Anal. Theory Appl.
Vol. 27, No. 3 (2011), 239–250
DOI10.1007/s10496-011-0239-8

# WEIGHTED APPROXIMATION OF $r$-MONOTONE FUNCTIONS ON THE REAL LINE BY BERNSTEIN OPERATORS

Laiyi Zhu
(*Renmin University of China, China*)
Xiaojie Zhu
(*Peking University, China*)
Xing Liu
(*Renmin University of China, China*)

Received Feb. 22, 2011



**Abstract.** In this paper, we give error estimates for the weighted approximation of $r$-monotone functions on the real line with Freud weights by Bernstein-type operators.

**Key words:** *Freud weight, r-monotone function, Bernstein-type operator*

**AMS (2010) subject classification:** 41A10, 41A25, 41A36

## 1 Introduction

For an integer $r \geq 0$, let $C^r(S)$ denote the set of all $r$-times continuously differentiable functions on $S$, where $C^0(S) = C(S)$ is the usual set of all continuous functions on $S$.

Let

$$w(x) = e^{-Q(x)}, \qquad x \in (-\infty, +\infty)$$

be a Freud weight, with the continuous function $Q(x)$ satisfying the following conditions:

(a) $Q \in C^2(0,\infty)$ is a positive even function;

(b) $\lim\limits_{x\to\infty} x\frac{Q''(x)}{Q'(x)} = \gamma > 0$;

*Supported by the National Natural Science Foundation, 10601065.

(c) if $\gamma = 1$ or 3, then $Q''$ is nondecreasing. (see [2, Definition 11.3.1, p.184]).

Evidently, we have the following proposition (see [7, Lemma 1]).

*Proposition A.* Let the continuous function $Q(x)$ satisfying the conditions (a),(b),(c). Then $\lim_{x\to\infty} Q'(x) = \infty$, and there exist $t_0 > 0$ and $A > 1$ such that

$$\begin{cases} Q'(x) & > & 0, \\ Q''(x) & > & 0, \\ Q'(2x) & \le & AQ'(x) \end{cases}$$

hold for $x > t_0$.

For a Freud weight $w(x)$, denote by $C_w$ the space of all $f \in C(R)$ such that $\lim_{|x|\to\infty} (wf)(x) = 0$ and equipped with the norm $\|wf\|_{C_w} = \sup_{x\in R} |(wf)(x)|$. We also put

$$\|wf\|_{[c,d]} = \sup_{x\in[c,d]} |(wf)(x)|.$$

For $f \in C_w$ the weighted modulus of smoothness is

$$\begin{aligned} \omega_2(f,t)_w &= \sup_{0<h\le t} \|w\Delta_h^2 f\|_{[-h^*,h^*]} + \inf_{\ell\in\mathcal{P}_1} \|w(f-\ell)\|_{[t^*,\infty)} \\ &\quad + \inf_{\ell\in\mathcal{P}_1} \|w(f-\ell)\|_{(-\infty,-t^*]}, \end{aligned} \tag{1.1}$$

where $h^*$ and $t^*$ are defined by $hQ'(h^*) = 1$ and $tQ'(t^*) = 1$ respectively (see [2, Definition 11.2.2, p.182]), $\mathcal{P}_n, n \in \mathbf{N}$, is the set of algebraic polynomials of degree at most n, and

$$\Delta_h^r f(x) = \sum_{i=0}^{r} (-1)^i \binom{r}{i} f\left(x + \frac{rh}{2} - ih\right)$$

is the $r$-th symmetric difference of $f$ (see [2, p. 7]).

Let the sequence of positive real numbers $\{\lambda_n\}$ be monotone increasing and defined by

$$\lambda_n Q'(\lambda_n) = \sqrt{n}, \qquad n > n_0, \tag{1.2}$$

with $n_0$ sufficiently large (see [2, p. 7]).It follows from $(1.2)$ that $\lim_{n\to\infty} \frac{\lambda_n}{\sqrt{n}} = 0$.

In the following $c, c_1, c_2$ denote positive constants which may assume different values in different formulas.

