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A NOTE ON H^p_w-BOUNDEDNESS OF RIESZ TRANSFORMS AND θ-CALDERÓN-ZYGMUND OPERATORS THROUGH MOLECULAR CHARACTERIZATION

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Abstract. Let $0 < p \leq 1$ and w in the Muckenhoupt class A_1. Recently, by using the weighted atomic decomposition and molecular characterization, Lee, Lin and Yang[11] established that the Riesz transforms $R_j, j = 1, 2, \cdots, n$, are bounded on $H^p_w(\mathbf{R}^n)$. In this note we extend this to the general case of weight w in the Muckenhoupt class A_∞ through molecular characterization. One difficulty, which has not been taken care in [11], consists in passing from atoms to all functions in $H^p_w(\mathbf{R}^n)$. Furthermore, the H^p_w-boundedness of θ-Calderón-Zygmund operators are also given through molecular characterization and atomic decomposition.

Key words: *Muckenhoupt weight, Riesz transform, Calderón-Zygmund operator*

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1 Introduction and Preliminaries

Calderón-Zygmund operators and their generalizations on Euclidean space $\mathbf{R}^n$ have been extensively studied, see for example[7,14,18,15]. In particular, Yabuta[18] introduced certain θ-Calderón-Zygmund operators to facilitate his study of certain classes of pseudo-differential operator.

Definition 1.1. *Let θ be a nonnegative nondecreasing function on $(0,\infty)$ satisfying*

$$\int_0^1 \frac{\theta(t)}{t}\mathrm{d}t < \infty.$$

A continuous function $K : \mathbf{R}^n \times \mathbf{R}^n \setminus \{(x,x) : x \in \mathbf{R}^n\} \to \mathbf{C}$ is said to be a θ-Calderón-Zygmund *singular integral kernel if there exists a constant* $C > 0$ *such that*

$$|K(x,y)| \leq \frac{C}{|x-y|^n}$$

for all $x \neq y$,

$$|K(x,y) - K(x',y)| + |K(y,x) - K(y,x')| \leq C\frac{1}{|x-y|^n}\theta\Big(\frac{|x-x'|}{|x-y|}\Big)$$

for all $2|x-x'| \leq |x-y|$.

A linear operator $T : \mathcal{S}(\mathbf{R}^n) \to \mathcal{S}'(\mathbf{R}^n)$ is said to be a θ-Calderón-Zygmund operator if T can be extended to a bounded operator on $L^2(\mathbf{R}^n)$ and there exists a θ-Calderon-Zygmund singular integral kernel K such that for all $f \in C_c^\infty(\mathbf{R}^n)$ and all $x \notin \mathrm{supp}\, f$, we have

$$Tf(x) = \int_{\mathbf{R}^n} K(x,y)f(y)\mathrm{d}y.$$

When

$$K_j(x,y) = \pi^{-(n+1)/2}\Gamma\Big(\frac{n+1}{2}\Big)\frac{x_j - y_j}{|x-y|^{n+1}}, \qquad j = 1,2,\cdots,n,$$

then they are the classical Riesz transforms denoted by R_j.

It is well-known that the Riesz transforms $R_j, j = 1,2,\cdots,n$, are bounded on unweighted Hardy spaces $H^p(\mathbf{R}^n)$. There are many different approaches to prove this classical result (see [11, 9]). Recently, by using the weighted molecular theory (see [10]) and combined with García-Cuerva's atomic decomposition [5] for weighted Hardy spaces $H^p_w(\mathbf{R}^n)$, the authors in [11] established that the Riesz transforms $R_j, j = 1,2,\cdots,n$, are bounded on $H^p_w(\mathbf{R}^n)$. More precisely, they proved that $\|R_j f\|_{H^p_w} \leq C$ for every w-$(p,\infty,ts-1)$-atom where $s,t \in \mathbf{N}$ satisfy $n/(n+s) < p \leq n/(n+s-1)$ and $((s-1)r_w + n)/(s(r_w-1))$ with r_w is the *critical index of w for the reverse Hölder condition.* Remark that this leaves a gap in the proof. Similar gaps exist in some litteratures, for instance in [10, 15] when the authors establish H^p_w-boundedness of Calderón-Zygmund type operators. Indee d, it is now well-known that (see [1]) the argument *"the operator T is uniformly bounded in $H^p_w(\mathbf{R}^n)$ on w-(p,∞,r)-atoms, and hence it extends to a bounded operator on $H^p_w(\mathbf{R}^n)$"* is wrong in general. However, Meda, Sjögren and Vallarino [13] establishes that (in the setting of unweighted Hardy spaces) this is correct if one replaces L^∞-atoms by L^q-atoms with $1 < q < \infty$. Later, the authors in [2] extended these results to the weighted anisotropic Hardy spaces. More precisely, it is claimed in [2] that the operator T can be extended to a bounded operator on $H^p_w(\mathbf{R}^n)$ if it is uniformly bounded on w-(p,q,r)-atoms for $q_w < q < \infty, r \geq [n(q_w/p-1)]$ where q_w is the *critical index of w.*

Motivated by [11, 10, 15, 1, 2], in this paper, we extend *Theorem 1* in [11] to A_∞ weights (see *Theorem 1.1*); *Theorem 4* in [10] (see *Theorem 1.2*), *Theorem 3* in [15] (see *Theorem 3.1*) to θ-Calderón-Zygmund operators; and fill the gaps of the proofs by using the atomic decomposition and molecular characterization of $H^p_w(\mathbf{R}^n)$ as in [11].

Throughout the whole paper, C denotes a positive geometric constant which is independent of the main parameters, but may change from line to line. In $\mathbf{R}^n$, we denote by $B = B(x,r)$ an open ball with center x and radius $r > 0$. For any measurable set E, we denote by $|E|$ its Lebesgue measure, and by E^c the set $\mathbf{R}^n \setminus E$.

