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BMO BOUNDEDNESS FOR BANACH SPACE VALUED SINGULAR INTEGRALS

Chunjie Zhang

(*Hangzhou Dianzi University, China*)

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Abstract. In this paper, we consider a class of Banach space valued singular integrals. The L^p boundedness of these operators has already been obtained. We shall discuss their boundedness from BMO to BMO. As applications, we get BMO boundedness for the classic g-function and the Marcinkiewicz integral. Some known results are improved.

Key words: BMO, *Banach space valued singular integral*

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1 Introduction

Let H be a Banach space. We denote by $L^p_H, 1 \le p \le +\infty$ the space of H-valued strongly measurable functions g on $\mathbf{R}^n$ such that

$$\|g\|_{L^p_H} = \left(\int_{\mathbf{R}^n} \|g\|_H^p \mathrm{d}x\right)^{1/p} < +\infty, \quad 1 \le p < \infty$$

and when $p = \infty$,

$$\|g\|_{L^\infty_H} = \text{ess sup}\|g\|_H < +\infty.$$

The corresponding sharp function is defined as

$$g^\sharp(x) = \sup_{x \in B} \frac{1}{|B|} \int_B \|g(y) - g_B\|_H \mathrm{d}y,$$

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where B denotes any ball in $\mathbf{R}^n$ and g_B is the average of g over B. Finally we define BMO(H) to be the space of all H-valued locally integrable functions g such that

$$\|g\|_{\mathrm{BMO}(H)} = \|g^{\sharp}\|_{L^\infty(\mathbf{R}^n)} < +\infty.$$

Now we introduce the concept of H-valued singular integral. Let $K(x)$ be an H-valued strongly measurable function defined on $\mathbf{R}^n\backslash\{0\}$, which is also locally integrable in this domain. As we shall take $K(x)$ as the kernel of singular integrals, we present the following continuity requirements which are first introduced by Rubio de Francia, Ruiz and Torrea in [8].

Given $1 \le r \le +\infty$, we call K satisfies the condition D_r if there is a sequence $\{c_k\}_{k=1}^{\infty} \in l^1$ such that for all $k \ge 1$ and $y \in \mathbf{R}^n$,

$$\left(\int_{S_k(y)} \|K(x-y)-K(x)\|_H^r \mathrm{d}x\right)^{1/r} \le c_k |S_k(y)|^{\frac{1}{r}-1}.$$

Here $S_k(y)$ denotes the spherical shell $\{x \in \mathbf{R}^n : \ 2^k|y| \le |x| \le 2^{k+1}|y|\}$. It is not hard to check that if

$$\|K(x-y)-K(x)\|_H \le C\frac{|y|}{|x|^{n+1}}, \quad |x| > 2|y|,$$

then K satisfies D_∞. And D_1 condition is equivalent to the familiar Hömander's condition

$$\int_{|x|>2|y|} \|K(x-y)-K(x)\|_H \mathrm{d}x < +\infty.$$

Besides, D_{r_1} implies D_{r_2} if $r_1 > r_2$. Finally, we call a linear operator T mapping functions into H-valued functions a singular integral operator if

(i) T is bounded from $L^2(\mathbf{R}^n)$ to $L^2_H(\mathbf{R}^n)$;

(ii) There exists a kernel K satisfying D_1 such that

$$Tf(x) = \int_{\mathbf{R}^n} K(x-y)f(y)\mathrm{d}y$$

for every compactly supported f and a.e. $x \notin \mathrm{supp}(f)$.

In [8], the authors proved that such operator can be extended to bounded operators on all $L^p(\mathbf{R}^n), 1 < p < \infty$ and satisfy

(a) $\|Tf\|_{L^p_H(\mathbf{R}^n)} \le C\|f\|_{L^p(\mathbf{R}^n)}, \ 1 < p < \infty$;

(b) $\|Tf\|_{L^1_H} \le C\|f\|_{H^1}$;

(c) $\|Tf\|_{\mathrm{BMO}(H)} \le C\|f\|_{L^\infty}, \ f \in L^\infty_c(\mathbf{R}^n)$.

