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ON DOUBLE SINE AND COSINE TRANSFORMS, LIPSCHITZ AND ZYGMUND CLASSES

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Abstract. We consider complex-valued functions $f \in L^1(\mathbf{R}^2_+)$, where $\mathbf{R}_+ := [0,\infty)$, and prove sufficient conditions under which the double sine Fourier transform $\hat{f}_{ss}$ and the double cosine Fourier transform $\hat{f}_{cc}$ belong to one of the two-dimensional Lipschitz classes $\mathrm{Lip}(\alpha,\beta)$ for some $0<\alpha,\ \beta\le 1$; or to one of the Zygmund classes $\mathrm{Zyg}(\alpha,\beta)$ for some $0<\alpha,\ \beta\le 2$. These sufficient conditions are best possible in the sense that they are also necessary for nonnegative-valued functions $f \in L^1(\mathbf{R}^2_+)$.

Key words: *double sine and cosine Fourier transform, Lipschitz class* $\mathrm{Lip}(\alpha,\beta)$*,* $0<\alpha,$ $\beta\le 1$*, Zygmund class* $\mathrm{Zyg}(\alpha,\beta)$*,* $0<\alpha,\ \beta\le 2$.

AMS (2010) subject classification: 42B10, 26A16, 26B35

1 Known Results: Single Sine and Cosine Transforms

We consider complex-valued functions $f:\mathbf{R}_+\to\mathbf{C}$ that are integrable in Lebesgue sense over $\mathbf{R}_+ := [0,\infty)$, in symbol: $f\in L^1(\mathbf{R}_+)$. We recall that the sine (Fourier) transform of f is defined by

$$\hat{f}_s(u) := \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin ux\,\mathrm{d}x,$$

while the cosine (Fourier) transform of f is defined by

$$\hat{f}_c(u) := \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos ux\,\mathrm{d}x, \qquad u\in\mathbf{R}.$$

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Both $\hat{f}_s$ and $\hat{f}_c$ are uniformly continuous on $\mathbf{R}$ and vanish at infinity. For details, we refer to [6, Ch. 1].

In the cases when we do not distinguish between $\hat{f}_s$ and $\hat{f}_c$, we simply use the notation $\hat{f}$. We recall that $\hat{f}$ is said to satisfy the Lipschitz condition of order $\alpha > 0$, in symbol: $\hat{f} \in \mathrm{Lip}(\alpha)$, if

$$|\hat{f}(u+h) - \hat{f}(u)| \le Ch^{\alpha} \quad \text{for all} \quad u \in \mathbf{R} \quad \text{and} \quad h > 0,$$

where the constant C does not depend on u and h. Furthermore, $\hat{f}$ is said to satisfy the Zygmund condition of order $\alpha > 0$, in symbol: $\hat{f} \in \mathrm{Zyg}(\alpha)$, if

$$|\hat{f}(u+h) - 2\hat{f}(u) + \hat{f}(u-h)| \le Ch^{\alpha} \quad \text{for all} \quad u \in \mathbf{R} \quad \text{and} \quad h > 0,$$

where the constant C does not depend on u and h.

It is well known (see, e.g., [1, Ch. 2] or [7, Ch. 2, §3] that if $\hat{f} \in \mathrm{Lip}(\alpha)$ for some $\alpha > 1$, or if $\hat{f} \in \mathrm{Zyg}(\alpha)$ for some $\alpha > 2$, then $\hat{f} \equiv 0$.

The following four theorems were proved in [4] by the second named author of the present paper.

Theorem A. (i) *Let $f : \mathbf{R}_+ \to \mathbf{C}$ be such that $f \in L^1_{\mathrm{loc}}(\mathbf{R}_+)$. If for some $0 < \alpha \le 1$,*

$$\int_0^s x|f(x)| = O(s^{1-\alpha}) \quad \textit{for all} \quad s > 0, \tag{1.1}$$

then $f \in L^1(\mathbf{R}_+)$ and $\hat{f}_s \in \mathrm{Lip}(\alpha)$.

(ii) *Let $f : \mathbf{R}_+ \to \mathbf{R}_+$ be such that $f \in L^1(\mathbf{R}_+)$. If $\hat{f}_s \in \mathrm{Lip}(\alpha)$ for some $0 < \alpha \le 1$, then* (1.1) *holds.*

Theorem B. *In case $0 < \alpha < 1$, Theorem A remains valid when $\hat{f}_s$ is replaced by $\hat{f}_c$.*

Theorem C. (i) *Let $f : \mathbf{R}_+ \to \mathbf{C}$ be such that $f \in L^1_{\mathrm{loc}}(\mathbf{R}_+)$. If for some $0 < \alpha \le 2$,*

$$\int_0^s x^2|f(x)| = O(s^{2-\alpha}) \quad \textit{for all} \quad s > 0, \tag{1.2}$$

then $f \in L^1(\mathbf{R}_+)$ and $\hat{f}_c \in \mathrm{Zyg}(\alpha)$.

(ii) *Let $f : \mathbf{R}_+ \to \mathbf{R}_+$ be such that $f \in L^1(\mathbf{R}_+)$. If $\hat{f}_c \in \mathrm{Zyg}(\alpha)$ for some $0 < \alpha \le 2$, then* (1.2) *holds.*

Theorem D. *In case $0 < \alpha < 2$, Theorem C remains valid when $\hat{f}_c$ is replaced by $\hat{f}_s$.*

Our goal in this paper is to extend these results from single to double sine and cosine transform.