Let us first recall some notations, definitions and well-known results.

Let $1 \le p < \infty$. A nonnegative locally integrable function w belongs to the *Muckenhoupt class* A_p, say $w \in A_p$, if there exists a positive constant C so that

$$\frac{1}{|B|}\int_B w(x)\mathrm{d}x\left(\frac{1}{|B|}\int_B (w(x))^{-1/(p-1)}\mathrm{d}x\right)^{p-1} \le C, \quad \text{if } 1<p<\infty,$$

and

$$\frac{1}{|B|}\int_B w(x)\mathrm{d}x \le C\,\underset{x\in B}{\text{ess-inf}}\, w(x), \quad \text{if } p=1,$$

for all balls B in $\mathbf{R}^n$. We say that $w \in A_\infty$ if $w \in A_p$ for some $p \in [1,\infty)$.

It is well known that $w \in A_p$, $1 \le p < \infty$, implies $w \in A_q$ for all $q > p$. Also, if $w \in A_p$, $1 < p < \infty$, then $w \in A_q$ for some $q \in [1,p)$. We thus write $q_w := \inf\{p \ge 1 : w \in A_p\}$ to denote the critical index of w. For a measurable set E, we note $w(E) = \int_E w(x)dx$ its weighted measure.

The following lemma gives a characterization of the class A_p, $1 \le p < \infty$. It can be found in [6].

Lemma A. *The function $w \in A_p$, $1 \le p < \infty$, if and only if, for all nonnegative functions and all balls B,*

$$\left(\frac{1}{|B|}\int_B f(x)\mathrm{d}x\right)^p \le C\frac{1}{w(B)}\int_B f(x)^p w(x)\mathrm{d}x.$$

A close relation to A_p is the reverse Hölder condition. If there exist $r > 1$ and a fixed constant $C > 0$ such that

$$\left(\frac{1}{|B|}\int_B w^r(x)\mathrm{d}x\right)^{1/r} \le C\left(\frac{1}{|B|}\int_B w(x)\mathrm{d}x\right) \qquad \textit{for every ball } B \subset \mathbf{R}^n,$$

we say that w satisfies reverse Hölder condition of order r and write $w \in RH_r$. It is known that if $w \in RH_r$, $r > 1$, then $w \in RH_{r+\varepsilon}$ for some $\varepsilon > 0$. We thus write $r_w := \sup\{r > 1 : w \in RH_r\}$ to denote the critical index of w for the reverse Hölder condition.

The following result provides us the comparison between the Lebesgue measure of a set E and its weighted measure $w(E)$. It also can be found in [6].

Lemma B. *Let $w \in A_p \cap RH_r$, $p \geq 1$ and $r > 1$. Then there exist constants $C_1, C_2 > 0$ such that*

$$C_1\Big(\frac{|E|}{|B|}\Big)^p \leq \frac{w(E)}{w(B)} \leq C_2\Big(\frac{|E|}{|B|}\Big)^{(r-1)/r},$$

for all balls B and measurable subsets $E \subset B$.

Given a weight function w on $\mathbf{R}^n$, as usual we denote by $L^q_w(\mathbf{R}^n)$ the space of all functions f satisfying

$$\|f\|_{L^q_w} := \Big(\int_{\mathbf{R}^n} |f(x)|^q w(x)\mathrm{d}x\Big)^{1/q} < \infty.$$

When $q = \infty$, $L^\infty_w(\mathbf{R}^n)$ is $L^\infty(\mathbf{R}^n)$ and $\|f\|_{L^\infty_w} = \|f\|_{L^\infty}$. Analogously to the classical Hardy spaces, the *weighted Hardy spaces* $H^p_w(\mathbf{R}^n), p > 0$, can be defined in terms of maximal functions. Namely, let ϕ be a function in $\mathcal{S}(\mathbf{R}^n)$, the Schwartz space of rapidly decreasing smooth functions, satisfying $\int_{\mathbf{R}^n} \phi(x)\mathrm{d}x = 1$. Define

$$\phi_t(x) = t^{-n}\phi(x/t), \quad t > 0, x \in \mathbf{R}^n,$$

and the maximal function f^* by

$$f^*(x) = \sup_{t>0} |f * \phi_t(x)|, \quad x \in \mathbf{R}^n.$$

Then $H^p_w(\mathbf{R}^n)$ consists of those tempered distributions $f \in \mathcal{S}'(\mathbf{R}^n)$ for which $f^* \in L^p_w(\mathbf{R}^n)$ with the (quasi-)norm

$$\|f\|_{H^p_w} = \|f^*\|_{L^p_w}.$$

In order to show the H^p_w-boundedness of Riesz transforms, we characterize weighted Hardy spaces in terms of atoms and molecules in the following way.

Definition of a weighted atom. Let $0 < p \leq 1 \leq q \leq \infty$ and $p \neq q$ such that $w \in A_q$. Let q_w be the critical index of w. Set $[\cdot]$ the integer function. For $s \in \mathbf{N}$ satisfying $s \geq [n(q_w/p - 1)]$, a function $a \in L^q_w(\mathbf{R}^n)$ is called *w-(p,q,s)-atom centered at x_0* if

(i) supp $a \subset B$ for some ball B centered at x_0,

(ii) $\|a\|_{L^q_w} \leq w(B)^{1/q-1/p}$,

(iii) $\int_{\mathbf{R}^n} a(x)x^\alpha \mathrm{d}x = 0$ for every multi-index α with $|\alpha| \leq s$.

