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THE BOUNDEDNESS OF TOEPLITZ-TYPE OPERATORS ON VANISHING-MORREY SPACES

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Abstract. In this note, we prove that the Toeplitz-type Operator Θ_α^b generated by the generalized fractional integral, Calderón-Zygmund operator and VMO funtion is bounded from $L^{p,\lambda}(R^n)$ to $L^{q,\mu}(R^n)$. We also show that under some conditions $\Theta_\alpha^b f \in VL^{q,\mu}(B_R)$, the vanishing-Morrey space.

Key words: *Toeplitz-type operator, generalized fractional integral, vanishing-Morrey space*

AMS (2010) subject classification: 42B25, 42B30

1 Introduction and Main Result

Suppose that L is a linear operator on $L^2(\mathbf{R}^n)$, which generates an analytic semigroup e^{-tL} with a kernel $p_t(x,y)$ satisfying a Gaussian kernel bound, that is,

$$|p_t(x,y)| \leq \frac{C}{t^{\frac{n}{2}}} e^{-c\frac{|x-y|^2}{t}}, \tag{1.1}$$

for $x,y \in \mathbf{R}^n$ and all $t > 0$.

For $0 < \alpha < n$, the generalized fractional integral $L^{-\alpha/2}$ generated by the operator L is defined by

$$L^{-\alpha/2}f(x) = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty e^{-tL}(f) \frac{\mathrm{d}t}{t^{-\alpha/2+1}}(x). \tag{1.2}$$

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When $L = \triangle$ is the Laplacian operator on $\mathbf{R}^n$, $L^{-\alpha/2}$ is the classical fractional integral I_α, for example see [1], which is given by

$$I_\alpha f(x) = \frac{\Gamma((n-\alpha)/2)}{\pi^{n/2}2^\alpha\Gamma(\alpha/2)}\int_{R^n}\frac{f(y)}{|x-y|^{n-\alpha}}\mathrm{d}y.$$

In 1982, S. Chanillo[2] showed that for all $0<\alpha<n$ and $b\in \mathrm{BMO}(\mathbf{R}^n)$, the commutator $[b,I_\alpha]$ is bounded from $L^p(\mathbf{R}^n)$ to $L^q(\mathbf{R}^n)$ with $1<p<n/\alpha, 1/q=1/p-\alpha/n$. In 2004, Duong and Yan [3] proved that for all $0<\alpha<n$ and $b\in\mathrm{BMO}$, both $L^{-\alpha/2}$ and the commutator $[b,L^{-\alpha/2}]$ are bounded from $L^p(\mathbf{R}^n)$ to $L^q(\mathbf{R}^n)$, where $1<p<n/\alpha$, $1/q=1/p-\alpha/n$. If $b\in\mathrm{BMO}(\mathbf{R}^n)$, the commutator $T^b f = bTf - T(bf)$, T is a Calderón-Zygmund operator with a standard kernel K , we know that T^b is (L^p,L^p)-boundedness for $1<p<\infty$.

In fact, since the kernel of $L^{-\alpha/2}$ is $K_\alpha(x,y)$ and the kernel of e^{-tL} is $p_t(x,y)$, which satisfies (1.1), we have

$$L^{-\alpha/2}f(x) = \int_{R^n}K_\alpha(x,y)f(y)\mathrm{d}y,$$

thus

$$K_\alpha(x,y) = \frac{1}{\Gamma(\alpha/2)}\int_0^\infty p_t(x,y)\frac{\mathrm{d}t}{t^{-\alpha/2+1}}. \tag{1.3}$$

And using(1.1),

$$|K_\alpha(x,y)| \le C\frac{\Gamma(n/2-\alpha/2)}{\Gamma(\alpha/2)}\frac{1}{|x-y|^{n-\alpha}}, \tag{1.4}$$

for $x\neq y$ and if $|x-z|\ge 2|y-z|$,

$$|K_\alpha(x,y)-K_\alpha(x,z)|+|K_\alpha(y,x)-K_\alpha(z,x)| \le C\frac{\Gamma(n/2-\alpha/2)}{\Gamma(\alpha/2)}\frac{|y-z|}{|x-z|}|x-z|^{\alpha-n}. \tag{1.5}$$

Let $B=B(x,\rho)$ be a ball in R^n of radius ρ at the point x.

Definition 1.1. Given $f\in L^1_{\mathrm{loc}}(\mathbf{R}^n)$, let us set

$$Mf(x) = \sup_{x\in B}\frac{1}{|B|}\int_B|f(y)|\mathrm{d}y, \qquad for \qquad \text{a. e. } x\in\mathbf{R}^n.$$

M is the Hardy-Littlewood maximal operator.