For every $f \in C_w$ let

$$B_n(f,x) = \sum_{k=0}^{n} p_{n,k}(x) f(x_k) \tag{1.3}$$

with

$$p_{n,k}(x)=\frac{1}{2n}\binom{n}{k}\left(1+\frac{x}{2\lambda_n}\right)^k\left(1-\frac{x}{2\lambda_n}\right)^{n-k}, \qquad x_k=x_{k,n}=2\lambda_n\frac{2k-n}{n}. \tag{1.4}$$

In [7], B. D. Vecchia et al. considered the Bernstein-type operator

$$B_n^*(f,x)=\begin{cases} B_n(f,x), & \text{if} \quad |x|\le\lambda_n,\\ B_n(f,\lambda_n)+B_n'(f,\lambda_n)(x-\lambda_n), & \text{if} \quad x\ge\lambda_n,\\ B_n(f,-\lambda_n)+B_n'(f,-\lambda_n)(x+\lambda_n), & \text{if} \quad x\le-\lambda_n. \end{cases} \tag{1.5}$$

and obtained the following error estimate.

*Theorem VMS.* If $f\in C_w$, then

$$\|w[f-B_n^*(f)]\|\le c\omega_2\left(f,\frac{\lambda_n}{\sqrt{n}}\right)_w. \tag{1.6}$$

A function $f:\mathbf{R}\to\mathbf{R}$ is said to be $r$-monotone if the $r$-th order divided difference

$$[x_0,x_1,\cdots,x_r,f]=\sum_{i=0}^{r}\frac{f(x_i)}{\prod\limits_{j=0,j\ne i}^{r}(x_i-x_j)}\ge 0 \tag{1.7}$$

for any collection of $r+1$ distinct points $x_0,x_1,\cdots,x_r$. It is well-known (see [6, p. 238]) that the usual monotone non-decreasing and convex functions are 1- and 2-monotone respectively, and that if $f$ is $r$-monotone, then $f^{(r-2)}$ exists and is convex and $f^{(r-1)}$ exists almost everywhere. In particular, if $f\in C^{r-1}(\mathbf{R})$ is $r$-monotone, then $f^{(r-1)}$ is non-decreasing and the divided difference $[x_0,x_1,\cdots,x_r,f]$ is a non-decreasing function of each of its arguments.

It is often important for mathematical objects which approximate a given function to preserve some of its properties such as monotonicity, convexity, etc. This direction in Approximation Theory is called Shape Preserving Approximation (see [3]). In this paper,we consider the following Bernstein-type operators.

For an integer $r\ge 2$ and $f\in C_w$, we define

$$B_{n,r}(f,x)=\begin{cases} B_n(f,x), & \text{if} \quad |x|\le\lambda_n,\\ \sum\limits_{i=0}^{r-1}\frac{B_n^{(i)}(f,\lambda_n)}{i!}(x-\lambda_n)^i, & \text{if} \quad x\ge\lambda_n,\\ \sum\limits_{i=0}^{r-1}\frac{B_n^{(i)}(f,-\lambda_n)}{i!}(x+\lambda_n)^i, & \text{if} \quad x\le-\lambda_n. \end{cases} \tag{1.8}$$

and

$$B_{n,r}^*(f,x)=\frac{\sqrt{n}}{2\lambda_n}\int_{x-\frac{\lambda_n}{\sqrt{n}}}^{x+\frac{\lambda_n}{\sqrt{n}}}B_{n,r}(f,t)\mathrm{d}t. \tag{1.9}$$

By (1.5) and (1.8), we know that the operator $B_n^*$ is the operator $B_{n,2}$.

It is well-known that Bernstein operators preserve $r$-monotonicity on closed intervals (see [3]). Thus if $f \in C(\mathbf{R})$ is $r$-monotone then $B_n^*(f,x)$, $B_{n,r}(f,x)$ and $B_{n,r}^*(f,x)$ are also $r$-monotone and $B_n^*(f,x) \in C^1(\mathbf{R})$, $B_{n,r}(f,x) \in C^{r-1}(\mathbf{R})$ and $B_{n,r}^*(f,x) \in C^r(\mathbf{R})$ respectively.

*Remark* 1. Note that $B_{n,r}$ and $B_{n,r}^*$ are linear operators, which reproduces linear functions $\ell$, i.e., $B_{n,r}(\ell,x) \equiv \ell(x), B_{n,r}^*(\ell,x) \equiv \ell(x)$.

Our main results are the following.

**Theorem 1.** *Let the integer $r \geq 2$. If $f \in C_w$ is r-monotone, then*

$$\|w[f - B_{n,r}(f)]\| \leq c\omega_2\left(f, \frac{\lambda_n}{\sqrt{n}}\right)_w. \tag{1.10}$$

**Theorem 2.** *Let the integer $r \geq 2$. If $f \in C_w$ is r-monotone, then $B_{n,r}^*(f,x) \in C_w$ is r-monotone, $B_{n,r}^*(f,x) \in C^r(R)$ and*

$$\|w[f - B_{n,r}^*(f)]\| \leq c\omega_2\left(f, \frac{\lambda_n}{\sqrt{n}}\right)_w. \tag{1.11}$$

*Remark* 2. In [4], O. Maizlish obtained the following result.

