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ON CERTAIN PROPERTIES OF THE COMBINATIONS OF SZÁSZ-DURRMEYER OPERATORS

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Abstract. In the present paper we study properties of Szász-Durrmeyer operators. These operators are introduced in [5] and generalize the integral operators proposed by S.M.Mazhar and V.Totik in [12]. We also generalize some results obtained by M. Heilmann[6] and D.-X. Zhou[16].

Key words: *modified Szász-Mirakyan operator, Baskakov-Durremeyer operator*

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1 Introduction

In this paper we study Szász-Durrmeyer operators defined on functions $f \in L_p$ in the following form:

$$M_{n,\nu}(f,x) = \int_0^\infty n \sum_{k=0}^\infty p_{n,k}(x) p_{n,k+\nu}(t) f(t) \mathrm{d}t,$$

where x, $n \in [0,\infty)$, $k \in I\!N$, $\nu \in (-1,\infty)$ and $p_{n,l}(t) = e^{-nx} \dfrac{(nx)^l}{\Gamma(l+1)}$ for $l \in [0,\infty)$. The term

$$K_{n,\nu}(t,x) = n \sum_{k=0}^\infty p_{n,k}(x) p_{n,k+\nu}(t)$$

is called the kernel of Szász-Durrmeyer operator. This family of operators was introduced by A. Ciupa and I.Gavrea[15] and was also independently proposed by E. Wachnicki[5]. Some typical results could be found in papers [5, 13, 14, 7]. In the first section we placed the results which are useful in the proof of the further theorems, some of them generalize the properties which are known for the particular case (for $\nu = 0$, $n \in \mathbf{N}$) but others couldn't even be formulated for the previously considered families of operators. As the main results in the second section we present certain theorems for combinations of considered operators. Similar results (theorems 3,4 and 5) were obtained by M. Heilmann[6] and later by D. Zhou[16] who used the same method for particular case of Szász-Durrmeyer operators (for $\nu = 0$, $n \in \mathbf{N}$). In our proofs we use the same ideas, but they are slightly simplified compared to the mentioned special case [16] thanks to Theorem 1 and Lemma 4.

2 Auxiliary Results

By simple induction with respect to r we obtain the following

Lemma 1. *Let $r \in \mathbf{N}$, $x, n, k \in [0, \infty)$, then*

$$p_{n,k}^{(r)}(x) = n^r \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} p_{n,k-j}(x), \quad r \leq k, \tag{1}$$

$$x p_{n,k}^{(r+1)}(x) = (k - nx) p_{n,k}^{(r)}(x) - r \left(p_{n,k}^{(r)}(x) + n p_{n,k}^{(r-1)}(x) \right), \tag{2}$$

$$p_{n,k}^{(r+1)}(x) = n[p_{n,k-1}^{(r)}(x) - p_{n,k}^{(r)}(x)]. \tag{3}$$

Moreover, $p_{n,k}^{(r)}(x)$ are in the form

$$x^r p_{n,k}^{(r)}(x) = p_{n,k}(x) \sum_{i=0}^{r} \sum_{j=0}^{[\frac{i}{2}]} b_{r,i,j} (k - nx)^{r-i} (nx)^j, \tag{4}$$

where $i \in \{0, 1, ..., r\}$, $j \in \{0, 1, ..., [\frac{i}{2}]\}$ and $b_{r,i,j}$ are coefficients independent of x, n, k.

Using properties of the Gamma-Euler function, we get

$$\int_0^\infty t^s p_{n,k+\nu}(t) \mathrm{d}t = n^{-(s+1)} \frac{\Gamma(k + \nu + s + 1)}{\Gamma(k + \nu + 1)}, \qquad \text{for } k \in \mathbf{N},\ s, n \in \mathbf{N}^* \tag{5}$$

and

$$n \int_0^\infty p_{n,k+\nu}(t) \mathrm{d}t = 1. \tag{6}$$

Let us note that

$$\frac{\Gamma(k + \nu + s + 1)}{\Gamma(k + \nu + 1)} = (k + \nu + 1) \cdot \ldots \cdot (k + \nu + s) = \sum_{i=0}^{s} k^i B_{s,i}^{\nu}, \tag{7}$$

where $B_{s,i}^{\nu}$ are coefficients of the polynomial independent of i, s and ν, moreover $B_{s,s}^{\nu} = 1$.

Taking $B_{s,i}^{\nu} = 0$ for $i < 0$ or $i > s$ we obtain

$$B_{s+1,i}^{\nu} = (\nu + 1) B_{s,i}^{\nu+1} + B_{s,i-1}^{\nu+1}. \tag{8}$$

for $\nu\in(-1,\infty)$, $s,i\in\mathbf{N}$. In particular, $B^{\nu}_{s+1,0}=(\nu+1)B^{\nu+1}_{s,0}$ and $B^{\nu}_{s,s}=1$.

The next lemma shows the recurrent dependency between moments of the operator. Let us notice that in this formula the derivative does not appear, in contrast to the results obtained by V. Totik[6]. It is clear that V. Totik could not get (9) since he considered only $M_{n,0}$. Let us also observe that in the proof of the formula (9) the essential role plays the result of Lemma 1.