Define the Sharp maximal function by

$$f^\sharp(x) = \sup_{x\in B}\frac{1}{|B|}\int_B|f(y)-f_B|\mathrm{d}y, \qquad for \qquad \text{a. e.} \quad x\in\mathbf{R}^n.$$

Definition 1.2. Let $f\in L^1_{\mathrm{loc}}(\mathbf{R}^n)$ and $0<\eta<1$, we set

$$M_\eta f(x) = \sup_{x\in B}\frac{1}{|B|^{1-\eta}}\int_B|f(y)|\mathrm{d}y, \qquad for \qquad \text{a. e.} \quad x\in\mathbf{R}^n.$$

Definition 1.3. Let $1 \le p < \infty$, $0 \le \lambda < n$. A measurable function $f \in L^p(\mathbf{R}^n)$ belongs to the Morry space $L^{p,\lambda}(\mathbf{R}^n)$ if

$$\|f\|^p_{L^{p,\lambda}(R^n)} = \sup_{x\in R^n,\rho>0} \frac{1}{\rho^\lambda}\int_{B(x,\rho)} |f(y)|^p \mathrm{d}y < \infty.$$

For simplieity, we will denote by $\|f\|_{\rho,\lambda} = \|f\| =_{L^{\rho,\lambda}(\mathbf{R}^n)}$.

Definition 1.4. Let $1 \le p < \infty$, $0 \le \lambda < n$. We say that $f \in L^{p,\lambda}(\mathbf{R}^n)$ belongs to the Vanshing-Morry space $VL^{p,\lambda}(\mathbf{R}^n)$ if the function

$$\zeta^p(r) = \sup_{x\in R^n,\rho\le r} \frac{1}{\rho^\lambda}\int_{B(x,\rho)} |f(y)|^p \mathrm{d}x,$$

satisfies

$$\lim_{r\to 0}\zeta(r) = 0.$$

In a similar way we obtain the definition of $VL^{p,\lambda}(X)$, $X \subset \mathbf{R}^n$ an open set with sufficiently smooth boundary, replacing $\mathbf{R}^n$ by X and the ball $B(x,\rho)$ by $B(x,\rho)\cap X$.

Definition 1.5. Let f be a locally integral function defined on $\mathbf{R}^n$. We say that f is in the space BMO($\mathbf{R}^n$) if

$$\|f\|_* = \sup_{x_0\in \mathbf{R}^n,\rho>0} \frac{1}{|B(x_0,\rho)|}\int_{B(x_0,\rho)} |f(y)-f_B|\mathrm{d}y < \infty,$$

where $B = B(x_0,\rho)$, $f_B = \dfrac{1}{|B|}\displaystyle\int_B f(y)\mathrm{d}y$.

Let $f \in$BMO($\mathbf{R}^{n+1}$) and $r > 0$. We define the VMO modulus of f by the rule

$$\eta(r) = \sup_{\rho\le r}\frac{1}{|B_\rho|}\int_{B_\rho} |f(y)-f_{B_\rho}|\mathrm{d}y,$$

where B_ρ is a generic ball having radius ρ.

BMO is a Banach space with the norm $\|f\|_* = \sup\limits_{r>0}\eta(r)$.

Definition 1.6. We say that a function $f \in$BMO($\mathbf{R}^n$) is in the Sarason class VMO($\mathbf{R}^n$) if

$$\lim_{r\to 0}\eta(r) = 0.$$

Next we examine an important class of operators in analysis, Calderón-Zygmund operators. We say that a function $K(x,y)$ defined away from the diagonal of $\mathbf{R}^n\times\mathbf{R}^n$ is of standard kernel, if it satisfies the size condition

$$|K(x,y)| \le C|x-y|^{-n}, \tag{1.6}$$

and the regularity condition

$$|K(x,y+h)-K(x,y)|+|K(x+h,y)-K(x,y)|\leq C\frac{|h|^{\delta}}{|x-y|^{n+\delta}}, \tag{1.7}$$

for some $0<\delta\leq 1$, whenever $|x-y|\geq 2|h|$.

Definition 1.7. An operator T is a Calderón-Zygmund operator, if T is bounded on $L^q(\mathbf{R}^n)$ for some $1<q<\infty$ and is associated with a standard kernel $K(x,y)$, in the sense that

$$Tf(x)=\int_{\mathbf{R}^n}K(x,y)f(y)\mathrm{d}y,$$

whenever $f\in L^q(\mathbf{R}^n)$ has compact support and x is not in the support of f.