*Theorem M.* Let $f \in C_{w_\alpha}$ be $r$-monotone, with $w_\alpha = e^{-|x|^\alpha}, \alpha \geq 1$, and $r \geq 1$. Then, for any $\varepsilon > 0$, there exists an $r$-monotone function $g \in C^r(R)$ such that $\|w[f-g]\| < \varepsilon$, and $g^{(r)}$ is identically zero outside some finite interval.

It follows from (1.8) and (1.9) that

$$B_{n,r}^{*(r)}(f,x) \equiv 0, |x| \geq \lambda_n + \frac{\lambda_n}{\sqrt{n}}. \tag{1.12}$$

Thus Theorem 2 extends Theorem M in a sense.

## 2 Auxiliary Lemmas

The proof of Theorem 1 and Theorem 2 is based on several lemmas.

**Lemma 1.** *For $f \in C_w$, let $\ell_1(x)$ be the linear function which realizes the infinite in* (1.1) *with respect to f and for $t^* = \lambda_n - \frac{\lambda_n}{\sqrt{n}}$ or $t^* = -\lambda_n + \frac{\lambda_n}{\sqrt{n}}$*, i. e.,

$$\inf_{\ell \in \mathcal{P}_1} \|w(f-\ell)\|_{\left[\lambda_n - \frac{\lambda_n}{\sqrt{n}}, +\infty\right)} = \|w(f-\ell_1)\|_{\left[\lambda_n - \frac{\lambda_n}{\sqrt{n}}, +\infty\right)} \tag{2.1}$$

*or*

$$\inf_{\ell \in \mathcal{P}_1} \|w(f-\ell)\|_{\left(-\infty, -\lambda_n + \frac{\lambda_n}{\sqrt{n}}\right]} = \|w(f-\ell_1)\|_{\left(-\infty, -\lambda_n + \frac{\lambda_n}{\sqrt{n}}\right]}. \tag{2.1$'$}$$

*Then there exists $\xi_n \in I_n$ such that*

$$\sup_{x\in I_n} |B_n(f-\ell_1,x)| \le c\omega_2\left(f,\frac{\lambda_n}{\sqrt{n}}\right)_w e^{Q(\xi_n)}, \tag{2.2}$$

*where $I_n = [\lambda_n - \frac{\lambda_n}{\sqrt{n}}, \lambda_n]$ or $I_n = [-\lambda_n, -\lambda_n + \frac{\lambda_n}{\sqrt{n}}]$.*

*Proof.* It is sufficient to prove (2.2) in the case $I_n = [\lambda_n - \frac{\lambda_n}{\sqrt{n}}, \lambda_n]$ and $t^* = \lambda_n - \frac{\lambda_n}{\sqrt{n}}$.

Let $\xi_n \in I_n$ be the point such that

$$|B_n(f-\ell_1,\xi_n)| = \sup_{x\in I_n} |B_n(f-\ell_1,x)|.$$

By (1.6), we have

$$\begin{aligned} |B_n(f-\ell_1,\xi_n)| &\le [|B_n(f,\xi_n)-f(\xi_n)|w(\xi_n) + |f(\xi_n)-\ell_1(\xi_n)|w(\xi_n)]e^{Q(\xi_n)} \\ &\le \left[c\omega_2\left(f,\frac{\lambda_n}{\sqrt{n}}\right)_w + \inf_{\ell\in\mathcal{P}_1}\|w[f-\ell]\|_{\left[\lambda_n-\frac{\lambda_n}{\sqrt{n}},+\infty\right)}\right] e^{Q(\xi_n)}. \end{aligned} \tag{2.3}$$

It is clear from Proposition A and (1.2) that there exists only one $t_n \in \left(0, \frac{1}{Q'(t_0)}\right)$ such that

$$t_n Q'\left(\lambda_n - \frac{\lambda_n}{\sqrt{n}}\right) = 1 \tag{2.4}$$

and

$$\frac{\sqrt{n}}{\lambda_n} = Q'(\lambda_n) \le Q'\left[2\left(\lambda_n - \frac{\lambda_n}{\sqrt{n}}\right)\right] \le AQ'\left(\lambda_n - \frac{\lambda_n}{\sqrt{n}}\right) = \frac{A}{t_n}. \tag{2.5}$$