Lemma 2. *Let us assume that* $x,n\in(0,\infty)$, $\nu\in(-1,\infty)$, $r\in\mathbf{N}$. *Then*

$$T^{r}_{n,\nu}(x)=M_{n,\nu}((.-x)^{r},x),$$

$$T^{r+1}_{n,\nu}(x)=x\left(T^{r}_{n,\nu+2}(x)-T^{r}_{n,\nu}(x)\right)+\frac{\nu+1}{n}T^{r}_{n,\nu+1}(x), \tag{9}$$

$$T^{2r}_{n,\nu}(x)=\sum_{i=0}^{r}q_{i,2r}(\nu)(\frac{x}{n})^{r-i}n^{-2i},$$
$$T^{2r+1}_{n,\nu}(x)=\sum_{i=0}^{r}q_{i,2r+1}(\nu)(\frac{x}{n})^{r-i}n^{-2i-1},$$

where $q_{i,j}(\nu)$ *are polynomials of* ν*, satisfying additional conditions:*

$$\begin{aligned} q_{0,2r+2}(\nu) &= q_{0,2r+1}(\nu+2)-q_{0,2r+1}(\nu),\\ q_{r+1,2r+2}(\nu) &= (\nu+1)q_{r,2r+1}(\nu+1),\\ q_{i,2r+2}(\nu) &= q_{i+1,2r+1}(\nu+2)-q_{i+1,2r+1}(\nu)+\frac{\nu+1}{n}q_{i,2r+1}(\nu+1),\end{aligned}$$

for $r\in\mathbf{N}$, $i\in\{1,2,\cdots,r\}$, *and*

$$\begin{aligned} q_{r,2r+1}(\nu) &= (\nu+1)\,q_{r,2r}(\nu+1),\\ q_{i,2r+1}(\nu) &= q_{i+1,2r}(\nu+2)-q_{i+1,2r}(\nu)+\frac{\nu+1}{n}q_{i,2r}(\nu+1),\end{aligned}$$

for $r\in\mathbf{N}$, $i\in\{0,1,\cdots,r-1\}$.

Moreover, $deg\ q_{i,2r}(\nu)\le 2i$ and $deg\ q_{i,2r+1}(\nu)\le 2i+1$.

Remark 1. If $r\in\mathbf{N}$, $n\in[0,\infty)$, $\nu\in(-1,\infty)$, then

$$|T^{2r}_{n,\nu}(x)|\le Cn^{-r}\left(x+\frac{1}{n}\right)^{r},$$

where C is a constant independent of x,i,n.

Now we give the properties of the derivatives of Szász-Durremeyer operators. We will use these properties in the second part of the paper. Independently they state the interesting result describing the connection between the operator and it's r-th derivative. The next lemma is a simple generalization of the result obtained by M. Heilmann[6].

Lemma 3. *Let $n,x \in [0,\infty)$, $\nu \in (-1,\infty)$, $r \in \mathbf{N}$ and $f \in L_p, p \in [1,\infty]$. Then*

$$M_{n,\nu}^{(r)}(f,x) = (-1)^r n \int_0^\infty \sum_{k=0}^\infty p_{n,k}(x) p_{n,k+\nu+r}^{(r)}(t) f(t) \mathrm{d}t, \tag{10}$$

$$M_{n,\nu}^{(r)}(f,x) = n^r \sum_{i=0}^r (-1)^{r-i} \begin{pmatrix} r \\ i \end{pmatrix} M_{n,\nu+i}(f,x). \tag{11}$$

Proof. Using Lemma 1 by induction we get (10) and consequently

$$\begin{aligned} M_{n,\nu}^{(r)}(f,x) &= \sum_{k=0}^\infty (-1)^r p_{n,k}(x) n \int_0^\infty p_{n,k+\nu+r}^{(r)}(t) f(t) \mathrm{d}t \\ &= \sum_{k=0}^\infty (-1)^j p_{n,k}(x) n^{r+1} \int_0^\infty \sum_{j=0}^r \begin{pmatrix} r \\ j \end{pmatrix} p_{n,k+\nu+r-j}(t) f(t) \mathrm{d}t \\ &= n^r \sum_{i=0}^r (-1)^{r-i} \begin{pmatrix} r \\ i \end{pmatrix} n \int_0^\infty \sum_{k=0}^\infty p_{n,k}(x) p_{n,k+\nu+i}(t) f(t) \mathrm{d}t. \end{aligned}$$

Thus we obtain (11).