The aim of this paper is to obtain BMO to BMO boundedness for such singular operator. If T is the usual scalar valued singular integral, then it in fact already maps BMO to BMO with

$\|Tf\|_{\mathrm{BMO}} \le C\|f\|_{\mathrm{BMO}}$, see [9], page 179. And for H-L maximal operator, $f \in \mathrm{BMO}$ still implies $Mf \in \mathrm{BMO}$, but

$$\|Mf\|_{\mathrm{BMO}} \le C\,(\|f\|_{\mathrm{BMO}} + |f_{B_1}|)$$

where f_{B_1} is the average of f over the unit ball, see [2]. Thus it seems that the BMO boundedness for the above H-valued operator T requires more continuity than D_1 on the kernel K. Below is our main theorem.

Theorem 1. *Let T be a singular integral associated to $K(x)$. Suppose*

$$\int_{\mathbf{R}^n} K(x)\mathrm{d}x = 0$$

and

$$\left(\int_{S_k(y)} \|K(x-y)-K(x)\|_H^r \mathrm{d}x\right)^{1/r} \le c_k |S_k(y)|^{\frac{1}{r}-1}, \quad r > 1 \tag{1}$$

for a sequence $\{c_k\}_{k=1}^{\infty}$ with $\sum\limits_{k=1}^{+\infty} kc_k < +\infty$. Let $f \in \mathrm{BMO}$. If $\|Tf\|_H$ is finite in a measurable set E with $|E| > 0$, then $\|Tf\|_H < +\infty$ almost everywhere in $\mathbf{R}^n$. Furthermore,

$$\|Tf\|_{\mathrm{BMO}(H)} \le C\|f\|_{\mathrm{BMO}}.$$

Remark. In [11], Wang found that the value of the classical g-functions, when acting on L^∞ or BMO, might be infinite everywhere in $\mathbf{R}^n$. He also proved the BMO boundedness for g-function assuming the existence of $g(f)$ on a set of positive measure. The operators we consider here obviously contain g-function, thus suffer the same existence restriction.

In the following sections we shall first prove Theorem 1, and then discuss two special cases of the operator T, the g-function and the Marcinkiewicz integral.

2 Proof of Theorem 1

Proof. Let x_0' be a density point of E. Denote by B the ball centered at x_0' with radius d, and B^* the same centerd ball with radius $3d$. We shall first show $\|Tf(x)\|_H < \infty$, a.e. $x \in B$. For any $x \in B$ and $f \in \mathrm{BMO}$,

$$f(x) = f_B + (f(x) - f_{B^*})\chi_{B^*} + (f(x) - f_{B^*})(1-\chi_{B^*}) = f_1 + f_2 + f_3.$$

Since $\int_{\mathbf{R}^n} K(x)\mathrm{d}x = 0$, $Tf_1 \equiv 0$. For $f_2(x)$, noting that

$$\int_{\mathbf{R}^n} |f_2(x)|^2 \mathrm{d}x = \int_{B^*} |f(x) - f_{B^*}|^2 \mathrm{d}x \le C|B|\|f\|_{\mathrm{BMO}}^2,$$

we get

$$\int_{\mathbf{R}^n} \|Tf_2(x)\|_H^2 \mathrm{d}x \le C\int_{\mathbf{R}^n} |f(x)|^2 \mathrm{d}x \le C|B|\|f\|_{\mathrm{BMO}}^2. \tag{2}$$

Therefore $\|Tf_2(x)\|_H < +\infty$ almost everywhere in $\mathbf{R}^n$. Now we choose $x_0 \in B\bigcap E$ (sufficiently close to x_0') such that $\|Tf_2(x_0)\|_H < \infty$ and $\|Tf(x_0)\|_H < \infty$. Then

$$\begin{aligned}\|Tf_3(x_0)\|_H &= \|Tf(x_0) - Tf_1(x_0) - Tf_2(x_0)\|_H \\ &\le \|Tf(x_0)\|_H + \|Tf_2(x_0)\|_H < +\infty.\end{aligned}$$

So if we can prove

$$\|Tf_3(x) - Tf_3(x_0)\|_H < +\infty, \quad \text{a.e. } x \in B, \tag{3}$$

then we reach $\|Tf(x)\|_H < +\infty$, a.e. $x \in B$.