Some of the main examples of Calderón-Zygmund operators are ones that are given as convolution, $K(x,y)=K(x-y)$, where k is locally integrable away from zero and satisfies the corresponding estimates (1.6) and (1.7).

Moreover, let $b\in \mathrm{VMO}(\mathbf{R}^n)$, M_b a multiplication operator, $T_{j,1}$ a Calderón-Zygmund operator[4] with a standard kernel K or $T_{j,1}=I$, $T_{j,3}=\pm I$, $T_{j,2}$ and $T_{j,4}$ linear operators bounded on $L^{p,\lambda}(\mathbf{R}^n)(1<p<\infty,\ 0<\lambda<n)$, where $j=1,2,\cdots,m$ and I is the identity operator. Define the Toeplitz-type operator related to the generalized fractional integral $L^{-\alpha/2}$ and Calderon-Zygmund operator with a standard kernel by

$$\Theta_{\alpha}^{b}f=\sum_{j=1}^{m}(T_{j,1}M_bL^{-\alpha/2}T_{j,2}+T_{j,3}L^{-\alpha/2}M_bT_{j,4}).$$

It is easy to see that when $m=1$, $T_{j,1}=T_{j,2}=-T_{j,3}=T_{j,4}=I$, $\Theta_{\alpha}^{b}f=[b,L^{-\alpha/2}]f$. We will discuss the boundedness of Toeplitz-type Operator on Morrey spaces and that on vanishing-Morrey spaces.

We can formulate our results as follows:

Theorem 1.1. *Assume that the condition (1.1) hods. Let* $b\in\mathrm{VMO}(\mathbf{R}^n)$ *,* $T_{j,i}(j=1,2,\cdots,m,i=1,2,3,4)$ *and* $\Theta_{\alpha}^{b}f$ *be defined as above. If for any* $f\in L^{p,\lambda}(\mathbf{R}^n)$ $(1<p<\infty,0<\lambda<n)$ *, we have* $\Theta_{\alpha}^{1}f=0$ *, then there exists a constant* $c>0$ *such that*

$$\|\Theta_{\alpha}^{b}f\|_{L^{q,\mu}(\mathbf{R}^n)}\leq c\|L^{-\alpha/2}\|\left(\left(\sum_{j=1}^{m}\|T_{j,1}\|\right)\left(\sum_{j=1}^{m}\|T_{j,2}\|\right)+\sum_{j=1}^{m}\|T_{j,4}\|\right)\|b\|_{*}\|f\|_{L^{p,\lambda}(\mathbf{R}^n)},$$

where $0<\beta<n$, $1<p<\frac{n}{\beta}$, $0<\lambda<n-\sqrt{n\beta p}$, $q>0$, $\frac{1}{q}=\frac{1}{p}-\frac{\beta}{n-\lambda}$ *and*

$$\mu=\frac{\lambda(n-\lambda)}{n-\lambda-\beta p}\quad (i.\ e.\quad \frac{\lambda}{p}=\frac{\mu}{q}).$$

Theorem 1.2. *Let* $0<\beta<n$*,* $1<p<\dfrac{n}{\beta}$*,* $0<\lambda<n-\sqrt{n\beta p}$ *,* $q>0$*,* $\dfrac{1}{q}=\dfrac{1}{p}-\dfrac{\beta}{n-\lambda}$ *and* $\mu=\dfrac{\lambda(n-\lambda)}{n-\lambda-\beta p}$*(i. e.*$\dfrac{\lambda}{p}=\frac{\mu}{q}$*). Suppose b is in* $\mathrm{VMO}(\mathbf{R}^n)$*, if for any* $f\in L^{p,\lambda}(\mathbf{R}^n)$ *, we have* $\Theta^1_\alpha f=0$*.*

Then $\exists\rho_0>0$, $\forall\mathbf{R}<\rho_0$, we have $\Theta^b_\alpha f\in VL^{q,\mu}(B_R)$.

2 Proof of Theorems

To prove our theorems, we need the following lemmas.