Lemma 4. *If $n,x \in [0,\infty)$, $\nu \in (-1,\infty)$, $r \in \mathbf{N}$ and $f \in D_p^r$, then the following recurrent formula holds:*

$$M_{n,\nu}^{(r)}(f,x) = M_{n,\nu+r}(f^{(r)},x).$$

Proof. We perform induction proof with respect to the order r. For $r = 0$ the identity is obvious. Let $r \in \mathbf{N}$. Using the induction assumption, the property (10) and Lemma 1, we obtain

$$\begin{aligned} M_{n,\nu}^{(r+1)}(f,x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left[M_{n,\nu}^{(r)}(f,x)\right] = \frac{\mathrm{d}}{\mathrm{d}x}\left[M_{n,r+\nu}(f^{(r)},x)\right] \\ &= (-1)^r n^2 \sum_{k=0}^\infty p_{n,k}^{(1)}(x) \int_0^\infty p_{n,k+r+\nu}(t) f^{(r)}(t) \mathrm{d}t \\ &= (-1)^r n^2 \sum_{k=0}^\infty [p_{n,k-1}(x) - p_{n,k}(x)] \int_0^\infty p_{n,k+r+\nu}(t) f^{(r)}(t) \mathrm{d}t \\ &= (-1)^r n^2 \sum_{k=0}^\infty p_{n,k}(x) \int_0^\infty f^{(r)}(t) [p_{n,k+r+\nu+1}(t) - p_{n,k+r+\nu}(t)] \mathrm{d}t \\ &= (-1)^{r+1} \sum_{k=0}^\infty p_{n,k}(x) n \int_0^\infty f^{(r)}(t) p_{n,k+r+\nu+1}^{(1)}(t) \mathrm{d}t \\ &= (-1)^{r+1} \sum_{k=0}^\infty p_{n,k}(x) n \int_0^\infty f^{(r+1)}(t) p_{n,k+r+\nu+1}(t) \mathrm{d}t \\ &= M_{n,\nu+r+1}(f^{(r+1)},x). \end{aligned}$$

Let us notice that Lemmas 3,4, as well as the previous ones, can not be formulated for modification of Szász-Mirakyan operators proposed by V.Totik et al., then we have only $M_{n,0}(f,.)$.

To prove the Bernstein type inequality we need the following lemma.

Lemma 5. *If* $r\in \mathbf{N}$, $\nu\in(-1,\infty)$, $p\in[1,\infty)$ *and* $g\in D_\infty^r$ *then*

$$\| x^{\frac{r}{2}} M_{n,\nu}^{(r)}(g,x) \|_{L_\infty} \le C \| t^{\frac{r}{2}} g^{(r)}(t) \|_{L_\infty} .$$

Proof. From (4) we get

$$\begin{aligned} \| x^{\frac{r}{2}} M_{n,\nu}^{(r)}(g,x) \|_{L_\infty} &\le \| x^{\frac{r}{2}} M_{n,\nu+r}(g^{(r)},x) \|_{L_\infty} \\ &= \| n \int_0^\infty x^{\frac{r}{2}} \sum_{k=0}^{\infty} p_{n,k}(x) p_{n,k+r+\nu}(t) g^{(r)}(t) \mathrm{d}t \|_{L_\infty} . \end{aligned} \tag{12}$$

Next using (12) and the known property of Gamma function

$$\frac{\Gamma(k+\frac{r}{2}+\nu+1)\Gamma(k+\frac{r}{2}+1)}{\Gamma(k+1)\Gamma(k+r+\nu+1)} \le 1,$$

we obtain

$$\begin{aligned} \| x^{\frac{r}{2}} M_{n,\nu}^{(r)}(g,x) \|_{L_\infty} &\le \| n \int_0^\infty \sum_{k=0}^{\infty} p_{n,k+\frac{r}{2}}(x) p_{n,k+\frac{r}{2}+\nu}(t) t^{\frac{r}{2}} g^{(r)}(t) \mathrm{d}t \|_{L_\infty} \\ &\le C \| t^{\frac{r}{2}} g^{(r)}(t) \|_{L_\infty} . \end{aligned}$$

3 Main Results

Since neither Bernstein type operators nor their Kantorowicz type modifications can be used for examination of higher orders of smoothness of functions and also since the order of convergency can not be better then $O(\frac{1}{n})$, P. L. Butzer[2] introduced the combination of Bernstein operators defined by

$$(2^r-1)B_n(f,r,x) = 2^r B_{2n}(f,r-1,x) - B_n(f,r-1,x),$$

where $B_n(f,0,x) = B_n(f,x)$ is the Bernstein operator. Similar combinations for other operators of exponential type were later used by C. P. May[10,11].

Now we present the definition of combination of Szász-Durrmeyer operators. Definition of this type was proposed by Z. Ditzan[3] for Bernstein operators. It is a generalization of combination defined by P. L. Butzer. For the other types operators similar definitions can be found in [4, 9]. As it was mentioned before we use the ideas introduced by M. Heilmann[6] , and later applied by D. Zhou[6] and Guo S., Li C., Sun Y., Yang G., Yue S[8]. Some of presented results generalize the theorems from this last paper.

Definition 1. Let $r \in \mathbf{N}$. The combination Szász-Durrmeyer operator is that of the form:

$$M_{n,\nu,r}(f,x) = \sum_{i=0}^{r-1} a_i(n) M_{n_i,\nu}(f,x),$$

where constants A, n_i, $a_i(n)$ satisfy the following conditions:

(a) $n = n_0 < n_1 < \cdots < n_{r-1} \le An$,

(b) $\sum_{i=0}^{r-1} |a_i(n)| \le A$,

(c) $\sum_{i=0}^{r-1} a_i(n) = 1$,

(d) $\sum_{i=0}^{r-1} a_i(n) n_i^{-k} = 0$ for $k \in \{1,2,\cdots,r-1\}$.