2 New Results: Double Sine and Cosine Transforms

We consider complex-valued functions $f:\mathbf{R}^2_+\to\mathbf{C}$ that are integrable in Lebesgue's sense over $\mathbf{R}^2_+$, in symbol: $f\in L^1(\mathbf{R}^2_+)$. We recall that , the *double sine* (*Fourier*)*transform* of f is defined by

$$\hat{f}_{ss}(u,v):=\frac{2}{\pi}\int_0^\infty\int_0^\infty f(x,y)\sin ux\sin vy\mathrm{d}x\mathrm{d}y, \tag{2.1}$$

while the *doublecosine*(*Fourier*)*transform* is defined by

$$\hat{f}_{cc}(u,v):=\frac{2}{\pi}\int_0^\infty\int_0^\infty f(x,y)\cos ux\cos vy\mathrm{d}x\mathrm{d}y, \quad (u,v)\in\mathbf{R}^2. \tag{2.2}$$

Both $\hat{f}_{ss}(u,v)$ and $\hat{f}_{cc}(u,v)$ are uniformly continuous on $\mathbf{R}^2$ and vanish as $\max\{u,v\}\to\infty$ (see, e.g., [5, Ch. 1]). Clearly, $\hat{f}_{ss}(u,v)$ is odd in each variable, while $\hat{f}_{cc}(u,v)$ is even in each variable.

In the cases when we do not distinguish between $\hat{f}_{ss}$ and $\hat{f}_{cc}$, we simply write $\hat{f}(u,v)$. We recall that $\hat{f}(u,v)$ is said to satisfy the *Lipschitz condition* of order $\alpha>0$ in u, and of order $\beta>0$ in v, in symbol: $\hat{f}\in\mathrm{Lip}(\alpha,\beta)$, if

$$\begin{aligned}|\Delta_{1,1}\hat{f}(u,v;h,k)|:=&|\hat{f}(u+h,v+k)-\hat{f}(u,v+k)\\ &-\hat{f}(u+h,v)+\hat{f}(u,v)|\le Ch^\alpha k^\beta \quad \text{for all}\quad (u,v)\in\mathbf{R}^2\quad\text{and}\quad h,k>0;\end{aligned} \tag{2.3}$$

where the constant C does not depend on u,v,h, and k (see, e.g., [3], where the term "multiplicative Lipschitz class" is used).

Furthermore, we recall that $\hat{f}(u,v)$ is said to satisfy the *Zygmund condition* of order $\alpha>0$ in u, and of order $\beta>0$ in v, in symbols: $\hat{f}\in\mathrm{Zyg}(\alpha,\beta)$, if

$$\begin{aligned}&|\Delta_{2,2}\hat{f}(u,v;h,k)|\\ &\quad=|\hat{f}(u+h,v+k)+\hat{f}(u-h,v+k)+\hat{f}(u+h,v-k)+\hat{f}(u-h,v-k)\\ &\qquad-2f(u+h,v)-2f(u-h,v)-2f(u,v+k)-2f(u,v-k)\\ &\qquad+4f(u,v)|\le Ch^\alpha k^\beta\quad\text{for all}\quad(u,v)\in\mathbf{R}^2\quad\text{and}\quad h,k>0;\end{aligned} \tag{2.4}$$

where the constant C does not depend on u,v,h, and k (see, e.g., [2], where the class $\mathrm{Zyg}(1,1)$ is introduced and denoted by $\Lambda_*(2)$).

Remark 1. We note that

$$\mathrm{Lip}(\alpha,\beta)\subset\mathrm{Zyg}(\alpha,\beta)\quad\text{for all}\quad\alpha,\beta>0,$$

due to the following identity: for all $(u,v) \in \mathbf{R}^2$ and $h,k > 0$, we have

$$\begin{aligned}\Delta_{2,2}\hat{f}(u,v;h,k) &= (\hat{f}(u+h,v+k) - \hat{f}(u,v+k) - \hat{f}(u+h,v) + \hat{f}(u,v)) \\ &+ (\hat{f}(u-h,v+k) - \hat{f}(u,v+k) - \hat{f}(u-h,v) + \hat{f}(u,v)) \\ &+ (\hat{f}(u+h,v-k) - \hat{f}(u,v-k) - \hat{f}(u+h,v) + \hat{f}(u,v)) \\ &+ (\hat{f}(u-h,v-k) - \hat{f}(u,v-k) - \hat{f}(u-h,v) + \hat{f}(u,v)) \\ &= \Delta_{1,1}\hat{f}(u,v;h,k) - \Delta_{1,1}\hat{f}(u-h,v;h,k) \\ &- \Delta_{11}\hat{f}(u,v-k;h,k) + \Delta_{1,1}\hat{f}(u-h,v-k;h,k).\end{aligned}$$

Now, we extend Theorems A-D for double sine and cosine transforms as follows. In Theorems 1-4 below we give the best possible sufficient condition in terms of f under which the double sine transform $\hat{f}_{ss}$ and the double cosine transform $\hat{f}_{cc}$ belong to one of the Lipschitz classes $\mathrm{Lip}(\alpha,\beta)$ for some $0 < \alpha,\beta \le 1$; or to one of the Zygmund classes $\mathrm{Zyg}(\alpha,\beta)$ for some $0 < \alpha,\beta \le 2$. We will prove in Theorems 1-4 that these sufficient conditions are also necessary for nonnegative - valued functions $\hat{f} \in L^1(\mathbf{R}^2_+)$.

