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A NEW BLO ESTIMATE FOR MAXIMAL SINGULAR INTEGRAL OPERATORS

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Abstract. In this paper, we extend Hu and Zhang's results in [2] to different case.

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1 Introduction

We will work on $\mathbf{R}^n$, $n \geq 2$. Let Ω be homogeneous of degree zero, integrable on the unit sphere S^{n-1} and have mean value zero. Define the singular integral operator T by

$$Tf(x) = p.v.\int_{R^n} \frac{\Omega(x-y)}{|x-y|^n} f(y)\mathrm{d}y \tag{1.1}$$

and the corresponding maximal operator T^* by

$$T^*f(x) = \sup_{0<\varepsilon<N<\infty} |T_{\varepsilon,N}f(x)|, \tag{1.2}$$

where $T_{\varepsilon,N}f(x)$ is the truncated operator defined by

$$T_{\varepsilon,N}f(x) = \int_{\varepsilon<|x-y|\leq N} \frac{\Omega(x-y)}{|x-y|^n} f(y)\mathrm{d}y. \tag{1.3}$$

Definition 1. The space $\mathrm{BLO}(R^n)$ consists of all $f \in L^1_{\mathrm{Loc}}(\mathbf{R}^n)$ such that

$$\|f\|_{\mathrm{BLO}(R^n)} = \sup_B (m_B(f) - \inf_{x\in B} f(x)) < \infty,$$

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where the supremum is taken over all balls B and $m_B(f)$ denotes the mean value of f on the ball B, that is, $m_B(f) = \frac{1}{|B|}\int_B f(x)\mathrm{d}x$.

Definition 2. Let $\Omega \in L^1(S^{n-1})$, define the L^1 modulus of continuity of Ω as

$$\omega(\delta) = \sup_{|\rho|\leq\delta}\int_{S^{n-1}}|\Omega(\rho x)-\Omega(x)|\mathrm{d}\sigma(x),$$

where $|\rho|$ denotes the distance of ρ from the identity rotation, and the supremum is taken over all rotations on the unit sphere with $|\rho|\leq\delta$.

Definition 3. As usual, a function $A:[0,\infty)\to[0,\infty)$ is a Young function if it is continuous, conex and increasing satisfying $A(0)=0$ and $A(t)\to\infty$ as $t\to\infty$. We define the A-average of a function f over a ball B by means of the following Luxemburg norm

$$\|f\|_{A,B} = \inf\{\lambda>0 : \frac{1}{|B|}\int_B A\left(\frac{|f(y)|}{\lambda}\right)\mathrm{d}y \leq 1\}. \tag{1.4}$$

The following generalized Hölder's inequality holds:

$$\frac{1}{|B|}\int_B |f(y)g(y)|\mathrm{d}y \leq \|f\|_{A,B}\|g\|_{A_1,B}, \tag{1.5}$$

where A_1 is the complementary function associated to A (see[4][5]).

Definition 4. For a suitable Young function A and its complementary function A_1, we say f satisfies A_1^q-condition if it satisfies

$$\frac{1}{|B|}\int_B A_1\left(\frac{|f(y)-m_B(f)|^q}{C}\right)\mathrm{d}y \leq C_1,$$

where $q\geq 1$, C and C_1 are positive constants.

For a Young function $A(t)=t\log(2+t)$, its complementary function $A_1(t)\approx \exp t$, Hu Guoen and Zhang Qihui[2] proved the following theorem:

Theorem A. Let T^* be the maximal singular integrable operator defined by (1.2), Ω be homogeneous of degree zero, integrable on the unit sphere S^{n-1} and have mean value zero. Suppose that for some $q>2$, $\Omega\in L(\log L)^q(S^{n-1})$, namely,

$$\int_{S^{n-1}}|\Omega|\log^q(2+|\Omega|)\mathrm{d}\sigma(x)<\infty,$$

and the L^1 modulus of continuity of Ω satisfies

$$\int_0^1 \omega(\delta)\log(2+\frac{1}{\delta})\frac{\mathrm{d}\delta}{\delta}<\infty.$$

Then for any $f\in$BMO$(\mathbf{R}^n)$, $T^*f(x)$ is either infinite everywhere or finite almost everywhere. More precise, if $f\in$BMO$(\mathbf{R}^n)$ such that $T^*f(x_0)<\infty$ for some $x_0\in\mathbf{R}^n$, then $T^*f(x)$ is finite almost everywhere, and

$$\|T^*f\|_{\mathrm{BLO}(R^n)} \leq C\|f\|_{\mathrm{BMO}(R^n)}.$$

In this paper, we consider the general case and $q>1$. Our main result is stated as follows.