Lemma 2.1[5]. *For* $0<\alpha<n$*, let* $L^{-\alpha/2}$ *and* I_α *be defined as above, then there exists a constant* $C>0$ *such that*

$$|L^{-\alpha/2}f|\le C_\alpha I_\alpha(|f|)(x).$$

Lemma 2.2. *Let* $1<p<\infty$ *and* $0\le\lambda<n$*. Then there exists a constant* $c>0$ *independent of f, such that*

$$\|Mf\|_{p,\lambda}\le c\|f^\sharp\|_{p,\lambda},$$

for every $f\in L^{p,\lambda}(\mathbf{R}^n)$*.*

Lemma 2.3(John-Nirenberg)[6]. *For* $1\le p<\infty$*,* $f\in L_{loc}(\mathbf{R}^n)$*, let*

$$\|f\|_{p,*}=\sup_B\left(\frac{1}{|B|}\int_B|f(y)-f_B|^p\mathrm{d}y\right)^{\frac{1}{p}},$$

then $\|f\|_*$ *and* $\|f\|_{p,*}$ *are equivalent.*

Lemma 2.4[7]. *Let* $0<\beta<n$*,* $1<p<\frac{n}{\beta}$*,* $0<\lambda<n-\beta p$ *. Set* $\frac{1}{q}=\frac{1}{p}-\frac{\beta}{n-\lambda}$ *and* $\mu=\frac{\lambda(n-\lambda)}{n-\lambda-\beta p}$(i. e. $\frac{\lambda}{p}=\frac{\mu}{q}$)*. Then there exists a constant* $c>0$ *independent of f such that*

$$\|I_{\frac{\beta n}{n-\lambda}}\|_{q,\mu}\le c\|f\|_{p,\lambda},\qquad\forall f\in L^{p,\lambda}(\mathbf{R}^n).$$

Lemma 2.5[7]. *Let* $0<\beta<n$*,* $1<p<+\infty$*,* $0<\lambda<n-\beta p$ *, then for* $\frac{1}{q}=\frac{1}{p}-\frac{\beta}{n-\lambda}$ *and* $\mu=\frac{\lambda\cdot q}{p}$ *, there exists* $c>0$ *independent on f such that*

$$\|M_{\frac{\beta}{n-\lambda}}f\|_{q,\mu}\le c\|f\|_{p,\lambda},\qquad\forall f\in L^{p,\lambda}(\mathbf{R}^n).$$

Proof of Theorem 1.1. We first prove Theorem 1.1 , for $0<\alpha<n$, choose $1<st,s_1t_1,\gamma,\gamma_1<p$. We will prove that there exists a constant $c>0$ such that for all $x\in\mathbf{R}^n$ and for all $x\in B$,

$$\begin{aligned}(\Theta^b_\alpha f)^\sharp(x)\quad\le\quad & c\sum_{j=1}^m\|b\|_*\left(M|L^{-\alpha/2}T_{j,2}f|^{st}\right)^{\frac{1}{st}}(x)+c\sum_{j=1}^m\|b\|_*\left(M_{\frac{\alpha s_1t_1}{n}}|T_{j,4}f|^{s_1t_1}\right)^{\frac{1}{s_1t_1}}(x)\\ & +c\sum_{j=1}^m\|b\|_*\left(M|L^{-\alpha/2}T_{j,2}f|^{\gamma}\right)^{\frac{1}{\gamma}}(x)+c\sum_{j=1}^m\|b\|_*\left(M_{\frac{\alpha\gamma_1}{n}}|T_{j,4}f|^{\gamma_1}\right)^{\frac{1}{\gamma_1}}(x).\end{aligned}$$

(2.1)

By Lemma 2.2, we have

$$\|\Theta_\alpha^b f\|_{q,\mu} \le \|M(\Theta_\alpha^b f)\|_{q,\mu} \le C\|(\Theta_\alpha^b f)^\sharp\|_{q,\mu}.$$

Therefore, in order to prove Theorem 1.1 , we only need to establish the $(L^{p,\lambda}, L^{q,\mu})$ boundedness of $(\Theta_\alpha^b f)^\sharp$. We now prove (2.1) , for any $x \in \mathbf{R}^n$, choose $B = B(x,r)$ such that $x \in B$. Let $\chi^1 = \chi_{2B}$ and $\chi^2 = 1 - \chi^1$. For any $f \in L^{p,\lambda}(\mathbf{R}^n)$, where $1 < p < \infty$ and $0 < \lambda < n$, since $\Theta_\alpha^1 f = 0$, then $\Theta_\alpha^{b_B} f = 0$. We write

$$\Theta_\alpha^b f = \Theta_\alpha^{(b-b_B)} f = \Theta_\alpha^{(b-b_B)\chi^1} f + \Theta_\alpha^{(b-b_B)\chi^2} f = f_1 + f_2,$$

where

$$f_1 = \sum_{j=1}^m T_{j,1}(b-b_B)\chi^1 L^{-\alpha/2} T_{j,2} f + \sum_{j=1}^m T_{j,3} L^{-\alpha/2}(b-b_B)\chi^1 T_{j,4} f = f_{11} + f_{12},$$

$$f_2 = \sum_{j=1}^m T_{j,1}(b-b_B)\chi^2 L^{-\alpha/2} T_{j,2} f + \sum_{j=1}^m T_{j,3} L^{-\alpha/2}(b-b_B)\chi^2 T_{j,4} f = f_{21} + f_{22}.$$