Now we are going to formulate the theorem which shows that the combination of operators $M_{n\nu}$ is not only commutative[5], but gives us something more: composition of Szász-Durrmeyer operators is also Szász-Durrmeyer operator.

Theorem 1. *Let $n,m \in (0,\infty)$, $p \in [0,\infty]$ and $f \in L_p$. Then*

$$M_{n,\nu}(M_{m,\nu}(f)) = M_{\frac{nm}{n+m},\nu}(f). \tag{13}$$

Proof. Let $n,m \in (0,\infty)$. First we assume that $x \neq 0$. In the calculation we use modified Bessel function I_ν in the form:

$$I_\nu(x) = \sum_{k=0}^{\infty} \frac{(\frac{x}{2})^{\nu+2k}}{k!\Gamma(k+\nu+1)}.$$

We have

$$\begin{aligned}
M_{n,\nu}(M_{m,\nu}(f,.),x) &= (nm)^{\nu+1} \int_0^\infty \int_0^\infty \sum_{k=0}^\infty \sum_{l=0}^\infty \frac{(n^2xt)^k}{k!\Gamma(k+\nu+1)} \frac{(m^2st)^l}{l!\Gamma(l+\nu+1)} \\
&\quad \times e^{-nx-ms} e^{-t(n+m)} (ts)^\nu f(s)\mathrm{d}s\mathrm{d}t \\
&= 2nm \int_0^\infty \int_0^\infty \sum_{k=0}^\infty \sum_{l=0}^\infty \frac{(2n\sqrt{x}u)^{2k+\nu}}{2^{2k+\nu}k!\Gamma(k+\nu+1)} \\
&\quad \times \frac{(2m\sqrt{s}u)^{2l+\nu}}{2^{2l+\nu}!\Gamma(l+\nu+1)} e^{-nx-ms} e^{-u^2(n+m)} (\frac{s}{x})^{\frac{\nu}{2}} f(s) u\mathrm{d}u\mathrm{d}s \\
&= 2nm \int_0^\infty (\int_0^\infty e^{-u^2(n+m)} I_\nu(2n\sqrt{x}u) I_\nu(2m\sqrt{s}u) u\mathrm{d}u) e^{-nx-ms} (\frac{s}{x})^{\frac{\nu}{2}} f(s)\mathrm{d}s.
\end{aligned}$$

The above transformations are available because of the uniform convergency of corresponding integrals and series. Using

$$\int_0^\infty e^{-u^2c^2} I_\nu(au) I_\nu(bu) u\mathrm{d}u = \frac{1}{2c^2} e^{\frac{a^2+b^2}{4c^2}} I_\nu(\frac{ab}{2c^2}),$$

$a > 0$, $b > 0$, $c > 0$, we get

$$\begin{aligned} M_{n,\nu}(M_{m,\nu}(f,.),x) &= \frac{nm}{n+m}\int_0^\infty e^{-\frac{nm}{n+m}(x+s)} I_\nu\left(\frac{2nm\sqrt{sx}}{n+m}\right)\left(\frac{s}{x}\right)^{\frac{\nu}{2}} f(s)\mathrm{d}s \\ &= \frac{nm}{n+m}\int_0^\infty e^{-\frac{nm}{n+m}(s+x)} \sum_{k=0}^\infty \frac{(\frac{nm}{n+m}x)^{2k}(\frac{nm}{n+m}s)^{2k+\nu}}{2^{2k+\nu}k!\Gamma(k+\nu+1)} f(s)\mathrm{d}s \\ &= M_{\frac{nm}{n+m},\nu}(f,x). \end{aligned}$$

Thus the proof of (13) is done for $x \neq 0$. If $x = 0$, then using

$$\int_0^\infty e^{-u^2 c} u^{\nu+1} I_\nu(bu)\mathrm{d}u = \frac{b^\nu}{(2c)^{\nu+1}} e^{\frac{b^2}{4c}}$$

$b > 0$, $c > 0$, in the similar way we obtain (13).

Observe that the composition of combination Szász-Durrmeyer operators is also the combination of the same type.

Remark 2. If $n, m \in (0,\infty)$, $r \in \mathbf{N}$ and $M_{n,\nu,r}(f,x)$ is the combination Szász-Durrmeyer operators, i. e. satisfy conditions of the definition 1, then $M_{\frac{mn}{n+m},\nu,r}(f,x)$ is the combination of the same type.

Now we present Bernstein type inequality.