Theorem 1. (i) *Let $f : \mathbf{R}^2_+ \to \mathbf{C}$ be such that $f \in L^1_{\mathrm{loc}}(\mathbf{R}^2_+)$. If for some $0 < \alpha,\beta \le 1$,*

$$\int_0^s \int_0^t xy|f(x,y)|\mathrm{d}x\mathrm{d}y = O(s^{1-\alpha}t^{1-\beta}) \quad \textit{for all} \quad s,t > 0, \tag{2.5}$$

then $f \in L^1(\mathbf{R}^2_+)$ and $\hat{f}_{ss} \in \mathrm{Lip}(\alpha,\beta)$.

(ii) *Let $f : \mathbf{R}^2_+ \to \mathbf{R}_+$ be such that $f \in L^1(\mathbf{R}^2_+)$. If $\hat{f}_{ss} \in \mathrm{Lip}(\alpha,\beta)$ for some $0 < \alpha,\beta \le 1$, then (2.5) holds.*

We note that for double sine series with nonnegative coefficients, an analogous theorem was proved in [3, Theorems 1-3] by the first named author.

Theorem 2. *In case $0 < \alpha,\beta < 1$, Theorem 1 remains valid when $\hat{f}_{ss}$ is replaced by $\hat{f}_{cc}$.*

Remark 2. It follows from Lemma 1 in Section 3 below that for $0 < \alpha,\beta < 1$, the condition (2.5) is equivalent to the following one:

$$\int_s^\infty \int_t^\infty |f(x,y)|\mathrm{d}x\mathrm{d}y = O(s^{-\alpha}t^{-\beta}) \quad \text{for all} \quad s,t > 0. \tag{2.6}$$

Theorem 3. (i) *Let $f : \mathbf{R}^2_+ \to \mathbf{C}$ be such that $f \in L^1_{\mathrm{loc}}(\mathbf{R}^2_+)$. If for some $0 < \alpha,\ \beta \le 2$,*

$$\int_0^s \int_0^t x^2y^2|f(x,y)|\mathrm{d}x\mathrm{d}y = O(s^{2-\alpha}t^{2-\beta}) \quad \textit{for all} \quad s,t > 0, \tag{2.7}$$

then $f \in L^1(\mathbf{R}^2_+)$ and $\hat{f}_{cc} \in \mathrm{Zyg}(\alpha,\beta)$.

(ii) *Let* $f:\mathbf{R}_+^2\to\mathbf{R}_+$ *be such that* $f\in L^1(\mathbf{R}_+^2)$. *If* $\hat{f}_{cc}\in \mathrm{Zyg}(\alpha,\beta)$ *for some* $0<\alpha,\beta\le 2$, *then* (2.7) *holds.*

Theorem 4. *In case* $0<\alpha$, $\beta<2$, *Theorem* 3 *remains valid when* $\hat{f}_{cc}$ *is replaced by* $\hat{f}_{ss}$.

We note that for double cosine series with nonnegative coeffients and the Zygmund class Zyg (1,1), an analogous theorem was proved in [2, Theorem 1, where the class Zyg (1,1) is denoted by $\Lambda_*(2)$] by the first named author.

Remark 3. It is obvious that if (2.5) is satisfied for some $0<\alpha$, $\beta\le 1$, then (2.7) is also satisfied. Furthermore, it follows from Lemma 1 in Section 3 that for $0<\alpha$, $\beta<2$, the condition (2.7) is equivalent to the condition (2.6). Consequently, the conditions (2.5) and (2.7) are equivalent for $0<\alpha$, $\beta<1$.

In connection with Theorems 2 and 4, we raise the following two problems.

Problem 1. How to find the best possible sufficient condition in terms of f under which its double cosine transform $\hat{f}_{cc}\in \mathrm{Lip}(\alpha,\beta)$, where $\alpha,\beta>0$ and $\max\{\alpha,\beta\}=1$.

Problem 2. How to find the best possible sufficient condition in terms of f under which its double sine transform $\hat{f}_{ss}\in \mathrm{Zyg}(\alpha,\beta)$, where $\alpha,\beta>0$ and $\max\{\alpha,\beta\}=2$.

3 Auxiliary Results

In this Section we consider functions $g:\mathbf{R}_+^2\to\mathbf{R}_+$ which are measurable in Lebesgue sense. The following two lemmas play key roles in the proof of Theorems 1-4. But they are also of interest in themselves.