First we estimate f_{11}, taking $1 < s < p < \infty$, $1 < t < \infty$ such that $1 < st < p$, by Hölder's inequality , the $L^s(\mathbf{R}^n)$-boundedness of $T_{j,1}$ and Lemma 2.3 we have

$$\begin{aligned}
\left(\frac{1}{|B|}\int_B |f_{11}(y)|^s \mathrm{d}y\right)^{\frac{1}{s}} &\le \sum_{j=1}^m |B|^{-1/s}\left(\int_B |T_{j,1}(b(y)-b_B)\chi^1 L^{-\alpha/2} T_{j,2} f(y)|^s \mathrm{d}y\right)^{\frac{1}{s}} \\
&\le \sum_{j=1}^m |B|^{-1/s}\left(\int_{2B} |(b(y)-b_B)|^s |L^{-\alpha/2} T_{j,2} f(y)|^s \mathrm{d}y\right)^{\frac{1}{s}} \\
&\le c\sum_{j=1}^m \left(\frac{1}{|2B|}\int_{2B} |b(y)-b_B|^{st'} \mathrm{d}y\right)^{\frac{1}{st'}} \left(\frac{1}{|2B|}\int_{2B} |L^{-\alpha/2} T_{j,2} f(y)|^{st} \mathrm{d}y\right)^{\frac{1}{st}} \\
&\le c\sum_{j=1}^m \|b\|_* \left(M|L^{-\alpha/2} T_{j,2} f|^{st}\right)^{\frac{1}{st}}(x).
\end{aligned}$$

Therefore

$$\frac{1}{|B|}\int_B |f_{11}(y)|\mathrm{d}y \le \left(\frac{1}{|B|}\int_B |f_{11}(y)|^s \mathrm{d}y\right)^{\frac{1}{s}} \le c\sum_{j=1}^m \|b\|_* \left(M|L^{-\alpha/2} T_{j,2} f|^{st}\right)^{\frac{1}{st}}(x). \qquad (2.2)$$

Let us now estimate f_{12}. Taking $1 < s_1 < n/\alpha$ and $1 < t_1 < \infty$ such that $1/s_0 = 1/s_1 - \alpha/n$ and $1 < s_1 t_1 < p < \infty$, by Hölder's inequality, the (L^{s_1}, L^{s_0})-boundedness of $L^{-\alpha/2}$ and Lemma

2.3, we have

$$
\begin{aligned}
\left(\frac{1}{|B|}\int_B |f_{12}(y)|^{s_0}\mathrm{d}y\right)^{\frac{1}{s_0}} &\le \sum_{j=1}^{m}|B|^{-\frac{1}{s_0}}\left(\int_B |L^{-\alpha/2}(b(y)-b_B)\chi^1 T_{j,4}f(y)|^{s_0}\mathrm{d}y\right)^{\frac{1}{s_0}}\\
&\le \sum_{j=1}^{m}|B|^{-\frac{1}{s_0}}\left(\int_{2B}|(b(y)-b_B)T_{j,4}f(y)|^{s_1}\mathrm{d}y\right)^{\frac{1}{s_1}}\\
&\le \sum_{j=1}^{m}\left(\frac{1}{|2B|}\int_{2B}|b(y)-b_B|^{s_1t_1'}\mathrm{d}y\right)^{\frac{1}{s_1t_1'}}\left(\frac{1}{|2B|^{1-\frac{\alpha s_1t_1}{n}}}\int_{2B}|T_{j,4}f(y)|^{s_1t_1}\mathrm{d}y\right)^{\frac{1}{s_1t_1}}\\
&\le c\sum_{j=1}^{m}\|b\|_*\left(M_{\frac{\alpha s_1t_1}{n}}|T_{j,4}f|^{s_1t_1}\right)^{\frac{1}{s_1t_1}}(x).
\end{aligned}
$$

Thus

$$
\frac{1}{|B|}\int_B |f_{12}(y)|dy \le \left(\frac{1}{|B|}\int_B |f_{12}(y)|^{s_0}\mathrm{d}y\right)^{\frac{1}{s_0}} \le c\sum_{j=1}^{m}\|b\|_*\left(M_{\frac{\alpha s_1t_1}{n}}|T_{j,4}f|^{s_1t_1}\right)^{\frac{1}{s_1t_1}}(x). \tag{2.3}
$$