Theorem 2. *Let $r \in \mathbf{N}$, x, $n \in [0,\infty)$ and $f \in L_\infty$. Then the inequality holds*

$$\| x^{\frac{r}{2}} M_{n,\nu}^{(r)}(f,x) \|_{L_\infty} \leq C n^{\frac{r}{2}} \| f \|_{L_\infty} .$$

Proof. Let $f \in L_\infty$, $r \in \mathbf{N}$, $n \in [0,\infty)$. We consider two cases. Let us start with $x \in [0, \frac{1}{n}]$. Then from Lemma 3 we obtain

$$\begin{aligned} \| x^{\frac{r}{2}} M_{n,\nu}^{(r)}(f,x) \|_{L_\infty}^{[0,\frac{1}{n}]} &\leq C \| x^{\frac{r}{2}} n^r \sum_{j=0}^r M_{n,\nu+j}(f,x) \|_{L_\infty}^{[0,\frac{1}{n}]} \\ &\leq C n^{\frac{r}{2}} \sum_{j=0}^r \| (nx)^{\frac{r}{2}} M_{n,\nu+j}(f,x) \|_{L_\infty}^{[0,\frac{1}{n}]} \\ &\leq C n^{\frac{r}{2}} \| f \|_{L_\infty} . \end{aligned}$$

Now let $x \in (\frac{1}{n},\infty)$. Then taking into account (10) we have

$$\begin{aligned} \| x^{\frac{r}{2}} M_{n,\nu}^{(r)}(f,x) \|_{L_\infty}^{(\frac{1}{n},\infty)} &\leq \| x^{\frac{r}{2}} \sum_{k=0}^\infty p_{n,k}^{(r)}(x) p_{n,k+\nu}(t) f(t)dt \|_{L_\infty}^{(\frac{1}{n},\infty)} \\ &\leq \| f \|_{L_\infty} \| x^{\frac{r}{2}} \sum_{k=0}^\infty p_{n,k}^{(r)}(x) \|_{L_\infty}^{(\frac{1}{n},\infty)}, \end{aligned}$$

and next from Lemma 1 and the formula (6) it follows that

$$\| x^{\frac{r}{2}} M_{n,\nu}^{(r)}(f,x) \|_{L_\infty}^{(\frac{1}{n},\infty)} \leq C \| f \|_{L_\infty} \| \sum_{i=0}^{r} \sum_{j=0}^{[\frac{i}{2}]} \sum_{k=0}^{\infty} x^{-\frac{r}{2}} (k-nx)^{r-i} (nx)^j p_{n,k}(x) \|_{L_\infty}^{(\frac{1}{n},\infty)} . \qquad (14)$$

Finally

$$\begin{aligned} \| x^{\frac{r}{2}} M_{n,\nu}^{(r)}(f,x) \|_{L_\infty}^{(\frac{1}{n},\infty)} &\leq C \sum_{i=0}^{r} \sum_{j=0}^{[\frac{i}{2}]} \| \sum_{k=0}^{\infty} n^{\frac{r}{2}} (nx)^{-\frac{i}{2}+j} p_{n,k}(x) \|_{L_\infty}^{(\frac{1}{n},\infty)} \| f \|_{L_\infty} \\ &\leq C n^{\frac{r}{2}} \| f \|_{L_\infty} . \end{aligned}$$

Remark 3.3. Let us notice that in the above proof the character C is used for different constants in order to simplify the notation. This agreement is valid later on.

Theorem 3. *Let $f \in L_\infty$, $n \in \mathbf{N}$. Then the following inequality holds*

$$|M_{n,\nu,r}(f,x) - f(x)| \leq C\omega_r(f, \sqrt{\frac{x}{n} + \frac{1}{n^2}}), \qquad (15)$$

where C is a constant depending on ν and r.

Proof. Let us start with an observation that

$$M_{n,\nu,r}((t-x)^s, x) = 0 \text{ for } s \in \{1, 2, \cdots, r-1\}$$

and

$$M_{n,\nu,r}(1,x) = 1$$

for the fixed function $g \in D_\infty^r$. Using Taylor's expansion

$$g(t) = \sum_{i=0}^{r-1} \frac{(t-x)^i}{i!} g^{(i)}(x) + \frac{1}{(r-1)!} \int_x^t (t-u)^{r-1} g^{(r)}(u) \mathrm{d}u$$

we obtain

$$\begin{aligned} |M_{n,\nu,r}(g,x) - g(x)| &= |M_{n,\nu,r}(g(t) - g(x), x)| \\ &\quad + M_{n,\nu,r} \left(\sum_{i=1}^{r-1} \frac{1}{i!} g^{(i)}(x)(t-x)^i + \int_x^t \frac{1}{(r-1)!} (t-u)^{r-1} g^{(r)}(u) \mathrm{d}u, x \right) \\ &\leq \sum_{i=1}^{r-1} |a_i(n)| M_{n_i,\nu}(|t-x|^r, x) \| g^{(r)} \|_{L_\infty} \\ &\leq \sum_{i=1}^{r-1} |a_i(n)| \| g^{(r)} \|_{L_\infty} \sqrt{T_{n_i,\nu}^{2r}(x)} \\ &\leq C \| g^{(r)} \|_{L_\infty} \left(\frac{x}{n} + \frac{1}{n^2} \right)^{\frac{r}{2}} . \end{aligned}$$