Next we are going to show (3). For $x \in B$,

$$\begin{aligned}&\|Tf_3(x) - Tf_3(x_0)\|_H \\ =& \left\|\int_{(B^*)^c} (K(x-y) - K(x_0-y)) f_3(y)\mathrm{d}\right\|_H \\ \le& \sum_{k=1}^{\infty} \int_{2^k|x_0-x|<|x_0-y|<2^{k+1}|x_0-x|} \|K(x-y) - K(x_0-y)\|_H |f_3(y)|\mathrm{d}y \\ \le& \sum_{k=1}^{\infty} \left(\int_{x_0-S_k(x_0-x)} \|K(x-y) - K(x_0-y)\|_H^r \mathrm{d}y\right)^{\frac{1}{r}} \left(\int_{x_0-S_k(x_0-x)} |f_3(y)|^{r'}\mathrm{d}y\right)^{\frac{1}{r'}} \\ \le& \sum_{k=1}^{\infty} c_k \left(\frac{1}{|S_k(x_0-x)|}\int_{x_0-S_k(x_0-x)} |f_3(y)|^{r'}\mathrm{d}y\right)^{\frac{1}{r'}} = \sum_{k=1}^{\infty} c_k I_k(x).\end{aligned}$$

Denote $d' = |x - x_0|$ and

$$T_k(x_0 - x) = x_0 - S_k(x_0 - x) = \{y : 2^k d' < |y - x_0| < 2^{k+1}d'\}.$$

Let k_0 be the integer such that $2^{k_0+2}d' \ge 4d$ and $2^{k_0+1}d' < 4d$. When $k > k_0$,

$$\begin{aligned}I_k(x) \le& \left(\frac{1}{|T_k(x_0-x)|}\int_{T_k(x_0-x)} |f(y) - f_{B^*}|^{r'}\mathrm{d}y\right)^{\frac{1}{r'}} \\ \le& \left(\frac{1}{|T_k(x_0-x)|}\int_{T_k(x_0-x)} |f(y) - f_{B(x_0,2^{k+1}d')}|^{r'}\mathrm{d}y\right)^{\frac{1}{r'}} \\ &+ \left(\frac{1}{|T_k(x_0-x)|}\int_{T_k(x_0-x)} |f_{B(x_0,2^{k+1}d')} - f_{B(x_0,2^k d')}|^{r'}\mathrm{d}y\right)^{\frac{1}{r'}} \\ &+ \cdots + \left(\frac{1}{|T_k(x_0-x)|}\int_{T_k(x_0-x)} |f_{B(x_0,2^{k_0+2}d')} - f_{B^*}|^{r'}\mathrm{d}y\right)^{\frac{1}{r'}} \\ \le& C\|f\|_{\mathrm{BMO}} + J_k + J_{k+1} + \cdots + J_{k_0+1}.\end{aligned}$$

The terms $J_k, \cdots, J_{k_0+1}$ can be estimated similarly. Take J_{k_0+1} for example,

$$\begin{aligned}|f_{B(x_0,2^{k_0+2}d')} - f_{B^*}| &\le \frac{1}{|B^*|}\int_{B^*}|f(y) - f_{B(x_0,2^{k_0+2}d')}|\mathrm{d}y \\ &\le \frac{2^n}{|B(x_0,2^{k_0+2}d')|}\int_{B(x_0,2^{k_0+2}d')}|f(y) - f_{B(x_0,2^{k_0+2}d')}|\mathrm{d}y \\ &\le 2^n\|f\|_{\mathrm{BMO}}.\end{aligned}$$

Thus

$$J_{k_0+1} = \left(\frac{1}{|T_k(x_0-x)|}\int_{T_k(x_0-x)}|f_{B(x_0,2^{k_0+2}d')} - f_{B^*}|^{r'}\mathrm{d}y\right)^{\frac{1}{r'}} \le 2^n\|f\|_{\mathrm{BMO}},$$

and

$$I_k(x) \le C(1+(k-k_0))\|f\|_{\mathrm{BMO}}.$$

When $k \le k_0$,

$$\begin{aligned}I_k(x) &\le \left(\frac{1}{|T_k(x_0-x)|}\int_{T_k(x_0-x)}|f(y) - f_{B(x_0,2^{k+1}d')}|^{r'}\mathrm{d}y\right)^{\frac{1}{r'}} \\ &\quad + \left(\frac{1}{|T_k(x_0-x)|}\int_{T_k(x_0-x)}|f_{B(x_0,2^{k+1}d')} - f_{B(x_0,2^{k}d')}|^{r'}\mathrm{d}y\right)^{\frac{1}{r'}} \\ &\quad + \cdots + \left(\frac{1}{|T_k(x_0-x)|}\int_{T_k(x_0-x)}|f_{B(x_0,2^{k_0}d')} - f_{B^*}|^{r'}\mathrm{d}y\right)^{\frac{1}{r'}} \\ &\le C\|f\|_{\mathrm{BMO}} + J'_k + J'_{k+1} + \cdots + J'_{k_0-1}.\end{aligned}$$