Next we deal with f_{21} , f_{22} respectively. Let T be a Calderón-Zygmund operator[8] with a standard kernel K, then when $y\in B(x,r)$, for any $1<\gamma<p$ we have

$$
|T[(b-b_B)\chi^2 f](y)-T[(b-b_B)\chi^2 f](x)| \le c\|b\|_*(M|f|^{\gamma})^{\frac{1}{\gamma}}(x). \tag{2.4}
$$

In fact,

$$
\begin{aligned}
&|T[(b-b_B)\chi^2 f](y)-T[(b-b_B)\chi^2 f](x)|\\
&= \left|\int_{R^n}(b(z)-b_B)f(z)\chi^2(z)(K(y,z)-K(x,z))\mathrm{d}z\right|\\
&\le \int_{R^n\setminus 2B}|b(z)-b_B||f(z)||K(y,z)-K(x,z)|\mathrm{d}z\\
&\le c\sum_{j=1}^{\infty}\int_{2^{j+1}B\setminus 2^jB}\frac{|y-x|^{\delta}}{|x-z|^{n+\delta}}|b(z)-b_B||f(z)|\mathrm{d}z\\
&\le c\sum_{j=1}^{\infty}2^{-j\delta}\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}|b(z)-b_B||f(z)|\mathrm{d}z\\
&\le c\sum_{j=1}^{\infty}2^{-j\delta}\left(\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}|b(z)-b_B|^{\gamma'}\mathrm{d}z\right)^{\frac{1}{\gamma'}}\left(\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}|f(z)|^{\gamma}\mathrm{d}z\right)^{\frac{1}{\gamma}}\\
&\le c\sum_{j=1}^{\infty}2^{-j\delta}j\|b\|_*(M|f|^{\gamma})^{\frac{1}{\gamma}}(x)\\
&\le c\|b\|_*(M|f|^{\gamma})^{\frac{1}{\gamma}}(x).
\end{aligned}
$$

Thus the estimates complete the proof of (2.4).

Let us get back to the proof of Theorem . Let $T_{j,1}(j=1,2,\cdots,m)$ be the identity operator. Since $y \in B(x,r)$, then $\chi^2(y)=0$, so we have

$$T_{j,1}(b-b_B)\chi^2 L^{-\alpha/2}T_{j,2}f(y)=0.$$

Since $T_{j,1}(j=1,2,\cdots,m)$ is a Calderón-Zygmund operator with a standard kernel K, and

$$\frac{1}{|B|}\int_B |f_{21}(y)-(f_{21})_B|\mathrm{d}y \le \frac{2}{|B|}\int_B |f_{21}(y)-f_{21}(x)|\mathrm{d}y,$$

for any $1<\gamma<p$, we obtain by (2.4)

$$\frac{1}{|B|}\int_B |f_{21}(y)-(f_{21})_B|\mathrm{d}y \le c\sum_{j=1}^{m}\|b\|_*\left(M|L^{-\alpha/2}T_{j,2}f|^\gamma\right)^{\frac{1}{\gamma}}(x). \tag{2.5}$$

We have showed that whether $T_{j,1}(j=1,2,\cdots,m)$ is the identity operator or the C-Z operator with a standard kernel K , the estimates on f_{21} complete the proof of (2.5).

To estimate f_{22} , we need the following lemma.

Lemma 2.6. *Let $L^{-\alpha/2}$ be the generalized fractional integral, then when $y \in B(x,r)$,*

$$|L^{-\alpha/2}[(b-b_B)\chi^2 g](y)-L^{-\alpha/2}[(b-b_B)\chi^2 g](x)| \le c\|b\|_*\left(M_{\frac{\alpha\gamma}{n}}|g|^\gamma\right)^{\frac{1}{\gamma}}(x), \tag{2.6}$$

where $0<\alpha\gamma<n$ and $1<\gamma<p$.