For the $K$-functional

$$K_2(f,t^2)_w = \inf_{g'\in AC_{loc}} [\|w(f-g)\| + t^2\|wg''\|], \tag{2.6}$$

We have the following equivalence relation:

$$c_1\omega_2(f,t)_w \le K_2(f,t^2)_w \le c_2\omega_2(f,t)_w \tag{2.7}$$

(cf. Theorem 11.2.3 in [2, p. 182]). Using (2.4)-(2.7),we have

$$\begin{aligned} \inf_{\ell\in\mathcal{P}_1}\|w(f-\ell)\|_{[\lambda_n-\frac{\lambda_n}{\sqrt{n}},+\infty)} &\le \omega_2(f,t_n)_w \\ &\le c\omega_2\left(f,\frac{\lambda_n}{\sqrt{n}}\right)_w. \end{aligned} \tag{2.8}$$

Combining this with (2.3), we obtain (2.2).

**Lemma 2.** *For an integer $r \geq 3$, let $f \in C_w$ be r-monotone. Then there exists $\eta_n \in \left[\lambda_n - \frac{\lambda_n}{\sqrt{n}}, \lambda_n + \frac{2\lambda_n}{\sqrt{n}}\right]$ such that*

$$|B_n^{(r-1)}(f,x)| \leq c \left(\frac{\sqrt{n}}{\lambda_n}\right)^{r-1} \omega_2\left(f, \frac{\lambda_n}{\sqrt{n}}\right)_w e^{Q(\eta_n)} \tag{2.9}$$

*holds true for $|x| \in \left[\lambda_n, \lambda_n + \dfrac{\lambda_n}{\sqrt{n}}\right]$.*

*Proof.* It is sufficient to prove (2.9) in the case $x \in [\lambda_n, \lambda_n + \dfrac{\lambda_n}{\sqrt{n}}]$. Let $\ell_1(x)$ and $\ell_2(x)$ be the linear functions respectively, such that

$$\inf_{\ell \in \mathcal{P}_1} \|w(f-\ell)\|_{\left[\lambda_n - \frac{\lambda_n}{\sqrt{n}}, +\infty\right)} = \|w(f-\ell_1)\|_{\left[\lambda_n - \frac{\lambda_n}{\sqrt{n}}, +\infty\right)},$$

$$\inf_{\ell \in \mathcal{P}_1} \|w(f-\ell)\|_{[\lambda_n, +\infty)} = \|w(f-\ell_2)\|_{[\lambda_n, +\infty)}.$$

Noting that

$$B_n^{(r-1)}(f,x) = B_n^{(r-1)}(f-\ell_1, x) = B_n^{(r-1)}(f-\ell_2, x)$$

holds true for $x \in \left[\lambda_n, \lambda_n + \dfrac{\lambda_n}{\sqrt{n}}\right]$ and $r \geq 3$.

For $x \in [\lambda_n, \lambda_n + \dfrac{\lambda_n}{\sqrt{n}}]$, if $B_n^{(r-1)}(f,x) \geq 0$, then using Petrov's result (see [5, Theorem 3.1]), we have

$$\begin{aligned}
|B_n^{(r-1)}(f,x)| &= |B_n^{(r-1)}(f-\ell_2, x)| \\
&\leq (r-1)! 2^{2r-3} \left(\lambda_n + \frac{2\lambda_n}{\sqrt{n}} - x\right)^{1-r} \|B_n(f-\ell_2)\|_{\left[x, \lambda_n + \frac{2\lambda_n}{\sqrt{n}}\right]} \\
&\leq c \left(\frac{\sqrt{n}}{\lambda_n}\right)^{r-1} \|B_n(f-\ell_2)\|_{\left[\lambda_n, \lambda_n + \frac{2\lambda_n}{\sqrt{n}}\right]} \\
&= c \left(\frac{\sqrt{n}}{\lambda_n}\right)^{r-1} |B_n(f-\ell_2, \eta_n)|
\end{aligned}$$

with suitable $\eta_n \in \left[\lambda_n, \lambda_n + \dfrac{2\lambda_n}{\sqrt{n}}\right]$.