Let $H^{p,q,s}_w(\mathbf{R}^n)$ denote the space consisting of tempered distributions admitting a decomposition $f = \sum_{j=1}^\infty \lambda_j a_j$ in $\mathcal{S}'(\mathbf{R}^n)$, where a_j's are w-(p,q,s)-atoms and $\sum_{j=1}^\infty |\lambda_j|^p < \infty$. And for every $f \in H^{p,q,s}_w(\mathbf{R}^n)$, we consider the (quasi-)norm

$$\|f\|_{H^{p,q,s}_w} = \inf\Big\{\Big(\sum_{j=1}^\infty |a_j|^p\Big)^{1/p} : f \overset{\mathcal{S}'}{=} \sum_{j=1}^\infty \lambda_j a_j, \ \{a_j\}_{j=1}^\infty \text{ are } w\text{-}(p,q,s)\text{-atoms}\Big\}.$$

Denote by $H^{p,q,s}_{w,\text{fin}}(\mathbf{R}^n)$ the *vector space of all finite linear combinations of* w-(p,q,s)*-atoms*, and the *(quasi-)norm* of f in $H^{p,q,s}_{w,\text{fin}}(\mathbf{R}^n)$ is defined by

$$\|f\|_{H^{p,q,s}_{w,\text{fin}}} := \inf\Big\{\Big(\sum_{j=1}^{k}|\lambda_j|^p\Big)^{1/p} : f=\sum_{j=1}^{k}\lambda_j a_j, k\in\mathbf{N}, \{a_j\}_{j=1}^{k} \text{ are } w\text{-}(p,q,s)\text{-atoms}\Big\}.$$

We have the following atomic decomposition for $H^p_w(\mathbf{R}^n)$. It can be found in [5] (see also [2, 8]).

Theorem A. *If the triplet* (p,q,s) *satisfies the conditions of* w-(p,q,s)*-atoms, then* $H^p_w(\mathbf{R}^n) = H^{p,q,s}_w(\mathbf{R}^n)$ *with equivalent norms.*

The molecules corresponding to the atoms mentioned above can be defined as follows.

Definition of a weighted molecule. For $0<p\le 1\le q\le\infty$ and $p\neq q$, let $w\in A_q$ with critical index q_w and critical index r_w for the reverse Hölder condition. Set $s\ge[n(q_w/p-1)]$, $\varepsilon>\max\{sr_w(r_w-1)^{-1}n^{-1}+(r_w-1)^{-1}, 1/p-1\}$, $a=1-1/p+\varepsilon$, and $b=1-1/q+\varepsilon$. A w-(p,q,s,ε)*-molecule centered at* x_0 is a function $M\in L^q_w(\mathbf{R}^n)$ satisfying

(i) $M.w(B(x_0,\cdot-x_0))^b\in L^q_w(\mathbf{R}^n)$,

(ii) $\|M\|^{a/b}_{L^q_w}\|M.w(B(x_0,\cdot-x_0))^b\|^{1-a/b}_{L^q_w}\equiv\mathfrak{N}_w(M)<\infty$,

(iii) $\int_{\mathbf{R}^n}M(x)x^\alpha\mathrm{d}x=0$ for every multi-index α with $|\alpha|\le s$.

The above quantity $\mathfrak{N}_w(M)$ is called the w-molecular norm of M.

In [10], Lee and Lin proved that every weighted molecule belongs to the weighted Hardy space $H^p_w(\mathbf{R}^n)$, and the embedding is continuous.

Theorem B. *Let* $0<p\le 1\le q\le\infty$ *and* $p\neq q$, $w\in A_q$, *and* (p,q,s,ε) *be the quadruple in the definition of molecule. Then, every* w-(p,q,s,ε)*-molecule* M *centered at any point in* $\mathbf{R}^n$ *is in* $H^p_w(\mathbf{R}^n)$, *and* $\|M\|_{H^p_w}\le C\mathfrak{N}_w(M)$ *where the constant* C *is independent of the molecule.*

Although, in general, one cannot conclude that an operator T *is bounded on* $H^p_w(\mathbf{R}^n)$ *by checking that their norms have uniform bound on all of the corresponding* w-(p,∞,s)*-atoms (cf. [1]). However, this is correct when dealing with* w-(p,q,s)*-atoms with* $q_w<q<\infty$. *Indeed, we have the following result (see [2, Theorem 7.2]).*

Theorem C. *Let* $0<p\le 1$, $w\in A_\infty$, $q\in(q_w,\infty)$ *and* $s\in\mathbf{Z}$ *satisfying* $s\ge[n(q_w/p-1)]$. *Suppose that* $T: H^{p,q,s}_{w,\text{fin}}(\mathbf{R}^n)\to H^p_w(\mathbf{R}^n)$ *is a linear operator satisfying*

$$\sup\{\|Ta\|_{H^p_w} : a \textit{ is any } w-(p,q,s)-\text{atom}\}<\infty.$$

Then T *can be extended to a bounded linear operator on* $H^p_w(\mathbf{R}^n)$.

Our first main result, which generalizes *Theorem 1* in [11], is as follows:

Theorem 1.1. *Let* $0<p\le 1$ *and* $w\in A_\infty$. *Then, the Riesz transforms are bounded on* $H^p_w(\mathbf{R}^n)$.

For the next result, we need the notion $T^*1=0$.

Definition 1.2. *Let T be a θ-Calderón-Zygmund operator. We say that $T^*1=0$ if $\int_{\mathbf{R}^n} Tf(x)dx = 0$ for all $f \in L^q(\mathbf{R}^n), 1<q\le\infty$, with compact support and $\int_{\mathbf{R}^n} f(x)\mathrm{d}x = 0$.*

We now can give the H^p_w-boundedness of θ-Calderón-Zygmund type operators, which generalizes *Theorem 4* in [10] by taking $q=1$ and $\theta(t)=t^\delta$, as follows:

Theorem 1.2. *Given $\delta\in(0,1]$, $n/(n+\delta)<p\le 1$, and $w\in A_q\cap RH_r$ with $1\le q< p(n+\delta)/n, (n+\delta)/(n+\delta-nq)<r$. Let θ be a nonnegative nondecreasing function on $(0,\infty)$ with $\int_0^1 \frac{\theta(t)}{t^{1+\delta}}\mathrm{d}t<\infty$, and T be a θ-Calderón-Zygmund operator satisfying $T^*1=0$. Then T is bounded on $H^p_w(\mathbf{R}^n)$.*

2 Proof of Theorem 1.1

In order to prove the main theorems, we need the following lemma (see [6, page 412]).