Proof.

$$\begin{aligned}
&|L^{-\alpha/2}(b-b_B)\chi^2 g](y)-L^{-\alpha/2}[(b-b_B)\chi^2 g](x)|\\
&= \left|\int_{R^n}(b(z)-b_B)g(z)\chi^2(z)(K_\alpha(y,z)-K_\alpha(x,z))\mathrm{d}z\right|\\
&\le \int_{R^n\setminus 2B}|b(z)-b_B||g(z)||K_\alpha(y,z)-K_\alpha(x,z)|\mathrm{d}z\\
&\le \sum_{j=1}^{\infty}\int_{2^{j+1}B\setminus 2^jB}|b(z)-b_B||g(z)||K_\alpha(y,z)-K_\alpha(x,z)|\mathrm{d}z\\
&\le c\sum_{j=1}^{\infty}\int_{2^{j+1}B\setminus 2^jB}\frac{1}{|x-z|^{n-\alpha}}\frac{|y-x|}{|x-z|}|b(z)-b_B||g(z)|\mathrm{d}z\\
&\le c\sum_{j=1}^{\infty}2^{-j}\frac{1}{|2^{j+1}B|^{1-\frac{\alpha}{n}}}\int_{2^{j+1}B}|b(z)-b_B||g(z)|\mathrm{d}z\\
&\le c\sum_{j=1}^{\infty}2^{-j}\left(\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}|b(z)-b_B|^{\gamma'}\mathrm{d}z\right)^{\frac{1}{\gamma'}}\left(\frac{1}{|2^{j+1}B|^{1-\frac{\alpha\gamma}{n}}}\int_{2^{j+1}B}|g(z)|^\gamma\mathrm{d}z\right)^{\frac{1}{\gamma}}\\
&\le c\sum_{j=1}^{\infty}2^{-j}j\|b\|_*\left(M_{\frac{\alpha\gamma}{n}}|g|^\gamma\right)^{\frac{1}{\gamma}}(x)\\
&\le c\|b\|_*\left(M_{\frac{\alpha\gamma}{n}}|g|^\gamma\right)^{\frac{1}{\gamma}}(x).
\end{aligned}$$

Then we complete the proof of Lemma 2.6.

We obtain by Lemma 2.6

$$\begin{aligned}\frac{1}{|B|}\int_B |f_{22}(y)-(f_{22})_B|\mathrm{d}y &\le \frac{2}{|B|}\int_B |f_{22}(y)-f_{22}(x)|\mathrm{d}y\\ &\le c\sum_{j=1}^m \|b\|_* \frac{1}{|B|}\int_B \left(M_{\frac{\alpha\gamma_1}{n}}|T_{j,4}f|^{\gamma_1}\right)^{\frac{1}{\gamma_1}}(x)\mathrm{d}y\\ &\le c\sum_{j=1}^m \|b\|_* \left(M_{\frac{\alpha\gamma_1}{n}}|T_{j,4}f|^{\gamma_1}\right)^{\frac{1}{\gamma_1}}(x),\end{aligned} \tag{2.7}$$

where $0<\alpha\gamma_1<n$ and $1<\gamma_1<p$.

Combining (2.2), (2.3), (2.5) and (2.7), there exists $1<s,s_0,s_1,t,t_1,\gamma,\gamma_1<\infty$, such that $1/s_0=1/s_1-\alpha/n$, $1<s_1t_1<p<n/\alpha$, $1<st<p$, $0<\alpha\gamma_1<n$ and $1<\gamma_1,\gamma<p$, we obtain (2.1).

For $\alpha=\frac{\beta n}{n-\lambda}$, $1<st,s_1t_1<p$ and $1<\gamma,\gamma_1<\min(p,\frac{n-\lambda}{\beta})$, we have

$$\begin{aligned}\|(\Theta_\alpha^b f)^\sharp\|_{q,\mu} \le\ & c\sum_{j=1}^m \|b\|_*\|\left(M|L^{-\frac{\beta n}{2(n-\lambda)}}T_{j,2}f|^{st}\right)^{\frac{1}{st}}\|_{q,\mu}+c\sum_{j=1}^m \|b\|_*\|\left(M_{\frac{\beta s_1t_1}{n-\lambda}}|T_{j,4}f|^{s_1t_1}\right)^{\frac{1}{s_1t_1}}\|_{q,\mu}\\ &+c\sum_{j=1}^m \|b\|_*\|\left(M|L^{-\frac{\beta n}{2(n-\lambda)}}T_{j,2}f|^{\gamma}\right)^{\frac{1}{\gamma}}\|_{q,\mu}+c\sum_{j=1}^m \|b\|_*\|\left(M_{\frac{\beta\gamma_1}{n-\lambda}}|T_{j,4}f|^{\gamma_1}\right)^{\frac{1}{\gamma_1}}\|_{q,\mu}.\end{aligned}$$

So, from Lemma 1.1, Lemma 2.4 and Lemma 2.5, by the $(L^{p,\lambda},L^{q,\mu})$-boundedness of $L^{-\alpha/2}$ and the $L^{p,\lambda}(\mathbf{R}^n)$-boundedness of $T_{j,2}$ and $T_{j,4}$, we get the desired result. Then we complete the proof of Theorem 1.1.

