\mathrm{d}x \\
&\leq C\left(\int_{\mathbf{R}^n} (\mathcal{M}(\vec{f})(x))^p M^{[\nu]+1} v_{\vec{w}}(x)\mathrm{d}x\right)^{1/\nu} \\
&\quad \times \left(\int_{\mathbf{R}^n} (Mg(x))^{\nu'} (M^{[\nu]+1} v_{\vec{w}}(x))^{1-\nu'}\mathrm{d}x\right)^{1/\nu'} \\
&\leq C\left(\prod_{j=1}^m \|f_j\|_{L^{p_j}(M^{[\nu]+2}w_j)}\right)^{p/\nu} \left(\int_{\mathbf{R}^n} |g(x)|^{\nu'} v_{\vec{w}}(x)^{1-\nu'}\mathrm{d}x\right)^{1/\nu'} \\
&\leq C\left(\prod_{j=1}^m \|f_j\|_{L^{p_j}(M^{[mp]+2}w_j)}\right)^{p/\nu},
\end{aligned}$$

in which we have used the following inequality (see [8]), that for any $p \in (1,\infty)$, weight u and $g \in L^{p'}(u^{1-p'})$,

$$\int_{\mathbf{R}^n} Mg(x)^{p'} M^{[p]+1}u(x)^{1-p'} \mathrm{d}x \leq C \int_{\mathbf{R}^n} |g(x)|^{p'} u(x)^{1-p'} \mathrm{d}x.$$

This completes the proof of the theorem.

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