Using the proof of Theorem VMS (see [7]), we have

$$\begin{aligned}
|B_n(f-\ell_2, \eta_n)| &\leq [|B_n(f,\eta_n) - f(\eta_n)| w(\eta_n) + |f(\eta_n) - \ell_2(\eta_n)| w(\eta_n)] e^{Q(\eta_n)} \\
&\leq c\omega_2\left(f, \frac{\lambda_n}{\sqrt{n}}\right)_w e^{Q(\eta_n)}.
\end{aligned} \tag{2.10}$$

If $B_n^{(r-1)}(f,x) \leq 0$, then $B_n^{(r-1)}(f-\ell_1, x) \leq 0$. Using Petrov's result again and the fact $B_n^{(r-1)}(f-$

$\ell_1,x)$ is non-decreasing, we obtain

$$\begin{aligned}|B_n^{(r-1)}(f,x)| &= |B_n^{(r-1)}(f-\ell_1,x)| \\ &\le |B_n^{(r-1)}(f-\ell_1,\lambda_n)| \\ &\le (r-1)!2^{2r-3}\left(\frac{\sqrt{n}}{\lambda_n}\right)^{r-1}\|B_n(f-\ell_1)\|_{I_n}.\end{aligned}$$

Thus Lemma 1 gives

$$|B_n^{(r-1)}(f,x)| \le c\left(\frac{\sqrt{n}}{\lambda_n}\right)^{r-1}\omega_2\left(f,\frac{\lambda_n}{\sqrt{n}}\right)_w e^{Q(\eta_n)} \tag{2.11}$$

with suitable $\eta_n \in I_n = \left[\lambda_n - \frac{\lambda_n}{\sqrt{n}}, \lambda_n\right]$.

The inequality (2.9) follows immediately from (2.10) and (2.11).

**Lemma 3.** *For an integer $r \ge 2$, let $f \in C_w$ be r-monotone. Then*

$$w(x)|B_{n,r}(f,x)-f(x)| \le c\omega_2\left(f,\frac{\lambda_n}{\sqrt{n}}\right)_w \tag{2.12}$$

*holds for $|x| \ge \lambda_n$.*

*Proof.* When $r=2$, the inequality (2.12) follows immediately from(1.6).

When $r \ge 3$, let $\ell(x)$ be the linear function such that

$$\inf_{\ell\in\mathcal{P}_1}\|w(f-\ell)\|_{[\lambda_n,\infty)} = \|w(f-\ell)\|_{[\lambda_n,\infty)}.$$

Since $B_{n,r}(f,x)$ reproduces linear functions, for $x \ge \lambda_n$, we have

$$\begin{aligned}& w(x)|B_{n,r}(f,x)-f(x)| \\ = \;& w(x)|B_{n,r}(f-\ell,x)-[f(x)-\ell(x)]| \\ \le \;& w(x)\sum_{i=0}^{r-1}\frac{|B_n^{(i)}(f-l,\lambda_n)|}{i!}(x-\lambda_n)^i + \omega_2\left(f,\frac{\lambda_n}{\sqrt{n}}\right)_w.\end{aligned} \tag{2.13}$$

It is clear from (1.6) that

$$\begin{aligned}& w(x)|B_n(f-\ell,\lambda_n)| \\ &\quad \le [w(\lambda_n)|B_n(f,\lambda_n)-f(\lambda_n)| + w(\lambda_n)|f(\lambda_n)-\ell(\lambda_n)|]e^{Q(\lambda_n)-Q(x)} \\ &\quad \le c\omega_2\left(f,\frac{\lambda_n}{\sqrt{n}}\right)_w.\end{aligned} \tag{2.14}$$

We now use for $0 < i < r-1$ the inequality

$$|B_n^{(i)}(f-\ell,\lambda_n)| \le c\left[\left(\frac{\sqrt{n}}{\lambda_n}\right)^i\|B_n(f-\ell)\|_{\left[\lambda_n,\lambda_n+\frac{\lambda_n}{\sqrt{n}}\right]} + \left(\frac{\lambda_n}{\sqrt{n}}\right)^{r-1-i}\|B_n^{(r-1)}(f-\ell)\|_{\left[\lambda_n,\lambda_n+\frac{\lambda_n}{\sqrt{n}}\right]}\right]$$

(see [1, p. 38 Theorem 5.6]) to obtain

$$\begin{aligned} & w(x)\sum_{i=1}^{r-2}\frac{|B_n^{(i)}(f-l,\lambda_n)|}{i!}(x-\lambda_n)^i \\ & \le c\left\{\sum_{i=1}^{r-2}\frac{1}{i!}\left(\frac{\sqrt{n}}{\lambda_n}\right)^i w(x)(x-\lambda_n)^i\|B_n(f-\ell)\|_{\left[\lambda_n,\lambda_n+\frac{\lambda_n}{\sqrt{n}}\right]}\right. \\ & \left.+\sum_{i=1}^{r-2}\frac{1}{i!}\left(\frac{\lambda_n}{\sqrt{n}}\right)^{r-1-i} w(x)(x-\lambda_n)^i\|B_n^{(r-1)}(f-\ell)\|_{\left[\lambda_n,\lambda_n+\frac{\lambda_n}{\sqrt{n}}\right]}\right\}. \end{aligned} \tag{2.15}$$