We use Lemma 2 to explain the last inequality. Let $f \in L_\infty$ and $g \in D^r_\infty$. Then

$$\begin{aligned}|M_{n,\nu,r}(f,x)-f(x)| &\leq |M_{n,\nu,r}(f-g,x)|+|M_{n,\nu,r}(g,x)-g(x)|+|f(x)-g(x)| \\ &\leq (A+1)\parallel f-g\parallel_{L_\infty}+C\parallel g^{(r)}\parallel_{L_\infty}\left(\frac{x}{n}+\frac{1}{n^2}\right)^{\frac{r}{2}} \\ &\leq C[\parallel f-g\parallel_{L_\infty}+\parallel g^{(r)}\parallel_{L_\infty}\left(\frac{x}{n}+\frac{1}{n^2}\right)^{\frac{r}{2}}].\end{aligned}$$

Finally we get

$$|M_{n,\nu,r}(f,x)-f(x)|\leq CK_r(f,\sqrt{\frac{x}{n}+\frac{1}{n^2}})$$

and the thesis by using properties of the K-functional.

Theorem 4. *Let $n,x\in[0,\infty)$, $\nu\in(-1,\infty)$, $r\in\mathbf{N}$ and $f\in L_\infty$, $\alpha\in(0,r)$. Then the following conditions are equivalent:*

(i) *There exists constant C independent of x and n such that*
$|M_{n,\nu,r}(f,x)-f(x)|\leq Cn^{-\frac{\alpha}{2}}\left(x+\frac{1}{n}\right)^{\frac{\alpha}{2}}$,

(ii) $\omega_r(f,h)=O(h^\alpha)$.

Proof.

(ii) $\Rightarrow$ (i)

Let $h=n^{-\frac{1}{2}}\left(x+\frac{1}{n}\right)^{\frac{1}{2}}$ and using Theorem 3, we obtain (i).

(i)$\Rightarrow$(ii)

Let $h\in(0,\infty)$ and $x\in(\frac{rh}{2},\infty)$. If $2rh>\frac{1}{4}$, then

$$\begin{aligned}|\Delta^r_h f(x)| &\leq |\sum_{i=0}^{r}(-1)^i\begin{pmatrix} r\\ i\end{pmatrix}f(x+(\frac{r}{2}-i)h)|\leq 2^r\parallel f\parallel_{L_\infty} \\ &\leq (8rh)^\alpha 2^r\parallel f\parallel_{L_\infty}\leq Ch^\alpha.\end{aligned}$$

Let us consider the case, $2rh\leq\frac{1}{4}$. Let $t\in(0,h)$, and by the definition the sequence

$$\delta_n(x,t)=\max\left\{\frac{1}{n},n^{-\frac{1}{2}}(x+\frac{rt}{2})^{\frac{1}{2}}\right\}.$$

From the assumption we have

$$\begin{aligned}|M_{n,\nu,r}(f,x+(\frac{r}{2}-i)t)-f(x+(\frac{r}{2}-i)t)| &\leq Cn^{-\frac{\alpha}{2}}\left(x+(\frac{r}{2}-i)t+\frac{1}{n}\right)^{\frac{\alpha}{2}} \\ &\leq Cn^{-\frac{\alpha}{2}}\left(x+\frac{rt}{2}+\frac{1}{n}\right)^{\frac{\alpha}{2}} \\ &\leq C\left(n^\alpha+n^{-\frac{\alpha}{2}}(x+\frac{rt}{2})^{\frac{\alpha}{2}}\right) \\ &\leq C2\delta_n(x,t),\end{aligned}$$

for $i \in \{0, 1, \cdots, r\}$. Using the above estimation we get

$$\begin{aligned} |\Delta_t^r(f,x)| &\le \sum_{i=0}^{r} \begin{pmatrix} r \\ i \end{pmatrix} |M_{n,\nu,r}(f, x+(\frac{r}{2}-i)t) - f(x+(\frac{r}{2}-i)t)| \\ &\quad + |\sum_{i=0}^{r} \begin{pmatrix} r \\ i \end{pmatrix} M_{n,\nu,r}(f, x+(\frac{r}{2}-i)t)| \\ &\le C4^r\, (\delta_n(x,t))^{\alpha} \\ &\quad + |\int_{-\frac{t}{2}}^{\frac{t}{2}} \cdots \int_{-\frac{t}{2}}^{\frac{t}{2}} M_{n,\nu,r}^{(r)}(f, x+\sum_{i=1}^{r} u_i) \mathrm{d}u_1 \cdots \mathrm{d}u_r|. \end{aligned} \tag{16}$$

For fixed $h > 0$, we choose $f_h \in D_p^r$ such that the following conditions hold:

$$\| f - f_h \|_{L_\infty} \le 2K_r(f, h^r) \le 2C\omega_r(f,h), \tag{17}$$

$$\| f_h^{(r)} \|_{L_\infty} \le h^{-r} K_r(f, h^r) \le 2C\omega_r(f,h). \tag{18}$$