Similar argument shows

$$I_k(x) \le C(1+(k_0-k))\|f\|_{\mathrm{BMO}}.$$

Collecting all, we have reached

$$\begin{aligned}\|Tf_3(x) - Tf_3(x_0)\|_H &\le C\sum_{k=1}^{\infty}c_k(1+|k-k_0|)\|f\|_{\mathrm{BMO}} \\ &\le C\left(\sum_{k=1}^{\infty}kc_k + k_0\sum_{k=1}^{\infty}c_k\right)\|f\|_{\mathrm{BMO}}.\end{aligned}$$

Bearing in mind that $k_0 \sim \log_2\frac{d}{d'} = \log_2\frac{d}{|x-x_0|}$, we get

$$\frac{1}{|B|}\int_B \log_2\frac{d}{|x-x_0|}\mathrm{d}x = C\int_{B(0,1)}\log_2\frac{1}{|z|}\mathrm{d}z = C_n.$$

And consequently,

$$\frac{1}{|B|}\int_B\|Tf_3(x) - Tf_3(x_0)\|_H < C\sum_{k=1}^{\infty}kc_k\|f\|_{\mathrm{BMO}}. \tag{4}$$

Finally we show $\|Tf(x)\|_{\mathrm{BMO}(H)} \le C\|f\|_{\mathrm{BMO}}$. Take any ball $B \subset \mathbf{R}^n$. Since now $Tf(x) < +\infty$ almost everywhere in $\mathbf{R}^n$, we can find an x_0 sufficiently close to the center of B. Decompose $f = f_1 + f_2 + f_3$ as above, then we find that (2), (3) and (4) hold by the same argument. Then

$$\begin{aligned} & \frac{1}{|B|}\int_B \|Tf(x) - (Tf)_B\|_H \mathrm{d}x \\ \le\ & \frac{1}{|B|}\int_B \|Tf(x) - Tf_3(x_0)\|_H \mathrm{d}x + \frac{1}{|B|}\int_B \|Tf_3(x_0) - (Tf)_B\|_H \mathrm{d}x \\ \le\ & \frac{2}{|B|}\int_B \|Tf(x) - Tf_3(x_0)\|_H \mathrm{d}x \\ \le\ & \frac{2}{|B|}\int_B \|Tf_2(x)\|_H \mathrm{d}x + \frac{2}{|B|}\int_B \|Tf_3(x) - Tf_3(x_0)\|_H \mathrm{d}x. \end{aligned}$$

By (4), the second term is less than $C_n \sum_{k=1}^{\infty} kc_k \|f\|_{\mathrm{BMO}}$ while by (2) and Hölder's inequality, the first term

$$\frac{1}{|B|}\int_B \|Tf_2(x)\|_H \mathrm{d}x \le \frac{1}{|B|^{\frac{1}{2}}}\left(\int_B \|Tf_2(x)\|_H^2 \mathrm{d}x\right)^{\frac{1}{2}} \le C\|f\|_{\mathrm{BMO}}.$$

3 Two Applications

In this section, we shall discuss two applications of Theorem 1. Or to be exact, we shall reprove the BMO boundedness for g-function and Marcinkiewicz integral in a uniform way.

Take $H = L^2(\mathbf{R}^+, dt/t)$ and $K_1(x) = t^{-1}\dfrac{\Omega(x)}{|x|^{n-1}}\chi_{|x|<t}(x)$, where Ω is homogeneous of degree zero and satisfies

$$\int_{\mathbf{S}^{n-1}} \Omega(x')\mathrm{d}\sigma(x') = 0. \tag{5}$$

Then the Marcinkiewicz integral

$$\mu_\Omega(f)(x) = \|K_1 * f(x)\|_{L^2(\mathbf{R}^+, dt/t)} = \|K_1 * f(x)\|_H.$$

Imposing certain restriction on Ω, we shall check that K_1 verifies the condition of Theorem 1. Thus we get the BMO boundedness for μ_Ω.