Lemma 1. (i) *Let* $\gamma>\mu\ge 0$ *and* $\delta>\nu\ge 0$. *If*

$$\int_0^s\int_0^t x^\gamma y^\delta g(x,y)\mathrm{d}x\mathrm{d}y=O(s^\mu t^\nu)\quad \textit{for all}\quad s,t>0, \tag{3.1}$$

then $g\in L^1((s,\infty)\times(t,\infty))$ *and*

$$\int_s^\infty\int_t^\infty g(x,y)\mathrm{d}x\mathrm{d}y=O(s^{\mu-\gamma}t^{\nu-\delta})\quad \textit{for all}\quad s,t>0. \tag{3.2}$$

(ii) *Conversely, let* $\gamma\ge\mu>0$ *and* $\delta\ge\nu>0$. *If* (3.2) *holds, then* (3.1) *also holds.*

Proof. Part (i). By (3.1), there exists a constant C such that

$$\int_0^s\int_0^t x^\gamma y^\delta g(x,y)\mathrm{d}x\mathrm{d}y\le Cs^\mu t^\nu\quad \text{for all}\quad s,t>0. \tag{3.3}$$

Let $s,t>0$ be arbitrary. In particular, we have

$$\begin{aligned}2^{m\gamma+n\delta}s^\gamma t^\delta\int_{2^m s}^{2^{m+1}s}\int_{2^n t}^{2^{n+1}t} g(x,y)\mathrm{d}x\mathrm{d}y &\le \int_0^{2^{m+1}s}\int_0^{2^{n+1}t} x^\gamma y^\delta g(x,y)\mathrm{d}x\mathrm{d}y\\ &\le C2^{(m+1)\mu+(n+1)\nu}s^\mu t^\nu,\quad m,n\in\mathbf{Z},\end{aligned}$$

whence it follows that

$$\int_{2^m s}^{2^{m+1}s}\int_{2^n t}^{2^{n+1}t} g(x,y)\mathrm{d}x\mathrm{d}y \le C2^{\mu+\nu}2^{m(\mu-\gamma)+n(\nu-\delta)}s^{\mu-\gamma}t^{\nu-\delta}.$$

Since $\gamma > \mu$ and $\delta > \nu$, we conclude that

$$\begin{aligned}\int_s^\infty\int_t^\infty g(x,y)\mathrm{d}x\mathrm{d}y &= \sum_{m=0}^\infty\sum_{n=0}^\infty\int_{2^m s}^{2^{m+1}s}\int_{2^n t}^{2^{n+1}t} g(x,y)\mathrm{d}x\mathrm{d}y\\ &\le C2^{\mu+\nu}s^{\mu-\gamma}t^{\nu-\delta}\sum_{m=0}^\infty\sum_{n=0}^\infty 2^{m(\mu-\gamma)+n(\nu-\delta)} = O(s^{\mu-\gamma}t^{\nu-\delta}),\end{aligned}$$

which is (3.2) to be proved.

Part (ii). By (3.2), there exists a constant C such that

$$\int_s^\infty\int_t^\infty g(x,y)\mathrm{d}x\mathrm{d}y \le Cs^{\mu-\gamma}t^{\nu-\delta} \quad \text{for all} \quad s,t>0.$$

Let $s,t > 0$ be arbitrary. In particular, we have

$$\begin{aligned}\int_{2^{m-1}s}^{2^m s}\int_{2^{n-1}t}^{2^n t} x^\gamma y^\delta g(x,y)\mathrm{d}x\mathrm{d}y &\le 2^{m\gamma+n\delta}s^\gamma t^\delta\int_{2^{m-1}s}^{2^m s}\int_{2^{n-1}t}^{2^n t} g(x,y)\mathrm{d}x\mathrm{d}y\\ &\le 2^{m\gamma+n\delta}s^\gamma t^\delta C2^{(m-1)(\mu-\gamma)+(n-1)(\nu-\delta)}s^{\mu-\gamma}t^{\nu-\delta}\\ &= C2^{\gamma+\delta}s^\mu t^\nu 2^{(m-1)\mu+(n-1)\nu}, \quad m,n\in\mathbf{Z}.\end{aligned}$$

Since $\mu > 0$ and $\nu > 0$, we conclude that

$$\begin{aligned}\int_0^s\int_0^t x^\gamma y^\delta g(x,y)\mathrm{d}x\mathrm{d}y &= \sum_{m=-\infty}^0\sum_{n=-\infty}^0\int_{2^{m-1}s}^{2^m s}\int_{2^{n-1}t}^{2^n t} g(x,y)\mathrm{d}x\mathrm{d}y\\ &\le C2^{\gamma+\delta}s^\mu t^\nu\sum_{m=-\infty}^0\sum_{n=-\infty}^0 2^{(m-1)\mu+(n-1)\nu} = O(s^\mu t^\nu),\end{aligned}$$

which is (3.1) to be proved.

The proof of Lemma 1 is complete.

Lemma 2. *Let $\gamma > \mu \ge 0$, and let δ and ν be arbitrary. If* (3.1) *holds, then*

$$\int_s^\infty\int_0^t y^\delta g(x,y)\mathrm{d}x\mathrm{d}y = O(s^{\mu-\gamma}t^\nu) \quad \textit{for all} \quad s,t>0. \tag{3.4}$$

Proof. Let $s,t > 0$ be arbitrary. By (3.3), we have

$$2^{m\gamma}s^\gamma\int_{2^m s}^{2^{m+1}s}\int_0^t y^\delta g(x,y)\mathrm{d}x\mathrm{d}y \le \int_{2^m s}^{2^{m+1}s}\int_0^t x^\gamma y^\delta g(x,y)\mathrm{d}x\mathrm{d}y \le C2^{(m+1)\mu}s^\mu t^\nu,$$

whence it follows that

$$\int_{2^m s}^{2^{m+1}s}\int_0^t y^\delta g(x,y)\mathrm{d}x\mathrm{d}y \le C2^\mu 2^{m(\mu-\gamma)}s^{\mu-\gamma}t^\nu, \quad m\in\mathbf{Z}.$$

Since $\mu>\gamma$, we conclude that

$$\begin{aligned}\int_s^\infty\int_0^t y^\delta g(x,y)\mathrm{d}x\mathrm{d}y &= \sum_{m=0}^\infty\int_{2^m s}^{2^{m+1}s}\int_0^t y^\delta g(x,y\mathrm{d}x\mathrm{d}y\\ &\le C2^\mu s^{\mu-\gamma}t^\nu\sum_{m=0}^\infty 2^{m(\mu-\gamma)}=O(s^{\mu-\gamma}t^\nu),\end{aligned}$$

which is (3.4) to be proved.