Then

$$\begin{aligned} |\int_{-\frac{t}{2}}^{\frac{t}{2}} \cdots \int_{-\frac{t}{2}}^{\frac{t}{2}} & M_{n,\nu,r}^{(r)}(f, x+\sum_{i=1}^{r} u_i) \mathrm{d}u_1 \cdots \mathrm{d}u_r| \\ &\le \sum_{i=0}^{r-1} |a_i(n)| \int_{-\frac{t}{2}}^{\frac{t}{2}} \cdots \int_{-\frac{t}{2}}^{\frac{t}{2}} [|M_{n_i,\nu}^{(r)}(f - f_h, x+\sum_{i=1}^{r} u_i)| \\ &\quad + |M_{n_i,\nu}^{(r)}(f_h, x+\sum_{i=1}^{r} u_i)|\,] \mathrm{d}u_1 \cdots \mathrm{d}u_r. \end{aligned} \tag{19}$$

From Lemma 4 we obtain

$$\begin{aligned} \sum_{i=0}^{r-1} |a_i(n)| \int_{-\frac{t}{2}}^{\frac{t}{2}} \cdots \int_{-\frac{t}{2}}^{\frac{t}{2}} |M_{n,\nu,r}^{(r)}(f_h, x+\sum_{i=1}^{r} u_i)| \mathrm{d}u_1 \cdots \mathrm{d}u_r &\le At^r \| f_h^{(r)} \|_{L_\infty} \\ &\le 2Ct^r h^{-r} \omega_r(f,h). \end{aligned} \tag{20}$$

Moreover, using the formula (11) from Lemma 3 we get

$$\begin{aligned} |M_{n_i,\nu}^{(r)}(f - f_h, x)| &\le n_i^r \sum_{j=0}^{r} (-1)^{r-j} \begin{pmatrix} r \\ i \end{pmatrix} M_{n_i,\nu+j}(|f - f_h|, x) \\ &\le (2n_i)^r \| f - f_h \|_{L_\infty}. \end{aligned} \tag{21}$$

Using (21) we obtain

$$\begin{aligned} I &= \sum_{i=0}^{r-1} |a_i(n)| \int_{-\frac{t}{2}}^{\frac{t}{2}} \cdots \int_{-\frac{t}{2}}^{\frac{t}{2}} M_{n,\nu,r}^{(r)}(f - f_h, x+\sum_{j=1}^{r} u_j) \mathrm{d}u_1 \cdots \mathrm{d}u_r \\ &\le A^{r+1}(2nt)^r \| f - f_h \|_{L_\infty}. \end{aligned} \tag{22}$$

On the other hand from Theorem 2 we also have

$$\begin{aligned} I &\le \sum_{i=0}^{r-1}|a_i(n)|\int_{-\frac{t}{2}}^{\frac{t}{2}}\cdots\int_{-\frac{t}{2}}^{\frac{t}{2}} n_i^{\frac{r}{2}} \parallel f-f_h \parallel_{L_\infty}(x+\sum_{j=1}^{r}u_j)\mathrm{d}u_1\cdots\mathrm{d}u_r \\ &\le A^{r+1}n^{\frac{r}{2}}\int_{-\frac{t}{2}}^{\frac{t}{2}}\cdots\int_{-\frac{t}{2}}^{\frac{t}{2}} \parallel f-f_h \parallel_{L_\infty}(x+\sum_{j=1}^{r}u_j)\mathrm{d}u_1\cdots\mathrm{d}u_r \end{aligned} \tag{23}$$

and we get

$$I \le A^{r+1}n^{\frac{r}{2}} \parallel f-f_h \parallel_{L_\infty} Ct^r\left(x+\frac{rt}{2}\right)^{-\frac{r}{2}}. \tag{24}$$

Using (22) and (24) we obtain

$$I \le CA^{r+1} \parallel f-f_h \parallel_{L_\infty} t^r \min\left\{n^r, n^{\frac{r}{2}}\left(x+\frac{rt}{2}\right)^{-\frac{r}{2}}\right\}. \tag{25}$$

Finally the formulas (16), (19), (20) and (25) imply

$$\Delta_t^r(f,x) \le C\left\{(\delta_n(x,t))^\alpha + t^r\omega_r(f,h) + t^r(\delta_n(x,t))^r\omega_r(f,h)\right\}. \tag{26}$$

Let $\delta \in (0,\frac{1}{8r})$. We consider $n \in \mathbf{N}$, such that

$$\delta_n(x,t) \le \delta \le 2\delta_n(x,t).$$

Thus

$$|\Delta_t^r(f,x)| \le C\left\{\delta^\alpha + h^r\delta^{-r}\omega_r(f,\delta)\right\}.$$

Taking supremum we obtain

$$\omega_r(f,h) \le C\{\delta^\alpha + h^r\delta^{-r}\omega_r(f,\delta)\}$$

and the condition (ii) by [1].

Theorem 5. *Let $f \in L_\infty, r \in \mathbf{N}, \alpha \in (0,r), \nu \in (-1,\infty)$. The following conditions are equivalent:*

(i) *There exists a constant C independent on x and n such that*

$$|M_{n,\nu}^{(r)}(f,x)| \le C\left(\min\{n,\sqrt{\frac{n}{x}}\}\right)^{r-\alpha},$$

(ii) $\omega_r(f,h) = O(h^\alpha)$.

Proof.