Using Proposition A, (1.6) and Lemma 2, we obtain

$$w(x)(x-\lambda_n)^i\|B_n(f-\ell)\|_{\left[\lambda_n,\lambda_n+\frac{\lambda_n}{\sqrt{n}}\right]} \le c\left(\frac{\lambda_n}{\sqrt{n}}\right)^i i!\omega_2\left(f,\frac{\lambda_n}{\sqrt{n}}\right)_w, \quad i=1,2,\cdots,r-2. \tag{2.16}$$

and

$$w(x)(x-\lambda_n)^i\|B_n^{(r-1)}(f-\ell)\|_{\left[\lambda_n,\lambda_n+\frac{\lambda_n}{\sqrt{n}}\right]} \le c\left(\frac{\sqrt{n}}{\lambda_n}\right)^{r-1-i} i!\omega_2\left(f,\frac{\lambda_n}{\sqrt{n}}\right)_w, i=1,2,\cdots,r-1. \tag{2.17}$$

The above estimates together with (2.13) yield

$$w(x)|B_{n,r}(f,x)-f(x)| \le c\omega_2\left(f,\frac{\lambda_n}{\sqrt{n}}\right)_w.$$

The case $x \le -\lambda_n$ is analogous. This completes the proof of Lemma 3.

## 3 Proof of Theorem 1 and Theorem 2

*Proof of Theorem* 1. By Theorem VMS, for $|x| \le \lambda_n$, we have

$$w(x)|B_{n,r}(f,x)-f(x)| \le c\omega_2\left(f,\frac{\lambda_n}{\sqrt{n}}\right)_w. \tag{3.1}$$

For $|x| \ge \lambda_n$, by Lemma 3, we have

$$w(x)|B_{n,r}(f,x)-f(x)| \le c\omega_2\left(f,\frac{\lambda_n}{\sqrt{n}}\right)_w. \tag{3.2}$$

Thus the proof of (1.10) is straightforward from (3.1) and (3.2).

*Proof of Theorem* 2. For $|x| \le \lambda_n$, we have

$$\begin{aligned} B_{n,r}^*(f,x)-f(x) &= \frac{\sqrt{n}}{2\lambda_n}\int_0^{\frac{\lambda_n}{\sqrt{n}}}[B_{n,r}(f,x+t)-f(x+t)]\mathrm{d}t \\ &+\frac{\sqrt{n}}{2\lambda_n}\int_0^{\frac{\lambda_n}{\sqrt{n}}}[B_{n,r}(f,x-t)-f(x-t)]\mathrm{d}t+\frac{\sqrt{n}}{2\lambda_n}\int_0^{\frac{\lambda_n}{\sqrt{n}}}\Delta_t^2 f(x)\mathrm{d}t. \end{aligned}$$

Thus

$$\begin{aligned} w(x)|B^*_{n,r}(f,x)-f(x)| &\le \frac{\sqrt{n}}{2\lambda_n}\int_0^{\frac{\lambda_n}{\sqrt{n}}} w(x+t)|B_{n,r}(f,x+t)-f(x+t)|e^{Q(x+t)-Q(x)}\mathrm{d}t \\ &\quad +\frac{\sqrt{n}}{2\lambda_n}\int_0^{\frac{\lambda_n}{\sqrt{n}}} w(x-t)|B_{n,r}(f,x-t)-f(x-t)|e^{Q(x-t)-Q(x)}\mathrm{d}t \\ &\quad +\frac{\sqrt{n}}{2\lambda_n}\int_0^{\frac{\lambda_n}{\sqrt{n}}} \|w\Delta_t^2 f\|_{[-t^*,t^*]}\mathrm{d}t \\ &= I_1+I_2+I_3 \end{aligned} \tag{3.3}$$

Using Proposition A, for $|x|\ge t_0$, we have

$$\begin{aligned} Q(x+t)-Q(x) &= Q(|x+t|)-Q(|x|) \\ &= Q^{'}(\xi)(|x+t|-|x|) \\ &\le Q^{'}(2\lambda_n)t\le A, \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} Q(x-t)-Q(x) &= Q(|x-t|)-Q(|x|) \\ &= Q^{'}(\eta)(|x-t|-|x|) \\ &\le Q^{'}(2\lambda_n)t\le A, \end{aligned} \tag{3.5}$$

where $\xi$ is between $|x+t|$ and $|x|$, and $\eta$ is between $|x-t|$ and $|x|$.