Lemma C. *Let $w\in A_r, r>1$. Then there exists a constant $C>0$ such that*

$$\int_{B^c} \frac{1}{|x-x_0|^{nr}} w(x)\mathrm{d}x \le C\frac{1}{\sigma^{nr}} w(B)$$

for all balls $B=B(x_0,\sigma)$ in $\mathbf{R}^n$.

Proof of Theorem 1.1. For $q=2(q_w+1)\in(q_w,\infty)$, then $s:=[n(q/p-1)]\ge[n(q_w/p-1)]$. We now choose (and fix) a positive number ε satisfying

$$\max\{sr_w(r_w-1)^{-1}n^{-1}+(r_w-1)^{-1}, q/p-1\}<\varepsilon<t(s+1)(nq)^{-1}+q^{-1}-1, \tag{2.1}$$

for some $t\in\mathbf{N}, t\ge 1$ and $\max\{sr_w(r_w-1)^{-1}n^{-1}+(r_w-1)^{-1}, q/p-1\}<t(s+1)(nq)^{-1}+q^{-1}-1$.

Clearly, $\ell:=t(s+1)-1\ge s\ge[n(q_w/p-1)]$. Hence, by Theorem B and Theorem C, it is sufficient to show that for every w-(p,q,ℓ)-atom f centered at x_0 and supported in ball $B=B(x_0,\sigma)$, the Riesz transforms $R_jf=K_j*f$, $j=1,2,\cdots,n$, are w-(p,q,s,ε)-molecules with the norm $\mathfrak{N}_w(R_jf)\le C$.

Indeed, as $w\in A_q$ by $q=2(q_w+1)\in(q_w,\infty)$. It follows from L^q_w-boundedness of Riesz transforms that

$$\|R_jf\|_{L^q_w}\le\|R_j\|_{L^q_w\to L^q_w}\|f\|_{L^q_w}\le Cw(B)^{1/q-1/p}. \tag{2.2}$$

To estimate $\|R_jf.w(B(x_0,|\cdot-x_0|))^b\|_{L^q_w}$ where $b=1-1/q+\varepsilon$, we write

$$\begin{aligned}\|R_jf.w(B(x_0,\cdot-x_0))^b\|^q_{L^q_w} &= \int_{|x-x_0|\le 2\sqrt{n}\sigma}|R_jf(x)|^q w(B(x_0,|x-x_0|))^{bq}w(x)\mathrm{d}x+\\ &\quad+\int_{|x-x_0|>2\sqrt{n}\sigma}|R_jf(x)|^q w(B(x_0,|x-x_0|))^{bq}w(x)\mathrm{d}x\\ &= I+II.\end{aligned}$$

By Lemma B, we have the following estimate,

$$\begin{aligned} I &= \int_{|x-x_0|\le 2\sqrt{n}\sigma} |R_j f(x)|^q w(B(x_0,|x-x_0|))^{bq} w(x)\mathrm{d}x \\ &\le w(B(x_0,2\sqrt{n}\sigma))^{bq} \int_{|x-x_0|\le 2\sqrt{n}\sigma} |R_j f(x)|^q w(x)\mathrm{d}x \\ &\le Cw(B)^{bq}\|R_j\|^q_{L^q_w\to L^q_w}\|f\|^q_{L^q_w} \le Cw(B)^{(b+1/q-1/p)q}. \end{aligned}$$

To estimate II, as f is w-(p,q,ℓ)-atom, by the Taylor's fomular and Lemma A, we get

$$\begin{aligned} |K_j * f(x)| &= \Big| \int_{|y-x_0|\le\sigma} \Big(K_j(x-y) - \sum_{|\alpha|\le\ell} \frac{1}{\alpha!} D^\alpha K_j(x-x_0)(x_0-y)^\alpha\Big) f(y)\mathrm{d}y \Big| \\ &\le C\int_{|y-x_0|\le\sigma} \frac{\sigma^{\ell+1}}{|x-x_0|^{n+\ell+1}} |f(y)|\mathrm{d}y \\ &\le C\frac{\sigma^{n+\ell+1}}{|x-x_0|^{n+\ell+1}} w(B)^{-1/q}\|f\|_{L^q_w}, \end{aligned}$$

for all $x\in(B(x_0,2\sqrt{n}\sigma))^c$. As $b=1-1/q+\varepsilon$, it follows from (2.1) that $(n+\ell+1)q-q^2nb>nq$. Therefore, by combining the above inequality, Lemma B and Lemma C, we obtain

$$\begin{aligned} II &= \int_{|x-x_0|>2\sqrt{n}\sigma} |R_j f(x)|^q w(B(x_0,|x-x_0|))^{bq} w(x)\mathrm{d}x \\ &\le C\sigma^{(n+\ell+1)q} w(B)^{-1}\|f\|^q_{L^q_w} \int_{|x-x_0|>2\sqrt{n}\sigma} \frac{1}{|x-x_0|^{(n+\ell+1)q}} w(B(x_0,|x-x_0|))^{bq} w(x)\mathrm{d}x \\ &\le C\sigma^{(n+\ell+1)q-q^2nb} w(B)^{(b-1/p)q} \int_{|x-x_0|>2\sqrt{n}\sigma} \frac{1}{|x-x_0|^{(n+\ell+1)q-q^2nb}} w(x)\mathrm{d}x \\ &\le Cw(B)^{(b+1/q-1/p)q}. \end{aligned}$$

Thus,

$$\|R_j f.w(B(x_0,|\cdot-x_0|))^b\|_{L^q_w} = (I+II)^{1/q} \le Cw(B)^{b+1/q-1/p}. \tag{2.3}$$

Remark that $a=1-1/p+\varepsilon$. Combining (2.2) and (2.3), we obtain

$$\mathfrak{N}_w(R_j f) \le Cw(B)^{(1/q-1/p)a/b} w(B)^{(b+1/q-1/p)(1-a/b)} \le C.$$

The proof will be concluded if we establish the vanishing moment conditions of $R_j f$. One first consider the following lemma.