Theorem. *Let $A(t)$ be a Young function and $A_1(t)$ be its complementary function. Suppose that $\int_{S^{n-1}} A(|\Omega(x)|)\mathrm{d}\sigma(x)<\infty$ and the L^1 modulus of continuity of Ω satisfies*

$$\int_0^1 \omega(\delta)\log^p(2+\frac{1}{\delta})\frac{\mathrm{d}\delta}{\delta}<\infty.$$

If $f\in \mathrm{BMO}(\mathbf{R}^n)$ and f satisfies A_1^q-condition such that $T^*f(x_0)<\infty$ for some $x_0\in\mathbf{R}^n$, then $T^*f(x)$ is finite almost everywhere, and

$$\|T^*f\|_{\mathrm{BLO}(R^n)}\le C\|f\|_{\mathrm{BMO}(R^n)},$$

where $p,q>1$ and $\frac{1}{p}+\frac{1}{q}=1$.

Remark 1. et us compare the above theorem with Theorem A. We consider the case where $q>1$ and the pair $(A(t),A_1(t))$ is a general complementary pair of Young functions. In Theorem A, the power $q>2$ and $(A(t),A_1(t))$ is a special pair of Young complement. But the assumption on $\omega(t)$ in our theorem is a little bit stronger than that of Theorem A. The following are two examples pairs of Young complements:

Example 1. $A(t)=t(1+\ln^+t)^\alpha$, $\alpha>0$. The complement of $A(t)$ is $A_1(t)\approx e^{t^{1/\alpha}}$.

Example 2. $A(t)=t\ln\ln(100+t)$. The complement of $A(t)$ is $A_1(t)\approx e^{e^t}$.

2 Proof of Theorem

We begin with some preliminary lemmas.

Lemma 1. *Let $p,q>1$ and $\frac{1}{p}+\frac{1}{q}=1$, $b,m>0$. Then we have*

$$b\le m^p+m^{-q}b^q.$$

Lemma 2[3][5]**.** *Let $A(t)$ be a Young function and $A_1(t)$ be its complementary function. Then for any $0\le t_1,t_2<\infty$,*

$$t_1t_2\le A(t_1)+A_1(t_2).$$

Lemma 3. *Suppose Ω is homogeneous of degree zero, and satisfies $\int_{S^{n-1}}A(|\Omega(x)|)\mathrm{d}\sigma(x)<\infty$. Then there is a positive constant C such that for any $f\in\mathrm{BMO}(\mathbf{R}^n)$, f satisfies A_1^q-condition and $r>0$,*

$$\sup_{R\ge 2r}\int_{R-r\le|x-y|<R}\frac{|\Omega(x-y)|}{|x-y|^n}|f(y)-m_{B_{(x,r)}}(f)|\mathrm{d}y\le C\|f\|_{\mathrm{BMO}}.$$

Proof. Without loss of generality, we may assume that $\|f\|_{BMO}=1$. For each fixed $R\ge 2r$, write

$$\begin{aligned}
&\int_{R-r\le|x-y|<R}\frac{|\Omega(x-y)|}{|x-y|^n}|f(y)-m_{B_{(x,r)}}(f)|\mathrm{d}y\\
&\qquad\le\int_{R-r\le|x-y|<R}\frac{|\Omega(x-y)|}{|x-y|^n}|f(y)-m_{B_{(x,R)}}(f)|\mathrm{d}y\\
&\qquad\quad+|m_{B_{(x,R)}}(f)-m_{B_{(x,r)}}(f)|\int_{R-r\le|x-y|<R}\frac{|\Omega(x-y)|}{|x-y|^n}\mathrm{d}y\\
&\qquad=B_1+B_2.
\end{aligned}$$

Recall that $|m_{B_{(x,R)}}(f)-m_{B_{(x,r)}}(f)|\le C\log\frac{R}{r}$, where C is a positive constant. Thus,

$$B_2\le C\frac{1}{(R-r)^n}\int_{R-r\le|x-y|<R}|\Omega(x-y)|\mathrm{d}y\log\frac{R}{r}\le C\frac{R^{n-1}r}{(R-r)^n}\log\frac{R}{r}\le C.$$

To estimate B_1, Lemma 2 gives that for $R\ge 2r$,

$$\begin{aligned}
B_1\quad&\le\frac{C}{(R-r)^n}\int_{|x-y|<R}A_1\left(\frac{|f(y)-m_{B_{(x,R)}}(f)|}{C_1}\right)\mathrm{d}y\\
&\quad+\frac{C}{(R-r)^n}\int_{|x-y|<R}A(|\Omega(x-y)|)\mathrm{d}y\\
&\le C\frac{(R)^n}{(R-r)^n}\le C.
\end{aligned}$$

This completes the proof of the lemma.