Corollary 1. *Let $\Omega \in L^r(\mathbf{R}^{n-1}), r > 1$ satisfy (5) and*

$$\int_0^1 \frac{\omega_r(\delta)}{\delta}\left(1 + \log\frac{1}{\delta}\right)\mathrm{d}\delta < +\infty,$$

where $\omega_r(\delta)$ is the r-module of Ω defined by

$$\omega_r(\delta) = \sup_{|\rho|<\delta}\left(\int_{\mathbf{S}^{n-1}} |\Omega(\rho x') - \Omega(x')|^r \mathrm{d}\sigma(x')\right)^{1/r}$$

and ρ *is a rotation on* $\mathbf{S}^{n-1}$. *If* $f \in BMO$ *and there exists a set* E *with* $|E| > 0$ *such that* $\mu_\Omega(f)(x) < +\infty$, *a.e.* $x \in E$, *then* $\mu_\Omega(f)(x) < \infty$, *a.e.* $x \in \mathbf{R}^n$ *and* $\|\mu_\Omega(f)\|_{\mathrm{BMO}} \le C\|f\|_{\mathrm{BMO}}$.

Proof. It is not hard to check

$$\|\mu_\Omega(f)\|_{\mathrm{BMO}} = \|\|K_1 * f\|_H\|_{\mathrm{BMO}} \le 2\|K_1 * f\|_{\mathrm{BMO}(H)}.$$

So we only have to show that $K_1(x)$ satisfies the requirement of Theorem 1. The pre-L^2 estimate has been proved in many literatures, see for example, [3, 4]. To check (1), we first bound $I(x,y) = \|K_1(x-y) - K_1(x)\|_H$, which can be written as

$$\begin{aligned}
I(x,y) &= \left(\int_0^\infty t^{-3}\left|\chi_B(\frac{x-y}{t})\frac{\Omega(x-y)}{|x-y|^{n-1}} - \chi_B(\frac{x}{t})\frac{\Omega(x)}{|x|^{n-1}}\right|^2 \mathrm{d}t\right)^{1/2} \\
&\le \left(\int_0^\infty t^{-3}\left|\chi_B(\frac{x-y}{t})\frac{\Omega(x-y)}{|x-y|^{n-1}} - \chi_B(\frac{x-y}{t})\frac{\Omega(x)}{|x-y|^{n-1}}\right|^2 \mathrm{d}t\right)^{1/2} \\
&\quad + \left(\int_0^\infty t^{-3}\left|\chi_B(\frac{x-y}{t})\frac{\Omega(x)}{|x-y|^{n-1}} - \chi_B(\frac{x}{t})\frac{\Omega(x)}{|x-y|^{n-1}}\right|^2 \mathrm{d}t\right)^{1/2} \\
&\quad + \left(\int_0^\infty t^{-3}\left|\chi_B(\frac{x}{t})\frac{\Omega(x)}{|x-y|^{n-1}} - \chi_B(\frac{x}{t})\frac{\Omega(x)}{|x|^{n-1}}\right|^2 \mathrm{d}t\right)^{1/2} \\
&= I_1(x,y) + I_2(x,y) + I_3(x,y).
\end{aligned}$$

By regular calculation (see also [12]), we find

$$I_2(x,y) \le C|\Omega(x)|\frac{|y|}{|x|^{n+1}},\ I_3(x,y) \le C|\Omega(x)|\frac{|y|^{1/2}}{|x|^{n+1/2}}$$

and

$$I_1(x,y) \le C\frac{|\Omega(x-y) - \Omega(x)|}{|x|^n}$$

whenever $|x| > 2|y|$. Thus

$$\begin{aligned}
\left(\int_{S_k(y)} |I_2(x,y)|^r \mathrm{d}x\right)^{1/r} &= C|y|\left(\int_{S_k(y)}\left|\frac{\Omega(x)}{|x|^{n+1}}\right|^r \mathrm{d}x\right)^{1/r} \\
&= C|y|\left(\int_{2^k|y|}^{2^{k+1}|y|}\int_{\mathbf{S}^{n+1}} |\Omega(x')|^r \mathrm{d}\sigma(x')\frac{\mathrm{d}t}{t^{(n+1)r+1-n}}\right)^{1/r} \\
&\le C\|\Omega\|_{L^r}|y|\left(\frac{2^k|y|}{(2^k|y|)^{(n+1)r+1-n}}\right)^{1/r} \\
&= C2^{-k}\frac{1}{(|2^k y|)^{n-n/r}} \sim C2^{-k}|S_k(y)|^{-1/r'}.
\end{aligned}$$

In a similar way

$$\left(\int_{S_k(y)} |I_3(x,y)|^r \mathrm{d}x\right)^{1/r} \le C2^{-k/2}|S_k(y)|^{-1/r'}.$$