Lemma. *For every classical atom $(p,2,\ell)$-atom g centered at x_0, we have*

$$\int_{\mathbf{R}^n} R_j g(x) x^\alpha \mathrm{d}x = 0 \quad \text{for } 0\le|\alpha|\le s, 1\le j\le n.$$

Proof of the Lemma. Since $b=1-1/q+\varepsilon<(\ell+1)(nq)^{-1}<(\ell+1)n^{-1}$, we obtain $2(n+\ell+1)-2nb>n$. It is similar to the previous argument, we also obtain that $R_j g$ and

$R_jg.|\cdot-x_0|^{nb}$ belong to $L^2(\mathbf{R}^n)$. Now, we establish that $R_jg.(\cdot-x_0)^\alpha\in L^1(\mathbf{R}^n)$ for every multi-index α with $|\alpha|\le s$. Indeed, since $\varepsilon>q/p-1$ by (2.1), implies that $2(s-nb)<(s-nb)q'<-n$ by $q=2(q_w+1)>2$, where $1/q+1/q'=1$. We use Schwartz inequality to get

$$\begin{aligned}\int_{B(x_0,1)^c}|R_jg(x)(x-x_0)^\alpha|\mathrm{d}x &\le \int_{B(x_0,1)^c}|R_jg(x)||x-x_0|^s\mathrm{d}x\\ &\le \Big(\int_{B(x_0,1)^c}|R_jg(x)|^2|x-x_0|^{2nb}\mathrm{d}x\Big)^{1/2}\Big(\int_{B(x_0,1)^c}|x-x_0|^{2(s-nb)}\mathrm{d}x\Big)^{1/2}\\ &\le C\|R_jg.|\cdot-x_0|^{nb}\|_{L^2}<\infty,\end{aligned}$$

and

$$\int_{B(x_0,1)}|R_jg(x)(x-x_0)^\alpha|\mathrm{d}x\le|B(x_0,1)|^{1/2}\Big(\int_{B(x_0,1)}|R_jg(x)|^2\mathrm{d}x\Big)^{1/2}<\infty.$$

Thus, $R_jg.(\cdot-x_0)^\alpha\in L^1(\mathbf{R}^n)$ for any $|\alpha|\le s$. Deduce that $R_jg(x)x^\alpha\in L^1(\mathbf{R}^n)$ for any $|\alpha|\le s$. Therefore,

$$(R_jg(x)x^\alpha)\hat{}(\xi)=C_\alpha.D^\alpha\widehat{(R_jg)}(\xi)$$

is continuous, with $|C_\alpha|\le C_s$ (C_s depends only on s) for any $|\alpha|\le s$, where $\hat{h}$ is used to denote the fourier transform of h. Consequently,

$$\int_{\mathbf{R}^n}R_jg(x)x^\alpha dx=C_\alpha.D^\alpha\widehat{(R_jg)}(0)=C_\alpha.D^\alpha(m_j\hat{g})(0),$$

where $m_j(x)=-ix_j/|x|$. Moreover, since g is a classical $(p,2,\ell)$-atom, it follows from [17, Lemma 9.1] that $\hat{g}$ is ℓth order differentiable and $\hat{g}(\xi)=O(|\xi|^{\ell+1})$ as $\xi\to0$. We write e_j to be the jth standard basis vector of $\mathbf{R}^n$, $\alpha=(\alpha_1,...,\alpha_n)$ a multi-index of nonnegative integers α_j, $\Delta_{he_j}\phi(x)=\phi(x)-\phi(x-he_j)$, $\Delta^{\alpha_j}_{he_j}\phi(x)=\Delta^{\alpha_j-1}_{he_j}\phi(x)-\Delta^{\alpha_j-1}_{he_j}\phi(x-he_j)$ for $\alpha_j\ge2$, $\Delta^0_{he_j}\phi(x)=\phi(x)$, and $\Delta^\alpha_h=\Delta^{\alpha_1}_{he_1}...\Delta^{\alpha_n}_{he_n}$. Then, the boundedness of m_j, and $|C_\alpha|\le C_s$ for $|\alpha|\le s$, implies

$$\begin{aligned}\Big|\int_{\mathbf{R}^n}R_jg(x)x^\alpha dx\Big| &= |C_\alpha|\Big|\lim_{h\to0}|h|^{-|\alpha|}\Delta^\alpha_h(m_j\hat{g})(0)\Big|\\ &\le C\lim_{h\to0}|h|^{\ell+1-|\alpha|}=0,\end{aligned}$$

for $|\alpha|\le s$ by $s\le\ell$. Thus, for any $j=1,2,\cdots,n$, and $|\alpha|\le s$,

$$\int_{\mathbf{R}^n}R_jg(x)x^\alpha\mathrm{d}x=0.$$

This complete the proof of the lemma.