4 Proof of Theorem 1

Part (i). Assume the condition (2.5) is satisfied for some $0<\alpha,\ \beta\le 1$. We will prove $\hat{f}_{ss}\in\mathrm{Lip}(\alpha,\beta)$, where $\hat{f}_{ss}$ is defined in (2.1). To this effect, let $u,v\ge 0$ and $h,k>0$ be arbitrarily given. Keeping (2.1) and (2.3) in mind, we estimate as follows:

$$\begin{aligned}\frac{\pi}{2}|\Delta_{1,1}\hat{f}_{ss}(u,v);h,k)| &= \Big|\int_0^\infty\int_0^\infty f(x,y)(\sin(u+h)x-\sin ux)(\sin(v+k)y-\sin vy)\mathrm{d}x\mathrm{d}y\Big|\\ &= 4\Big|\int_0^\infty\int_0^\infty f(x,y)\cos\big(u+\frac{h}{2}\big)x\sin\frac{hx}{2}\cos\big(v+\frac{k}{2}\big)y\sin\frac{ky}{2}\mathrm{d}x\mathrm{d}y\Big|\\ &\le 4\int_0^\infty\int_0^\infty|f(x,y)\sin\frac{hx}{2}\sin\frac{ky}{2}\Big|\mathrm{d}x\mathrm{d}y.\end{aligned}\tag{4.1}$$

We decompose the last double integral in (4.1) as follows:

$$\begin{aligned}&\frac{\pi}{2}|\Delta_{1,1}\hat{f}_{ss}(u,v;h,k)|\\ &\le 4\Big\{\int_0^{1/h}\int_0^{1/k}+\int_{1/h}^\infty\int_0^{1/k}+\int_0^{1/h}\int_{1/k}^\infty+\int_{1/h}^\infty\int_{1/k}^\infty\Big\}\Big|f(x,y)\sin\frac{hx}{2}\sin\frac{ky}{2}\Big|\mathrm{d}x\mathrm{d}y\\ &=: I_1+I_2+I_3+I_u,\end{aligned}\tag{4.2}$$

say. First, we use the obvious inequality

$$\Big|2\sin\frac{t}{2}\Big|\le\min\{2,|t|\},$$

and by (2.5) we obtain

$$\begin{aligned}I_1 &\le 4hk\int_0^{1/h}\int_0^{1/k}xy|f(x,y)|\mathrm{d}x\mathrm{d}y\\ &= hkO\Big(\big(\frac{1}{h}\big)^{1-\alpha}\big(\frac{1}{k}\big)^{1-\beta}\Big)=O(h^\alpha k^\beta).\end{aligned}\tag{4.3}$$

Second, we apply Part (i) in Lemma 1 in the case of (2.5) to obtain

$$\begin{aligned}I_4 &\le 16\int_{1/h}^\infty\int_{1/k}^\infty|f(x,y)|\mathrm{d}x\mathrm{d}y\\ &= O\Big(\big(\frac{1}{h}\big)^{-\alpha}\big(\frac{1}{k}\big)^{-\beta}\Big)=O(h^\alpha k^\beta).\end{aligned}\tag{4.4}$$

Third, we apply Part (i) in Lemma 2 in the case of (2.5) to obtain

$$\begin{aligned} I_2 &\le 8k\int_{1/h}^{\infty}\int_0^{1/k} y|f(x,y)|\mathrm{d}x\mathrm{d}y \\ &= kO\Big(\big(\frac{1}{h}\big)^{-\alpha}\big(\frac{1}{k}\big)^{1-\beta}\Big) = O(h^\alpha k^\beta). \end{aligned} \tag{4.5}$$

Fourth, we apply the symmetric counterpart of Lemma 2 in the case of (2.5) to obtain

$$\begin{aligned} I_3 &\le 8h\int_0^{1/h}\int_{1/k}^{\infty} x|f(x,y)|\mathrm{d}x\mathrm{d}y \\ &= hO\Big(\big(\frac{1}{h}\big)^{1-\alpha}\big(\frac{1}{k}\big)^{-\beta}\Big) = O(h^\alpha k^\beta). \end{aligned} \tag{4.6}$$

Combining (4.2) - (4.6) yields

$$|\Delta_{1,1}\hat{f}_{ss}(u,v;h,k)| = O(h^\alpha k^\beta).$$

Since $u,v \ge 0$ and $h,k>0$ are arbitrary, this proves $\hat{f}_{ss} \in \mathrm{Lip}(\alpha,\beta)$.