Proof of Theorem. It suffices to show that there is a positive constant C such that for any ball B,

$$\frac{1}{|B|}\int_B T^*f(x)\mathrm{d}x\le C\|f\|_{\mathrm{BMO}(R^n)}+\inf_{y\in B}T^*f(y).\tag{2.6}$$

We now prove (1.6). Let $f\in$BMO($\mathbf{R}^n$), without loss of generality, we may assume that $\|f\|_{BMO(R^n)}=1$. For each fixed ball $B=B(x_0,r)$, set

$f_1(x)=(f(x)-m_B(f))\chi_{6B}(x)$, $f_2(x)=(f(x)-m_B(f))\chi_{R^n\setminus 6B}(x)$.

The vanishing moment of Ω implies the following pointwise inequality

$$T^*f(x)\le T^*f_1(x)+T^*f_2(x).$$

The $L^2(\mathbf{R}^n)$ boundedness of T^* via the Hölder's inequality tells us that

$$\begin{aligned}
\frac{1}{|B|}\int_B T^*f_1(x)\mathrm{d}x\quad&\le C\left(\frac{1}{|B|}\int(T^*f_1(x))^2\mathrm{d}x\right)^{\frac12}\\
&\le C\left(\frac{1}{|B|}\int_B|f(x)-m_B(f)|^2\mathrm{d}x\right)^{\frac12}\le C.
\end{aligned}$$

It remains to deal with $T^*f_2(x)$. Set

$$T_{\varepsilon,\infty}f(x)=\int_{|x-y|>\varepsilon}\frac{\Omega(x-y)}{|x-y|^n}f(y)\mathrm{d}y.$$

Note that for $y\in B$,

$$\begin{aligned}T^*f_2(y) &= \sup_{0<\varepsilon<N<\infty}|T_{\varepsilon,N}f_2(y)|\\ &\le \sup_{\substack{\varepsilon\le r\\ 0<\varepsilon<N<\infty}}|T_{\varepsilon,N}f_2(y)|+\sup_{\substack{\varepsilon>r\\ 0<\varepsilon<N<\infty}}|T_{\varepsilon,N}f_2(y)|\end{aligned}$$

and

$$\sup_{\substack{\varepsilon\le r\\ 0<\varepsilon<N<\infty}}|T_{\varepsilon,N}f_2(y)|=\max\left\{\sup_{0<\varepsilon\le r<N<\infty}|T_{\varepsilon,N}f_2(y)|,\sup_{0<\varepsilon<N\le r}|T_{\varepsilon,N}f_2(y)|\right\}.$$

An easy computation shows that for $y\in B$,

$$\begin{aligned}\sup_{0<\varepsilon\le r<N<\infty}|T_{\varepsilon,N}f_2(y)| &\le \sup_{0<\varepsilon\le r<N<\infty}\left|\int_{\varepsilon<|x-y|\le r}\frac{\Omega(y-z)}{|y-z|^n}f_2(z)\mathrm{d}z\right|\\ &\quad+\sup_{0<\varepsilon\le r<N<\infty}\left|\int_{r<|y-z|\le N}\frac{\Omega(y-z)}{|y-z|^n}f_2(z)\mathrm{d}z\right|\\ &=\sup_{0<N<\infty}|T_{r,N}f_2(y)|,\end{aligned}$$

and if $0<\varepsilon<N\le r$, $T_{\varepsilon,N}f_2(y)=0$. Therefore for any $y\in B$,

$$\sup_{\substack{\varepsilon\le r\\ 0<\varepsilon<N<\infty}}|T_{\varepsilon,N}f_2(y)|\le\sup_{0<N<\infty}|T_{r,N}f_2(y)|.$$

Then,

$$\begin{aligned}T^*f_2(y) &\le \sup_{r\le\varepsilon<N<\infty}|T_{\varepsilon,N}f_2(y)|\\ &\le \sup_{r\le\varepsilon<N<\infty}|T_{\varepsilon,N}f(y)|+\sup_{r\le\varepsilon<N<\infty}|T_{\varepsilon,N}f_1(y)|\\ &\quad+\sup_{r\le\varepsilon<N<\infty}\left|\int_{\varepsilon<|y-z|\le N}\frac{\Omega(y-z)}{|y-z|^n}m_B(f)\mathrm{d}z\right|\\ &\le T^*f(y)+\sup_{r\le\varepsilon<N<\infty}|T_{\varepsilon,N}f_1(y)|\\ &\le T^*f(y)+2\sup_{\varepsilon\ge r}|T_{\varepsilon,\infty}f_1(y)|.\end{aligned}$$