And for $|x|\le t_0$, it is clear that

$$Q(x+t)-Q(x)\le c$$

$$Q(x-t)-Q(x)\le c.$$

Therefore, using Theorem 1 and (1.1), we have

$$|I_i|\le c\omega_2\left(f,\frac{\lambda_n}{\sqrt{n}}\right)_w, \qquad i=1,2,3$$

which implies

$$|w(x)[B^*_{n,r}(f,x)-f(x)]|\le c\omega_2\left(f,\frac{\lambda_n}{\sqrt{n}}\right)_w \tag{3.6}$$

holds true for $|x|\le\lambda_n$.

For $x>\lambda_n+\dfrac{\lambda_n}{\sqrt{n}}$, let $\ell_1(x)$ be the linear function such that

$$\inf_{\ell\in\mathcal{P}_1}\|w(f-\ell)\|_{[\lambda_n,\infty)}=\|w(f-\ell_1)\|_{[\lambda_n,\infty)}.$$

Since $B^*_{n,r}$ reproduces linear functions, we have

$$\begin{aligned} w(x)|B^*_{n,r}(f,x)-f(x)| &\le \frac{\sqrt{n}}{2\lambda_n}\int_{x-\frac{\lambda_n}{\sqrt{n}}}^{x+\frac{\lambda_n}{\sqrt{n}}} w(x)|B_{n,r}(f-\ell_1,t)|\mathrm{d}t+\omega_2\left(f,\frac{\lambda_n}{\sqrt{n}}\right)_w \\ &\le \frac{\sqrt{n}}{2\lambda_n}w(x)\sum_{i=0}^{r-1}\frac{|B_n^{(i)}(f-\ell_1,\lambda_n)|}{i!}\int_{x-\frac{\lambda_n}{\sqrt{n}}}^{x+\frac{\lambda_n}{\sqrt{n}}}(t-\lambda_n)^i\mathrm{d}t \\ &\quad +\omega_2\left(f,\frac{\lambda_n}{\sqrt{n}}\right)_w. \end{aligned} \tag{3.7}$$

Observing that for $x \ge \lambda_n + \frac{\lambda_n}{\sqrt{n}}$,

$$x-\lambda_n+\frac{\lambda_n}{\sqrt{n}} \le 2(x-\lambda_n)$$

and

$$x-\lambda_n-\frac{\lambda_n}{\sqrt{n}} \le x-\lambda_n,$$

we have

$$\left|\int_{x-\frac{\lambda_n}{\sqrt{n}}}^{x+\frac{\lambda_n}{\sqrt{n}}}(t-\lambda_n)^i\mathrm{d}t\right| \le \frac{2^{i+2}\lambda_n}{i+1}(x-\lambda_n)^i, \qquad i=0,1,\cdots,r-1$$

which implies

$$\begin{aligned} &\frac{\sqrt{n}}{2\lambda_n}w(x)\sum_{i=0}^{r-1}\frac{|B_n^{(i)}(f-\ell_1,\lambda_n)|}{i!}\int_{x-\frac{\lambda_n}{\sqrt{n}}}^{x+\frac{\lambda_n}{\sqrt{n}}}(t-\lambda_n)^i\mathrm{d}t \\ &\quad \le cw(x)\sum_{i=0}^{r-1}\frac{|B_n^{(i)}(f-\ell_1,\lambda_n)|}{i!}(x-\lambda_n)^i. \end{aligned} \tag{3.8}$$

Following the proof of Lemma 3, we obtain

$$w(x)\sum_{i=0}^{r-1}\frac{|B_n^{(i)}(f-\ell_1,\lambda_n)|}{i!}(x-\lambda_n)^i \le c\omega_2\left(f,\frac{\lambda_n}{\sqrt{n}}\right)_w. \tag{3.9}$$

Therefore, it follows from (3.7)-(3.9) that

$$|w(x)[B^*_{n,r}(f,x)-f(x)]| \le c\omega_2\left(f,\frac{\lambda_n}{\sqrt{n}}\right)_w \tag{3.10}$$

holds true for $x > \lambda_n + \dfrac{\lambda_n}{\sqrt{n}}$.