Part (ii). Assume $f \ge 0$ and $\hat{f}_{ss} \in \mathrm{Lip}(\alpha,\beta)$ for some $0<\alpha,\ \beta \le 1$. In particular, we have

$$\begin{aligned} &\tfrac{\pi}{2}|\Delta_{1,1}\hat{f}_{ss}(0,0;u,v)| \\ &= \Big|\int_0^{\infty}\int_0^{\infty} f(x,y)\sin ux \sin vy \mathrm{d}x\mathrm{d}y\Big| \le Cu^\alpha v^\beta \quad \text{for all} \quad u,v>0, \end{aligned} \tag{4.7}$$

where the constant C does not depend on u and v. We will integrate the double integral in (4.7) between the absolute value bars with respect to u over the interval $(0,h)$, where $h>0$ is arbitrary. Due to the fact that the convergence

$$\lim_{\xi\to\infty}\int_0^{\xi}\int_0^{\infty} f(x,y)\sin ux\sin vy\mathrm{d}x\mathrm{d}y = \int_0^{\infty}\int_0^{\infty} f(x,y)\sin ux \sin vy\mathrm{d}x\mathrm{d}y$$

is uniform in $u,v\ge 0$, we may change the order of integration with respect to x and u, and from (4.7) we conclude that

$$\Big|\int_0^{\infty}\int_0^{\infty} f(x,y)\frac{1-\cos hx}{x}\sin vy\mathrm{d}x\mathrm{d}y\Big| \le C\frac{h^{\alpha+1}}{\alpha+1}v^\beta \quad \text{for all} \quad h,v>0. \tag{4.8}$$

Next, we will integrate the double integral in (4.8) between the absolute value bars with respect to v over the interval $(0,k)$, where $k>0$ is arbitrary. By the same token as above, we may change the order of integration with respect to y and v, and from (4.8) we conclude that

$$\begin{aligned} &\Big|\int_0^{\infty}\int_0^{\infty} f(x,y)\frac{1-\cos hx}{x}\frac{1-\cos ky}{y}\mathrm{d}x\mathrm{d}y\Big| \\ &= 4\textstyle\int_0^{\infty}\int_0^{\infty}\frac{f(x,y)}{xy}\sin^2\frac{hx}{2}\sin^2\frac{ky}{2}\mathrm{d}x\mathrm{d}y \\ &\le C\frac{h^{\alpha+1}}{\alpha+1}\frac{k^{\beta+1}}{\beta+1}, \quad \text{for all} \quad h,k>0, \end{aligned} \tag{4.9}$$

where we have taken into account that $f\ge 0$.

Using the familiar inequality

$$\sin t\ge \frac{2}{\pi}t \quad \text{for} \quad 0\le t\le \frac{\pi}{2}, \tag{4.10}$$

it follows from (4.9) that

$$\frac{4h^2k^2}{\pi^4}\int_0^{1/h}\int_0^{1/k} xyf(x,y)\mathrm{d}x\mathrm{d}y\le C\frac{h^{\alpha+1}}{\alpha+1}\frac{k^{\beta+1}}{\beta+1} \quad \text{for all} \quad h,k>0,$$

or equivalently,

$$\int_0^{1/h}\int_0^{1/k} xyf(x,y)\mathrm{d}x\mathrm{d}y\le \frac{C\pi^4}{4(\alpha+1)(\beta+1)}h^{\alpha-1}k^{\beta-1}=O\Big(\big(\frac{1}{h}\big)^{1-\alpha}\big(\frac{1}{k}\big)^{1-\beta}\Big).$$

This proves (2.5) with $s=1/h$ and $t=1/k$, $h,k>0$.

The proof of Theorem 1 is complete.

5 Proof of Theorem 2

Part (i). Given $u,v\ge 0$ and $h,k>0$, by (2.2) we have (cf. (4.1))

$$\begin{aligned}\frac{\pi}{2}|\Delta_{1,1}\hat{f}_{cc}(u,v;h,k)| &= \Big|\int_0^\infty\int_0^\infty f(x,y)(\cos(u+h)x-\cos ux)(\cos(v+k)y-\cos vy)\mathrm{d}x\mathrm{d}y\Big|\\ &= 4\Big|\int_0^\infty\int_0^\infty f(x,y)\sin\big(u+\frac{h}{2}\big)x\sin\frac{hx}{2}\sin\big(v+\frac{k}{2}\big)y\sin\frac{ky}{2}\mathrm{d}x\mathrm{d}y\Big|\\ &\le 4\int_0^\infty\int_0^\infty \Big|f(x,y)\sin\frac{hx}{2}\sin\frac{ky}{2}\Big|\mathrm{d}x\mathrm{d}y.\end{aligned} \tag{5.1}$$

We observe that the right-most side of (5.1) is identical to that of (4.1). Thus, the proof of Part (i) in Theorem 1 in Section 4 can be repeated word by word, and it yields $\hat{f}_{cc}\in \mathrm{Lip}(\alpha,\beta)$ even in the case when $0<\alpha,\beta\le 1$.