Let us come back to the proof of Theorem 1.1. As $q/2=q_w+1>q_w$, by Lemma A,

$$\Big(\frac{1}{|B|}\int_B|f(x)|^2\mathrm{d}x\Big)^{q/2}\le C\frac{1}{w(B)}\int_B|f(x)|^qw(x)\mathrm{d}x.$$

Therefore, $g := C^{-1/q}|B|^{-1/p}w(B)^{1/p}f$ is a classical $(p,2,\ell)$-atom since f is w-(p,q,ℓ)-atom associated with ball B. Consequently, by the above lemma,

$$\int_{\mathbf{R}^n} R_j f(x)x^\alpha \mathrm{d}x = C^{1/q}|B|^{1/p}w(B)^{-1/p}\int_{\mathbf{R}^n} R_j g(x)x^\alpha \mathrm{d}x = 0$$

for all $j = 1,2,\cdots,n$ and $|\alpha| \le s$. Thus, the theorem is proved.

Following a similar but easier argument, we also have the following H^p_w-boundedness of Hilbert transform. We leave details to readers.

Theorem 2.1. *Let* $0 < p \le 1$ *and* $w \in A_\infty$. *Then, the Hilbert transform is bounded on* $H^p_w(\mathbf{R})$.

3 Proof of Theorem 1.2

We first consider the following lemma

Lemma 3.1. *Let* $p \in (0,1], w \in A_q, 1 < q < \infty$*, and* T *be a* θ*-Calderón-Zygmund operator satisfying* $T^*1 = 0$. *Then,* $\int_{\mathbf{R}^n} Tf(x)\mathrm{d}x = 0$ *for all* w*-*$(p,q,0)$*-atoms* f.

Proof of Lemma 3.1. Let f be an arbitrary w-$(p,q,0)$-ato m associated with ball B. It is well-known that there exists $1 < r < q$ such that $w \in A_r$. Therefore, it follows from Lemma A that

$$\int_B |f(x)|^{q/r}\mathrm{d}x \le C|B|w(B)^{1/r}\|f\|^{q/r}_{L^q_w} < \infty.$$

We deduce that f is a multiple of classical $(p,q/r,0)$-atom, and thus the condition $T^*1 = 0$ implies $\int_{\mathbf{R}^n} Tf(x)\mathrm{d}x = 0$.

Proof of Theorem 1.2. Because of the hypothesis, without loss of generality we can assume $q > 1$. Futhermore, it is clear that $[n(q_w/p-1)] = 0$, and there exists a positive constant ε such that

$$\max\left\{\frac{1}{r_w-1},\frac{1}{p}-1\right\} < \varepsilon < \frac{n+\delta}{nq}-1. \tag{3.1}$$

Similarly to the arguments in Theorem 1.1, it is sufficient to show that, for every w-$(p,q,0)$-atom f centered at x_0 and supported in ball $B = B(x_0,\sigma)$, Tf is a w-$(p,q,0,\varepsilon)$-molecule with the norm $\mathfrak{N}_w(Tf) \le C$. One first observe that $\int_{\mathbf{R}^n} Tf(x)dx = 0$ by Lemma 3.1, and

$$\sum_{k=0}^\infty \theta(2^{-k})2^{knbq} < \infty,$$

where $b = 1-1/q+\varepsilon$, by $\displaystyle\int_0^1 \frac{\theta(t)}{t^{1+\delta}}\mathrm{d}t < \infty$ and (3.1). We deduce that

$$\sum_{k=0}^\infty \left(\theta(2^{-k})2^{knbq}\right)^q < \infty. \tag{3.2}$$

As $w \subset A_q$, $1 < q < \infty$, it follows from [18, Theorem 2.4] that

$$\|Tf\|_{L^q_w} \leq C\|f\|_{L^q_w} \leq Cw(B)^{1/q-1/p}. \tag{3.3}$$

To estimate $\|Tf.w(B(x_0,|\cdot - x_0|))^b\|_{L^q_w}$, we write

$$\begin{aligned}\|Tf.w(B(x_0,\cdot - x_0))^b\|^q_{L^q_w} &= \int_{|x-x_0|\leq 2\sigma} |Tf(x)|^q w(B(x_0,|x-x_0|))^{bq} w(x)\mathrm{d}x + \\ &+ \int_{|x-x_0|>2\sigma} |Tf(x)|^q w(B(x_0,|x-x_0|))^{bq} w(x)\mathrm{d}x = I + II.\end{aligned}$$

By Lemma B, we have the following estimate,

$$\begin{aligned} I &= \int_{|x-x_0|\leq 2\sigma} |Tf(x)|^q w(B(x_0,|x-x_0|))^{bq} w(x)\mathrm{d}x \\ &\leq w(B(x_0,2\sigma))^{bq} \int_{|x-x_0|\leq 2\sigma} |Tf(x)|^q w(x)\mathrm{d}x \\ &\leq Cw(B)^{bq}\|f\|^q_{L^q_w} \leq Cw(B)^{(b+1/q-1/p)q}. \end{aligned}$$

To estimate II, since f is of mean zero, by Lemma A, we have

$$\begin{aligned} |Tf(x)| &= \Big|\int_{|y-x_0|\leq\sigma} (K(x,y)-K(x,x_0))f(y)\mathrm{d}y\Big| \\ &\leq C\int_{|y-x_0|\leq\sigma} \frac{1}{|x-x_0|^n}\theta\Big(\frac{|y-x_0|}{|x-x_0|}\Big)|f(y)|\mathrm{d}y \\ &\leq C\frac{\sigma^n}{|x-x_0|^n}\theta\Big(\frac{\sigma}{|x-x_0|}\Big)w(B)^{-1/q}\|f\|_{L^q_w}, \end{aligned}$$

for all $x \in (B(x_0,2\sigma))^c$. Therefore, by combining the above inequality, Lemma B and (3.2), we obtain