Proof of Theorem 1.2. Let B be a generic ball in $\mathbf{R}^n$, for $\alpha=\frac{\beta n}{n-\lambda}$, from Theorem 1.1 it follows that

$$\begin{aligned}&\left(\sup_{x\in B,\rho>0}\frac{1}{\rho^\mu}\int_{B(x,\rho)\cap B}|\Theta_\alpha^b f(y)|^q\mathrm{d}y\right)^{\frac{1}{q}}\\ \le\ &\left(\sup_{x\in\mathbf{R}^n,\rho>0}\frac{1}{\rho^\mu}\int_{B(x,\rho)}|\Theta_\alpha^b f(y)|^q\mathrm{d}y\right)^{\frac{1}{q}}\\ =\ &\|\Theta_\alpha^b f\|_{L^{q,\mu}(\mathbf{R}^n)}\le c\|b\|_*\|f\|_{L^{p,\lambda}\mathbf{R}R^n)}.\end{aligned}$$

For all B, we also have

$$\|\Theta_\alpha^b f\|_{L^{q,\mu}(B)}\le c\|b\|_*\|f\|_{L^{p,\lambda}(\mathbf{R}^n)}.$$

We observe, likewise Theorem 2.13 in [9], that for any $\varepsilon > 0$, $\exists \rho_0 > 0$ such that for any generic ball $B_R = B(x,R)$ with radius R, such that $0 < R < \rho_0$,

$$\|\Theta_\alpha^b f\|_{L^{q,\mu}(B_R)} \le c \cdot \varepsilon \cdot \|f\|_{L^{p,\lambda}(\mathbf{R}^n)},$$

then

$$\sup_{x \in B_R, 0<\rho<\mathrm{diam} B_R} \frac{1}{\rho^\mu} \int_{B(x,\rho)\cap B_R} |\Theta_\alpha^b f(y)|^q \mathrm{d}y \le c \cdot \varepsilon, \ \forall \varepsilon > 0.$$

Since we are interested in $\lim_{r\to 0} \zeta(r)$, let us now consider only $r < \mathrm{diam} B_R$, then

$$\begin{aligned} \zeta^q(r) &= \sup_{x\in B_R, \rho<r} \frac{1}{\rho^\mu} \int_{B(x,\rho)\cap B_R} |\Theta_\alpha^b f(y)|^q \mathrm{d}y \\ &\le \sup_{x \in B_R, 0<\rho<\mathrm{diam} B_R} \frac{1}{\rho^\mu} \int_{B(x,\rho)\cap B_R} |\Theta_\alpha^b f(y)|^q \mathrm{d}y \\ &\le c \cdot \varepsilon, \ \forall \varepsilon > 0. \end{aligned}$$

It follows that

$$\zeta^q(r) \le c \cdot \varepsilon, \forall r < \mathrm{diam}\ B_R, \qquad \forall \varepsilon > 0,$$

then

$$\lim_{r\to 0} \zeta(r) = 0.$$

We have prove that

$$\Theta_\alpha^b f \in VL^{q,\mu}(B_R), \ \forall R < \rho_0.$$

Then we complete the proof of Theorem 1.2.

References

[1] Stein, E. M., Singular Integrals and Differentiability Properties of Functions, Princeton University Press, Princeton N.J.1970.

[2] Chanillo, S., A Note on Commutators, Indiana Univ.Math.J., 44(1982), 7-16.

[3] Duong, X. T. and Yan, L. X., On Commutators of Fractional Integral, Proc. Amer. Math.Soc., 132(2004), 3549-3557.

[4] Christ, M., Lectures on Singular Integral Operator, CBMS Regional Conference Series in Mathematics, 1990, 77.

[5] Mo, H. X. and Lu, S. Z., Boundedness of Multilinear Commutators of Generalized Fractional Integrals, Math. Nachr., 9:281(2008), 1328-1340.

[6] Zhou, M. Q., Harmonic Analysis, Beijing: Beijing University Press, 2003.

[7] Ragusa, M. A., Commutators of Fraction Integral Operators on Vanishing-Morrey Spaces, J Glob Optim, 40(2008), 361-368.

[8] Garcia-Cuerva, J., Rubio de Francia J L. Weighted Norm Inequalities and Related Topics, Norm-Holland Math. Stud., 1985, 116, Amsterdam.

[9] Chiarenza, F., Frasca, M. and Longo, P., et al. Interior $W^{2,p}$ Estimates for Nondivergence Elliptic Equations with Discontinuous Coefficients, Ricerche di Matematica, 40(1991), 49-168.

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