Part (ii). Assume $f\ge 0$ and $\hat{f}_{cc}\in\mathrm{Lip}(\alpha,\beta)$ for some $0<\alpha,\ \beta<1$. In particular, we have

$$\begin{aligned}\frac{\pi}{2}|\Delta_{1,1}\hat{f}_{cc}(0,0;h,k)| &= \Big|\int_0^\infty\int_0^\infty f(x,y)(\cos hx-1)(\cos ky-1)\mathrm{d}x\mathrm{d}y\Big|\\ &=4\int_0^\infty\int_0^\infty f(x,y)\sin^2\frac{hx}{2}\sin^2\frac{ky}{2}\mathrm{d}x\mathrm{d}y\le Ch^\alpha k^\beta \quad \text{for all} \quad h,k>0,\end{aligned}$$

where the constant C does not depend on h and k. Making use of inequality (4.10) gives

$$\frac{4h^2k^2}{\pi^4}\int_0^{1/h}\int_0^{1/k} x^2y^2f(x,y)\mathrm{d}x\mathrm{d}y\le Ch^\alpha k^\beta,$$

or equivalently,

$$\begin{aligned}\int_0^{1/h}\int_0^{1/k} x^2y^2 f(x,y)\mathrm{d}x\mathrm{d}y &\le \frac{C\pi^4}{4}h^{\alpha-2}k^{\beta-2}\\ &= O\big(\big(\tfrac{1}{h}\big)^{2-\alpha}\big(\tfrac{1}{k}\big)^{2-\beta}\big) \quad \text{for all} \quad h,k>0.\end{aligned} \tag{5.2}$$

First, applying Part (i) in Lemma 1 with $\gamma=\delta=2$ and $\mu=2-\alpha$ and $\nu=2-\beta$, it follows from (5.2) that

$$\int_0^{1/h}\int_0^{1/k} f(x,y)\mathrm{d}x\mathrm{d}y = O\big(\big(\frac{1}{h}\big)^{-\alpha}\big(\frac{1}{k}\big)^{-\beta}\big). \tag{5.3}$$

Second, applying Part (ii) in Lemma 1 with $\gamma=\delta=1$ and $\mu=1-\alpha$ and $\nu=1-\beta$ (we must have $\mu, nu>0$, but this is the case since by assumption $0<\alpha,\beta<1$), it follows from (5.3) that

$$\int_0^{1/h}\int_0^{1/k} xyf(x,y)\mathrm{d}x\mathrm{d}y = O\Big(\big(\frac{1}{h}\big)^{1-\alpha}\big(\frac{1}{k}\big)^{1-\beta}\Big) \quad \text{for all} \quad h,k>0.$$

This proves (2.1) with $s=1/h$ and $t=1/k$, $h,k>0$.

The proof of Theorem 2 is complete.

6 Proof of Theorem 3

Part (i). Assume the condition (2.7) is satisfied for some $0<\alpha,\ \beta\le 2$. We will prove that $\hat{f}_{cc}\in \mathrm{Zyg}(\alpha,\beta)$, where $\hat{f}_{cc}$ is defined in (2.2). To this effect, let $u,v\ge 0$ and $h,k>0$ be arbitrarily given. Keeping (2.2) and (2.4) in mind, we estimate as follows (cf. (4.1)):

$$\begin{aligned}&\tfrac{\pi}{2}\Big|\Delta_{2,2}\hat{f}_{cc}(u,v;h,k)\Big|\\ &\Big|\int_0^{\infty}\int_0^{\infty} f(x,y)(\cos(u+h)x-\cos ux+\cos(u-h)x)\cdot(\cos(v+k)y\\ &\qquad -2\cos vy+\cos(v-k)y)\mathrm{d}x\mathrm{d}y\Big|\\ &=4\Big|\int_0^{\infty}\int_0^{\infty} f(x,y)\cos ux(\cos hx-1)\cos vy(\cos ky-1)\mathrm{d}x\mathrm{d}y\Big|\\ &\le 4\int_0^{\infty}\int_0^{\infty}|f(x,y)|(1-\cos hx)(1-\cos ky)\mathrm{d}x\mathrm{d}y.\end{aligned} \tag{6.1}$$

We decompose the last double integral in (6.1) as follows:

$$\begin{aligned}\frac{\pi}{2}|\Delta_{2,2}\hat{f}_{cc}(u,v;h,k) &\le 4\Big\{\int_0^{1/h}\int_0^{1/k}+\int_{1/h}^{\infty}\int_0^{1/k}+\int_0^{1/h}\int_{1/k}^{\infty}\\ &+\int_{1/h}^{\infty}\int_{1/k}^{\infty}\Big\}|f(x,y)|(1-\cos hx)(1-\cos ky)\mathrm{d}x\mathrm{d}y = J_1+J_2+J_3+J_4,\end{aligned} \tag{6.2}$$

say. First, we use the inequality

$$2(1-\cos t)=4\sin^2\frac{t}{2}\le \min\{4,t^2\},$$

and by (2.7) we obtain

$$\begin{aligned} J_1 &\le h^2k^2\int_0^{1/h}\int_0^{1/k} x^2y^2|f(x,y)|\mathrm{d}x\mathrm{d}y \\ &= h^2k^2 O\Big(\Big(\frac{1}{h}\Big)^{2-\alpha}\Big(\frac{1}{k}\Big)^{2-\beta}\Big) = O(h^\alpha k^\beta). \end{aligned} \tag{6.3}$$

Second, we apply Part (i) in Lemma 1 in the case of (2.7) to obtain

$$\begin{aligned} J_4 &\le 16\textstyle\int_{1/h}^{\infty}\int_{1/k}^{\infty}|f(x,y)|\mathrm{d}x\mathrm{d}y \\ &= O\Big(\big(\tfrac{1}{h}\big)^{-\alpha}\big(\tfrac{1}{k}\big)^{-\beta}\Big) = O(h^\alpha k^\beta). \end{aligned} \tag{6.4}$$