(ii)$\Rightarrow$ (i)

Let $x,\ n \in (0,\infty)$ and $g \in D^r_\infty$. By estimations similar to once used in the proof of Theorem 4 and by Lemma 4 we obtain

$$\begin{aligned}
|M^{(r)}_{n,\nu}(f,x)| &\leq |M^{(r)}_{n,\nu}(f-g,x)| + |M^{(r)}_{n,\nu}(g,x)| \\
&\leq C[\min\{n^r, n^{\frac{r}{2}}x^{-\frac{r}{2}}\} \parallel f-g \parallel_{L_\infty} + \parallel g^r \parallel_{L_\infty}] \\
&\leq Cn^{\frac{r}{2}}(x+\frac{1}{n})^{-\frac{r}{2}}[\parallel f-g \parallel_{L_\infty} + n^{-\frac{r}{2}}(x+\frac{1}{n})^{\frac{r}{2}} \parallel g^r \parallel_{L_\infty}] \\
&\leq Cn^{\frac{r}{2}}(x+\frac{1}{n})^{-\frac{r}{2}} K_r(f, n^{-\frac{r}{2}}(x+\frac{1}{n})^{\frac{r}{2}}).
\end{aligned}$$

Taking infimum for $g \in D^r_\infty$ we get

$$|M^{(r)}_{n,\nu}(f,x)| \leq Cn^{\frac{r}{2}}(x+\frac{1}{n})^{-\frac{r}{2}}\omega_r(f, n^{-\frac{r}{2}}(x+\frac{1}{n})^{\frac{r}{2}}). \tag{27}$$

And finally the formula (27) and assumption (ii) give us the thesis.

(i)⇒ (ii)

Using Theorem 1 we have

$$\begin{aligned}
\Delta^r_t(f,x) &\leq |\sum_{j=0}^{r} \begin{pmatrix} r \\ j \end{pmatrix} (-1)^j \{M_{\frac{nm}{n+m},\nu,r}(f, x+(j-\frac{r}{2})t) - f(x+(j-\frac{r}{2})t)| \\
&+ |\sum_{i=0}^{r-1} a_i(n)\Delta^r_t(M_{m,\nu}M_{n_i,\nu}(f,.),x)| = I_1 + I_2.
\end{aligned}$$

From Theorem 3 we obtain

$$\begin{aligned}
I_1 &\leq \sum_{j=0}^{r} \begin{pmatrix} r \\ j \end{pmatrix} C\omega_r(f, \sqrt{[x+(j-\frac{r}{2})t+n^{-1}]n^{-1}} \\
&\leq 4^r C\omega_r(f, \delta_n(x,t)).
\end{aligned}$$

Using Lemmas 3 and 4 by steps similar to the proof of Theorem 4 we get

$$\begin{aligned}
I_2 &\leq \sum_{i=0}^{r-1} |a_i(n)| \int_{-\frac{t}{2}}^{\frac{t}{2}} \cdots \int_{-\frac{t}{2}}^{\frac{t}{2}} |M^{(r)}_{m,\nu}\left(M_{n_i,\nu}(f,.), x+\sum_{j=1}^{r} u_j\right) \mathrm{d}u_1 \cdots \mathrm{d}u_r \\
&\leq \sum_{i=0}^{r-1} |a_i(n)| \min\{\parallel M^{(r)}_{n_i,\nu}(f,.) \parallel_{L_\infty} t^r, \\
&\int_{-\frac{t}{2}}^{\frac{t}{2}} \cdots \int_{-\frac{t}{2}}^{\frac{t}{2}} \left(x+\sum_{j=1}^{r} u_j\right)^{\frac{\alpha-r}{2}} \parallel x^{\frac{r-\alpha}{2}} M^{(r)}_{n_i,\nu}(f,x) \parallel_{L_\infty} \mathrm{d}u_1 \cdots \mathrm{d}u_r\}.
\end{aligned}$$

Consequently, we obtain

$$\begin{aligned}
I_2 &\leq \sum_{i=0}^{r-1} |a_i(n)| \min\{C(An)^{r-\alpha}t^r, C(An)^{\frac{r-\alpha}{2}} C^{\frac{r-\alpha}{2}} \left(x+\frac{rt}{2}\right)^{\frac{\alpha-r}{2}} t^r\} \\
&\leq Ct^r(\delta_n(x,t))^{\alpha-r}.
\end{aligned}$$

If $\delta \in (0, \frac{1}{8r})$, let us take $n \in \mathbb{N}$, that

$$\delta_n(x,t) \le \delta < 2\delta_n(x,t).$$

Then

$$\Delta_t^r(f,x) \le 2^r C_3 \left(\omega_r(f,\delta) + h^r \delta^{\alpha - r}\right),$$

hence

$$\omega_r(f,h) \le 2^r C \left(\omega_r(f,\delta) + h^r \delta^{\alpha - r}\right).$$

And finally using the results from [1] we obtain (ii).

4 Conclusions

In the paper we consider the generalized family of operators ($\nu \in (-1,\infty)$). Thanks to this approach we are able not only to prove the interesting results which can not be formulated for $\nu = 0$, but also simplify the already existing proof by reformulating the known theorems as special cases of the generalized versions. Further results involving other types of operators are in preparation.

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