$$\begin{aligned} II &= \int_{|x-x_0|>2\sigma} |Tf(x)|^q w(B(x_0,|x-x_0|))^{bq} w(x)\mathrm{d}x \\ &\leq Cw(B)^{-1}\|f\|^q_{L^q_w}\int_{|x-x_0|>2\sigma} \frac{\sigma^{nq}}{|x-x_0|^{nq}}\left(\theta\Big(\frac{\sigma}{|x-x_0|}\Big)\right)^q w(B(x_0,|x-x_0|))^{bq}w(x)\mathrm{d}x \\ &\leq Cw(B)^{-q/p}\sum_{k=1}^{\infty}\int_{2^k\sigma<|x-x_0|\leq 2^{k+1}\sigma} \frac{\sigma^{nq}}{|x-x_0|^{nq}}\left(\theta\Big(\frac{\sigma}{|x-x_0|}\Big)\right)^q w(B(x_0,|x-x_0|))^{bq}w(x)\mathrm{d}x \\ &\leq Cw(B)^{(b+1/q-1/p)q}\sum_{k=0}^{\infty}\left(\theta(2^{-k})2^{knbq}\right)^q \leq Cw(B)^{(b+1/q-1/p)q}. \end{aligned}$$

Thus,

$$\|Tf.w(B(x_0,|\cdot - x_0|))^b\|_{L^q_w} = (I+II)^{1/q} \leq Cw(B)^{b+1/q-1/p}. \tag{3.4}$$

Remark that $a = 1 - 1/p + \varepsilon$. Combining (3.3) and (3.4), we obtain

$$\mathfrak{N}_w(Tf) \leq Cw(B)^{(1/q-1/p)a/b}w(B)^{(b+1/q-1/p)(1-a/b)} \leq C.$$

This finishes the proof.

It is well-known that the molecular theory of (unweighted) Hardy spaces of Taibleson and Weiss [17] is one of useful tools to establish boundedness of operators in Hardy spaces (cf. [17, 12]). In the setting of Muckenhoupt weight, this theory has been considered by the authors in [10], since then, they have been well used to establish boundedness of operators in weighted Hardy spaces (cf. [10, 11, 3]). However in some cases, the *weighted molecular characterization*, which obtained in [10], does not give the best possible results. For Calderón-Zygmund type operators in Theorem 1.2, for instance, it involves assumption on the *critical index of w for the reverse Hölder condition* as the following theorem does not.

Theorem 3.1. *Given $\delta\in(0,1]$, $n/(n+\delta)<p\le 1$, and $w\in A_q$ with $1\le q<p(n+\delta)/n$. Let θ be a nonnegative nondecreasing function on $(0,\infty)$ with $\int_0^1\frac{\theta(t)}{t^{1+\delta}}dt<\infty$, and T be a θ-Calderón-Zygmund operator satisfying $T^*1=0$. Then T is bounded on $H^p_w(\mathbf{R}^n)$.*

The following corollary give the boundedness of the classical Calderón-Zygmund type operators on weighted Hardy spaces (see [15, Theorem 3]).

Corollary 3.1. *Let $0<\delta\le 1$ and T be the classical δ-Calderón-Zygmund operator, i.e. $\theta(t)=t^\delta$, satisfying $T^*1=0$. If $n/(n+\delta)<p\le 1$ and $w\in A_q$ with $1\le q<p(n+\delta)/n$, then T is bounded on $H^p_w(\mathbf{R}^n)$.*

Proof of Corollary 3.1. By taking $\delta'\in(0,\delta)$ which is close enough δ. Then, we apply Theorem 3.1 with δ' instead of δ.

Proof of Theorem 3.1. Without loss of generality we can assume $1<q<p(n+\delta)/n$. Fix $\phi\in\mathcal{S}(\mathbf{R}^n)$ with $\int_{\mathbf{R}^n}\phi(x)dx\neq 0$. By Theorem C, it is sufficient to show that for every w-$(p,q,0)$-atom f centered at x_0 and supported in ball $B=B(x_0,\sigma)$, $\|(Tf)^*\|_{L^p_w}\le C$. In order to do this, one write

$$\begin{aligned}\|(Tf)^*\|^p_{L^p_w} &= \int_{|x-x_0|\le 4\sigma}\Big((Tf)^*(x)\Big)^p w(x)dx+\int_{|x-x_0|>4\sigma}\Big((Tf)^*(x)\Big)^p w(x)dx\\ &= L_1+L_2.\end{aligned}$$

By Hölder inequality, L^q_w-boundedness of the maximal function and Lemma B, we get

$$\begin{aligned}L_1 &\le \left(\int_{|x-x_0|\le 4\sigma}\Big((Tf)^*(x)\Big)^q w(x)\mathrm{d}x\right)^{p/q}\left(\int_{|x-x_0|\le 4\sigma}w(x)\mathrm{d}x\right)^{1-p/q}\\ &\le C\|f\|^p_{L^q_w}w(B(x_0,4\sigma))^{1-p/q}\le C.\end{aligned}$$

To estimate L_2, we first estimate $(Tf)^*(x)$ for $|x-x_0|>4\sigma$. For any $t>0$, since $\displaystyle\int_{\mathbf{R}^n}Tf(x)\mathrm{d}=$

0 by Lemma 3.1, we get

$$\begin{aligned}
|Tf*\phi_t(x)| &= \left|\int_{\mathbf{R}^n} Tf(y)\frac{1}{t^n}\left(\phi\Big(\frac{x-y}{t}\Big)-\phi\Big(\frac{x-x_0}{t}\Big)\right)\mathrm{d}y\right| \\
&\leq \frac{1}{t^n}\int_{|y-x_0|<2\sigma}|Tf(y)|\left|\phi\Big(\frac{x-y}{t}\Big)-\phi\Big(\frac{x-x_0}{t}\Big)\right|\mathrm{d}y \\
&\quad +\frac{1}{t^n}\int_{2\sigma\leq|y-x_0|<\frac{|x-x_0|}{2}}\cdots+\frac{1}{t^n}\int_{|y-x_0|\geq\frac{|x-x_0|}{2}}\cdots \\
&= E_1(t)+E_2(t)+E_3(t).
\end{aligned}$$