For each ε with $r \le \varepsilon < \infty$ and $y \in B$, an application of Lemma 2 and the increasing of Young function shows that

$$
\begin{aligned}
|T_{\varepsilon,\infty} f_1(y)| &\le \int_{r\le|y-z|<8r} \frac{|\Omega(y-z)|}{|y-z|^n} |f(z) - m_{B_{(y,8r)}}(f)| \mathrm{d}z \\
&\quad + |m_{B_{(y,8r)}}(f) - m_B(f)| \int_{r\le|y-z|<8r} \frac{|\Omega(y-z)|}{|y-z|^n} dz \\
&\le \frac{C}{r^n} \int_{|y-z|<8r} A(|\Omega(y-z)|) \mathrm{d}z \\
&\quad + \frac{C}{r^n} \int_{|y-z|<8r} A_1\left(\frac{|f(z) - m_{B_{(y,R)}}(f)|}{C_1}\right) \mathrm{d}z \\
&\le \frac{C}{r^n} \int_{|y-z|<8r} \max\left\{A_1\left(\frac{|f(y) - m_{B_{(y,R)}}(f)|^q}{C_1}\right), A_1\left(\frac{1}{C_1}\right)\right\} \mathrm{d}z \\
&\quad + C \le C.
\end{aligned}
$$

We thus obtain that for $y \in B$,

$$
T^* f_2(y) \le T^* f(y) + C. \tag{2.7}
$$

The proof of the inequality (1.6) is now reduced to proving that for any $x, y \in B$,

$$
|T^* f_2(x) - T^* f_2(y)| \le C. \tag{2.8}
$$

To prove (1.8), note that

$$
\begin{aligned}
\sup_{\varepsilon>0} |T_{\varepsilon,\infty} f_2(x) - T_{\varepsilon,\infty} f_2(y)| &\le \sup_{\varepsilon>0} \int_{|x-z|\ge\varepsilon} \left|\frac{\Omega(x-z)}{|x-z|^n} - \frac{\Omega(y-z)}{|y-z|^n}\right| |f_2(z)| \mathrm{d}z \\
&\quad + \sup_{\varepsilon>0} \int_{|x-z|\le\varepsilon, |y-z|>\varepsilon} \frac{|\Omega(y-z)|}{|y-z|^n} |f_2(z)| \mathrm{d}z \\
&\quad + \sup_{\varepsilon>0} \int_{|x-z|>\varepsilon, |y-z|\le\varepsilon} \frac{|\Omega(y-z)|}{|y-z|^n} |f_2(z)| \mathrm{d}z \\
&= D_1 + D_2 + D_3
\end{aligned}
$$

It follows from Lemma 3 that for $x, y \in B$,

$$\begin{aligned} D_3 &\le \sup_{\varepsilon \ge 5r} \int_{\varepsilon - 2r < |y-z| \le \varepsilon} \frac{|\Omega(y-z)|}{|y-z|^n} |f(z) - m_B(f)| \mathrm{d}z \\ &\le \sup_{\varepsilon \ge 4r} \int_{\varepsilon - 2r < |y-z| \le \varepsilon} \frac{|\Omega(y-z)|}{|y-z|^n} |f(z) - m_{B_{(y,2r)}}(f)| \mathrm{d}z \\ &\quad + |m_{B_{(y,2r)}}(f) - m_B(f)| \sup_{\varepsilon \ge 4r} \int_{\varepsilon - 2r < |y-z| \le \varepsilon} \frac{|\Omega(y-z)|}{|y-z|^n} \mathrm{d}z \\ &\le C. \end{aligned}$$

Similarly, for any $x, y \in B$,

$$\begin{aligned} D_2 &\le \sup_{\varepsilon \ge 5r} \int_{\varepsilon < |y-z| \le \varepsilon + 2r} \frac{|\Omega(y-z)|}{|y-z|^n} |f(z) - m_B(f)| \mathrm{d}z \\ &= \sup_{\varepsilon \ge 7r} \int_{\varepsilon - 2r < |y-z| \le \varepsilon} \frac{|\Omega(y-z)|}{|y-z|^n} |f(z) - m_B(f)| \mathrm{d}z \le C. \end{aligned}$$