For $I_1(x,y)$, we may argue as Lemma 5 of [7] to get

$$\begin{aligned}\left(\int_{S_k(y)} |I_1(x,y)|^r \mathrm{d}x\right)^{1/r} &\le C(2^k|y|)^{n(\frac{1}{r}-1)} \int_{\frac{|y|}{2^k|y}}^{\frac{2|y|}{2^k|y}} \omega_r(\delta)\frac{\mathrm{d}\delta}{\delta} \\ &\le C|S_k(y)|^{-\frac{1}{r'}} \int_{2^{-k}}^{2^{-k+1}} \omega_r(\delta)\frac{\mathrm{d}\delta}{\delta}.\end{aligned}$$

Now it remains to check

$$D_r = \sum kc_k = \sum k \int_{2^{-k}}^{2^{-k+1}} \omega_r(\delta)\frac{\mathrm{d}\delta}{\delta} < +\infty.$$

But since $2^{-k} < \delta < 2^{-k+1}$ implies $k < 1 + \log\frac{1}{\delta}$, D_r does not exceed

$$\int_0^1 \frac{\omega_r(\delta)}{\delta}(1+\log\frac{1}{\delta})\mathrm{d}\delta.$$

The same result for μ_Ω is proved in [5] under the hypothesis that $\Omega \in L^r, r > 1$ and

$$\int_0^1 \frac{\omega_r(\delta)}{\delta}(1+\log\frac{1}{\delta})^\sigma \mathrm{d}\delta < +\infty$$

for some $\sigma > 2$. So Corollary 1 provides a slight improvement. However, a more general condition has been found by Hu, Meng and Yang in [6] to get the same result for μ_Ω.

Now let us turn to the classical g-function which is defined by

$$g(f)(x) = \left(\int_0^\infty |\psi_t * f(x)|^2 \frac{\mathrm{d}t}{t}\right)^{1/2},$$

where $\psi_t(x) = t^{-n}\psi(x/t)$ and $\psi(x)$ satisfies

(i) $\psi(x) \in L^1$ and $\int_{\mathbf{R}^n} \psi(x)\mathrm{d}x = 0$;

(ii) $|\psi(x)| + |\triangledown\psi(x)| \le C\frac{1}{(1+|x|)^{n+1}}$.

If we take $K_2(x) = \psi_t(x)$ and $H = L^2(\mathbf{R}^+, \mathrm{d}t/t)$, then

$$g(f)(x) = \|K_2 * f(x)\|_H = \|Tf(x)\|_H.$$

Again by Theorem 1, we can obtain the BMO boundedness of g if K_2 verifies the required condition.

The case is easier than that of μ_Ω, since (i) and (ii) already imply

$$\|K_2(x-y) - K_2(x)\|_H \le C\frac{|y|}{|x|^{n+1}}, \quad |x| > 2|y|,$$

see [9], p. 28. Arguing as we did for $I_2(x,y)$ in the proof of Theorem 2, we get

$$\left(\int_{S_k(y)} \|K_2(x-y)-K_2(x)\|_H^r \mathrm{d}x\right)^{1/r} \leq c_k|S_k(y)|^{\frac{1}{r}-1}$$

Thus we have proved the following corollary which is first obtained in Wang's work[11].

Corollary 2. Let $g(f)(x)$ be defined as above. If $f \in \mathrm{BMO}$ and there exists a set E with $|E|>0$ such that

$$g(f)(x) < +\infty,$$

a.e. $x \in E$, then $g(f)$ exists almost everywhere in $\mathbf{R}^n$ and furthermore, $\|g(f)\|_{\mathrm{BMO}} \leq C\|f\|_{\mathrm{BMO}}$.

Finally, let us point out that the BMO-boundedness of singular integral and some Littlewood-Paley operators was extended to Campanato-type spaces by many authors. The latest development along this direction belongs to Zhang and Tao (see [13, 10]) who considered three types of Littlewood-Paley functions and proved their boundedness on generalized Orlicz-Campanato spaces. The Banach space valued singular integrals discussed in this paper should also be bounded in generalized Orlicz-Campanato spaces and furthermore, using the techniques there, we can only assume that $\|Tf\| < \infty$ for one point rather than a measurable set E of $|E|>0$ in our main theorem.

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School of Science
Hangzhou Dianzi University
Hangzhou, 310018
P. R. China

E-mail: purezhang@hdu.edu.cn