For $\lambda_n \le x \le \lambda_n + \dfrac{\lambda_n}{\sqrt{n}}$, let $\ell_2(x)$ be the linear function such that

$$\inf_{\ell\in\mathcal{P}_1}\|w(f-\ell)\|_{\left[\lambda_n-\frac{\lambda_n}{\sqrt{n}},+\infty\right)} = \|w(f-\ell_2)\|_{\left[\lambda_n-\frac{\lambda_n}{\sqrt{n}},+\infty\right)}.$$

We now write

$$\begin{aligned}
w(x)|B^*_{n,r}(f,x)-f(x)| &\le \frac{\sqrt{n}}{2\lambda_n}w(x)\int_0^{\frac{\lambda_n}{\sqrt{n}}}|B_{n,r}(f,x+t)-f(x+t)|\mathrm{d}t\\
&\quad+\frac{\sqrt{n}}{2\lambda_n}w(x)\int_0^{\frac{\lambda_n}{\sqrt{n}}}|B_{n,r}(f,x-t)-f(x-t)|\mathrm{d}t\\
&\quad+\frac{\sqrt{n}}{2\lambda_n}w(x)\int_0^{\frac{\lambda_n}{\sqrt{n}}}|\Delta_t^2(f-\ell)(x)|\mathrm{d}t\\
&= I_1+I_2+I_3
\end{aligned}$$

Using (3.5), (3.6) and Theorem 1, we can easily get

$$|I_i|\le c\omega_2\left(f,\frac{\lambda_n}{\sqrt{n}}\right)_w, \qquad i=1,2. \tag{3.11}$$

To estimate $I_3$, we write

$$\begin{aligned}
I_3 &\le \frac{\sqrt{n}}{2\lambda_n}\int_0^{\frac{\lambda_n}{\sqrt{n}}}|f(x+t)-\ell_2(x+t)|w(x+t)e^{Q(x+t)-Q(x)}\mathrm{d}t\\
&\quad+\frac{\sqrt{n}}{2\lambda_n}\int_0^{\frac{\lambda_n}{\sqrt{n}}}|f(x-t)-\ell_2(x-t)|w(x-t)e^{Q(x-t)-Q(x)}\mathrm{d}t\\
&\quad+w(x)|f(x)-\ell_2(x)|.
\end{aligned}$$

Using (3.5), (3.6) and (2.8), we obtain

$$I_3\le c\omega_2\left(f,\frac{\lambda_n}{\sqrt{n}}\right)_w.$$

Combining this with (3.12), for $\lambda_n\le x\le\lambda_n+\dfrac{\lambda_n}{\sqrt{n}}$, we have

$$w(x)|B^*_{n,r}(f,x)-f(x)|\le c\omega_2\left(f,\frac{\lambda_n}{\sqrt{n}}\right)_w. \tag{3.12}$$

Similar estimate yields

$$w(x)|B^*_{n,r}(f,x)-f(x)|\le c\omega_2\left(f,\frac{\lambda_n}{\sqrt{n}}\right)_w, \qquad x\le-\lambda_n. \tag{3.13}$$

This completes the proof of Theorem 2.

## References

[1] DeVore, R. A. and Lorentz, G. G., Constructive Approximation, Springer, Berlin, 1993.

[2] Ditzian, Z. and Totik, V., Moduli of Smoothness, Springer New York, 1987.

[3] Levitan, D., Shape Preserving Approximation by Polynomials, J. Comput. Appl. Math., 121(2000), 73-94.

[4] Maizlish, O., Shape Preserving Approximation on the Real line with Exponential Weights, J. Approx. Theory, 157(2009), 127-133.

[5] Petrov, P. P., Some Sharp Inequalities for $n$-monotone Functions, m Acta Math. Hungar., 108:1-2(2005), 37-46.

[6] Robers, A. W. and Varberg, D. E., convex Functions, Academic Press, New York, 1973.

[7] Vecchia, B. D., Mastroianni, G. and Szabados, J., Weighted Approximation of Functions on the Real Line by Bernstein Polynomials, J. Approx. Theory, 127(2004), 223-239.

L. Y. Zhu
School of Information
Renmin University of China
Beijing, 100872
P. R. China

E-mail: zhulaiyi@ruc.edu.cn

X. J. Zhu
School of mathematical
P eking University
Beijing, 100871
P. R. China

E-mail: zhuxiaojie@pku.edu.cn

and

X. Liu
School of Information
Renmin University of China
Beijing, 100872
P. R. China

E-mail: yanshanliuxing@163.com