Observing that for any $x, y \in B$, we can write

$$\begin{aligned} D_1 &\le \int_{|x-z| \ge 5r} \left| \frac{\Omega(x-z)}{|x-z|^n} - \frac{\Omega(y-z)}{|y-z|^n} \right| |f_2(z)| \mathrm{d}z \\ &\le \int_{|x-z| \ge 2r} \frac{|\Omega(x-z) - \Omega(y-z)|}{|x-z|^n} |f_2(z)| \mathrm{d}z \\ &\quad + C \int_{|x-z| \ge 2r} \frac{|x-y|}{|x-z|^{n+1}} |\Omega(x-z) f_2(z)| \mathrm{d}z \\ &= D_{11} + D_{12}. \end{aligned}$$

The term D_{12} is easy to deal with. In fact,

$$\begin{aligned} D_{12} &\le Cr \sum_{k=1}^{\infty} \int_{2^k r \le |x-z| < 2^{k+1} r} \frac{|\Omega(x-z)|}{|x-z|^{n+1}} |f(z) - m_{B_{(x,2^{k+1}r)}}(f)| \mathrm{d}z \\ &\quad + Cr \sum_{k=1}^{\infty} |m_{B_{(x,2^{k+1}r)}}(f) - m_B(f)| \int_{2^k r \le |x-z| < 2^{k+1} r} \frac{|\Omega(x-z)|}{|x-z|^{n+1}} \mathrm{d}z \\ &\le C. \end{aligned}$$

On the other hand, invoking Lemma 1, a straightforward computation gives that for any $x,y\in B$ and some $q>1$,

$$\begin{aligned}
D_{11} &\le \sum_{k=1}^{\infty}\int_{2^kr\le|x-z|<2^{k+1}r}\frac{|\Omega(x-z)-\Omega(y-z)|}{|x-z|^n}|f(z)-m_{B_{(x,2^{k+1}r)}}(f)|\mathrm{d}z\\
&\quad+\sum_{k=1}^{\infty}|m_{B_{(x,2^{k+1}r)}}(f)-m_B(f)|\int_{2^kr\le|x-z|<2^{k+1}r}\frac{|\Omega(x-z)-\Omega(y-z)|}{|x-z|^n}\mathrm{d}z\\
&\le C\sum_{k=1}^{\infty}k^q\int_{2^kr\le|x-z|<2^{k+1}r}\frac{|\Omega(x-z)-\Omega(y-z)|}{|x-z|^n}\mathrm{d}z\\
&\quad+\sum_{k=1}^{\infty}k^{-q}\int_{2^kr\le|x-z|<2^{k+1}r}\frac{|\Omega(x-z)|}{|x-z|^n}|f(z)-m_{B_{(x,2^{k+1}r)}}(f)|^q\mathrm{d}z\\
&\quad+\sum_{k=1}^{\infty}k^{-q}\int_{2^kr\le|x-z|<2^{k+1}r}\frac{|\Omega(y-z)|}{|x-z|^n}|f(z)-m_{B_{(x,2^{k+1}r)}}(f)|^q\mathrm{d}z\\
&=E+F+G.
\end{aligned}$$

Note that by the same argument as used in [1], there is a positive constant D depending only on n such that for any $x,y\in B$,

$$\int_{2^{k+1}nB\backslash 2^knB}|\Omega(x-z)-\Omega(y-z)|\mathrm{d}z\le C|2^kB|\int_{D2^{-k-1}<\delta<D2^{-k}}\omega(\delta)\frac{\mathrm{d}\delta}{\delta}.$$

This in turn implies that

$$\begin{aligned}
E &\le C\sum_{k=1}^{\infty}k^p\int_{D2^{-k-1}<\delta<D2^{-k}}\omega(\delta)\frac{\mathrm{d}\delta}{\delta}\\
&\le C\sum_{k=1}^{\infty}\int_{D2^{-k-1}<\delta<D2^{-k}}\omega(\delta)\log^p(2+\delta^{-1})\frac{\mathrm{d}\delta}{\delta}\le C.
\end{aligned}$$

Applying the generalized Hölder's inequality (1.5) we deduce that for $x\in B$ and $q>1$,

$$\begin{aligned}
F &\le \sum_{k=1}^{\infty}\frac{k^{-q}}{(2^kr)^n}\int_{|x-z|<2^{k+1}r}|\Omega(x-z)||f(z)-m_{B_{(x,2^{k+1}r)}}(f)|^q\mathrm{d}z\\
&\le C\sum_{k=1}^{\infty}k^{-q}\|\Omega(x-\cdot)\|_{A,2^{k+1}B}\|(f(z)-m_{B_{(x,2^{k+1}r)}}(f))^q\|_{A_1,2^{k+1}B}\le C.
\end{aligned}$$

Similarly, we have $G\le C$ and then $D_1\le C$. Combining the estimates for D_1, D_2 and D_3 yields the inequality (1.6), and finishes the proof of Theorem.

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