As $|x-x_0|>4\sigma$, by the mean value theorem, Lemma A and Lemma B, we get

$$\begin{aligned}
E_1(t) &= \frac{1}{t^n}\int_{|y-x_0|<2\sigma}|Tf(y)|\left|\phi\Big(\frac{x-y}{t}\Big)-\phi\Big(\frac{x-x_0}{t}\Big)\right|\mathrm{d}y \\
&\leq \frac{1}{t^n}\int_{|y-x_0|<2\sigma}|Tf(y)|\frac{|y-x_0|}{t}\sup_{\lambda\in(0,1)}\left|\nabla\phi\Big(\frac{x-x_0+\lambda(y-x_0)}{t}\Big)\right|\mathrm{d}y \\
&\leq C\frac{\sigma}{|x-x_0|^{n+1}}\int_{|y-x_0|<2\sigma}|Tf(y)|\mathrm{d}y \\
&\leq C\frac{\sigma}{|x-x_0|^{n+1}}|B(x_0,2\sigma)|w(B(x_0,2\sigma))^{-1/q}\|Tf\|_{L^q_w} \\
&\leq C\frac{\sigma^{n+1}}{|x-x_0|^{n+1}}w(B)^{-1/q}\|f\|_{L^q_w}\leq C\frac{\sigma^{n+1}}{|x-x_0|^{n+1}}w(B)^{-1/p}.
\end{aligned}$$

Similarly, we also get

$$\begin{aligned}
E_2(t) &\leq \frac{1}{t^n}\int_{2\sigma\leq|y-x_0|<\frac{|x-x_0|}{2}}\left|\int_{\mathbf{R}^n}f(z)\Big(K(y,z)\right. \\
&\quad \left.-K(y,x_0)\Big)\mathrm{d}z\right|\frac{|y-x_0|}{t}\times\sup_{\lambda\in(0,1)}\left|\nabla\phi\Big(\frac{x-x_0+\lambda(y-x_0)}{t}\Big)\right|\mathrm{d}y \\
&\leq C\frac{1}{|x-x_0|^{n+1}}\int_{2\sigma\leq|y-x_0|<\frac{|x-x_0|}{2}}|y-x_0|\int_{|z-x_0|<\sigma}|f(z)|\frac{1}{|y-x_0|^n}\theta\Big(\frac{|z-x_0|}{|y-x_0|}\Big)\mathrm{d}z\mathrm{d}y \\
&\leq C\Big(\frac{\sigma}{|x-x_0|}\Big)^{n+1}\int_{2\sigma/|x-x_0|}^{1/2}\frac{\theta(t)}{t^2}\mathrm{d}t\,w(B)^{-1/p} \\
&\leq C\Big(\frac{\sigma}{|x-x_0|}\Big)^{n+1}\Big(\frac{|x-x_0|}{2\sigma}\Big)^{1-\delta}\int_{2\sigma/|x-x_0|}^{1/2}\frac{\theta(t)}{t^{1+\delta}}\mathrm{d}t\,w(B)^{-1/p} \\
&\leq C\Big(\frac{\sigma}{|x-x_0|}\Big)^{n+\delta}w(B)^{-1/p}.
\end{aligned}$$

Next, let us look at L_3. Similarly, we also have

$$\begin{aligned}E_3(t) &\leq \frac{1}{t^n}\int_{|y-x_0|\geq\frac{|x-x_0|}{2}}\left|\int_{\mathbf{R}^n} f(z)\Big(K(y,z)-K(y,x_0)\Big)\mathrm{d}z\right|\left(\left|\phi\Big(\frac{y-x_0}{t}\Big)\right|+2\left|\phi\Big(\frac{x-x_0}{t}\Big)\right|\right)\mathrm{d}y\\ &\leq C\frac{1}{|x-x_0|^n}\int_{|y-x_0|\geq\frac{|x-x_0|}{2}}\int_{|z-x_0|<\sigma}|f(z)|\frac{1}{|y-x_0|^n}\theta\Big(\frac{|z-x_0|}{|y-x_0|}\Big)\mathrm{d}z\mathrm{d}y\\ &\leq C\Big(\frac{\sigma}{|x-x_0|}\Big)^n\int_0^{2\sigma/|x-x_0|}\frac{\theta(t)}{t}\mathrm{d}t\,w(B)^{-1/p}\\ &\leq C\Big(\frac{\sigma}{|x-x_0|}\Big)^n\int_0^{2\sigma/|x-x_0|}\frac{\theta(t)}{t^{1+\delta}}\mathrm{d}t\Big(\frac{2\sigma}{|x-x_0|}\Big)^{\delta}w(B)^{-1/p}\\ &\leq C\Big(\frac{\sigma}{|x-x_0|}\Big)^{n+\delta}w(B)^{-1/p}.\end{aligned}$$

Therefore, for all $|x-x_0|>4\sigma$,

$$(Tf)^*(x)=\sup_{t>0}(E_1(t)+E_2(t)+E_3(t))\leq C\Big(\frac{\sigma}{|x-x_0|}\Big)^{n+\delta}w(B)^{-1/p}.$$

Combining this, Lemma C and Lemma B, we obtain that

$$\begin{aligned}L_2=\int_{|x-x_0|>4\sigma}\Big((Tf)^*(x)\Big)^p w(x)\mathrm{d}x &\leq C\int_{|x-x_0|>4\sigma}\frac{\sigma^{(n+\delta)p}}{|x-x_0|^{(n+\delta)p}}w(B)^{-1}w(x)\mathrm{d}x\\ &\leq Cw(B)^{-1}w(B(x_0,4\sigma))\leq C,\end{aligned}$$

since $(n+\delta)p>nq$. This finishes the proof.

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