Third, we apply Part (i) in Lemma 2 in the case of (2.7) to obtain

$$\begin{aligned} J_2 &\le 8k^2\int_{1/h}^{\infty}\int_0^{1/k} y^2|f(x,y)|\mathrm{d}x\mathrm{d}y \\ &= k^2 O\Big(\big(\tfrac{1}{h}\big)^{-\alpha}\big(\tfrac{1}{k}\big)^{2-\beta}\Big) = O(h^\alpha k^\beta). \end{aligned} \tag{6.5}$$

Fourth, we apply the symmetric counterpart of Lemma 2 in the case of (2.7) to obtain

$$\begin{aligned} J_3 &\le 8h^2\int_0^{1/h}\int_{1/k}^{\infty} x^2|f(x,y)|\mathrm{d}x\mathrm{d}y \\ &= h^2 O\Big(\big(\tfrac{1}{h}\big)^{2-\alpha}\big(\tfrac{1}{k}\big)^{-\beta}\Big) = O(h^\alpha k^\beta). \end{aligned} \tag{6.6}$$

Combining (6.2) - (6.6) yields

$$|\Delta_{2,2}\hat{f}_{cc}(u,v;h,k)| = O(h^\alpha k^\beta).$$

Since $u,v\ge 0$ and $h,k>0$ are arbitrary, this proves $\hat{f}_{cc}\in \mathrm{Zyg}(\alpha,\beta)$.

Part (ii). Assume $f\ge 0$ and $\hat{f}_{cc}\in\mathrm{Zyg}(\alpha,\beta)$ for some $0<\alpha,\ \beta\le 2$. In particular, we have (cf. (6.1))

$$\begin{aligned} \tfrac{\pi}{2}|\Delta_{2,2}\hat{f}_{cc}(0,0;h,k)| &= 4\Big|\int_0^{\infty}\int_0^{\infty} f(x,y)(\cos hx-1)(\cos ky-1)\mathrm{d}x\mathrm{d}y\Big| \\ &= 16\int_0^{\infty}\int_0^{\infty} f(x,y)\sin^2\frac{hx}{2}\sin^2\frac{ky}{2}\mathrm{d}x\mathrm{d}y \le Ch^\alpha k^\beta \quad \text{for all} \quad h,k>0, \end{aligned} \tag{6.7}$$

where the constant C does not depend on h and k.

Making use of the inequality (4.10), from (6.7) we conclude that

$$\frac{4h^2k^2}{\pi^4}\int_0^{1/h}\int_0^{1/k} x^2y^2 f(x,y)\mathrm{d}x\mathrm{d}y \le Ch^\alpha k^\beta,$$

or equivalently

$$\int_0^{1/h}\int_0^{1/k} x^2y^2 f(x,y)\mathrm{d}x\mathrm{d}y \le \frac{C\pi^4}{4}h^{\alpha-2}k^{\beta-2} \quad \text{for all} \quad h,k>0.$$

This proves (2.7) with $s=1/h$ and $t=1/k$, $h,k>0$.

The proof of Theorem 3 is complete.

7 Proof of Theorem 4

Part (i). Given $u,v\ge 0$ and $h,k>0$, by (2.1) we have (cf. (6.1))

$$\begin{aligned}\frac{\pi}{2}|\Delta_{2,2}\hat{f}_{ss}(u,v;h,k)| &= \Big|\int_0^\infty\int_0^\infty f(x,y)(\sin(u+h)x-2\sin ux+\sin(u-h)x)\cdot \\ &\quad\cdot(\sin(v+k)y-2\sin vy+\sin(v-k)y)\mathrm{d}x\mathrm{d}y\Big| \\ &= 4\Big|\int_0^\infty\int_0^\infty f(x,y)\sin ux(\cos hx-1)\sin vy(\cos ky-1)\mathrm{d}x\mathrm{d}y\Big| \\ &\le 4\int_0^\infty\int_0^\infty |f(x,y)|(1-\cos hx)(1-\cos ky)\mathrm{d}x\mathrm{d}y.\end{aligned} \tag{7.1}$$

We observe that the right-most side of (7.1) is identical to that of (6.1). Thus, the proof of Part (i) of Theorem 3 in Section 6 can be repeated word by word, and it yields $\hat{f}_{ss}\in \mathrm{Lip}(\alpha,\beta)$ even in the case when $0<\alpha,\ \beta\le 2$.

Part (ii). Assume $f\ge 0$ and $\hat{f}_{ss}\in \mathrm{Zyg}(\alpha,\beta)$ for some $0<\alpha,\ \beta<2$. Let $u,v\ge 0$ and $h,k>0$ be arbitrary. By (7.1), we have

$$\frac{\pi}{8}|\Delta_{2,2}\hat{f}_{ss}(u,v;h,k)| = \Big|\int_0^\infty\int_0^\infty f(x,y)\sin ux(\cos hx-1)\sin vy(\cos ky-1)\mathrm{d}x\mathrm{d}y| \le Ch^\alpha k^\beta, \tag{7.2}$$

where the constant C does not depend on u,v,h, and k.

We will integrate the double integral in (7.2) between the absolute value bars with respect to u over the interval $(0,h)$. Due to the fact that the convergence

$$\begin{aligned}&\lim_{\xi\to\infty}\int_0^\xi\int_0^\infty f(x,y)\sin ux(\cos hx-1)\sin vy(\cos ky-1)\mathrm{d}x\mathrm{d}y \\ &= \int_0^\infty\int_0^\infty f(x,y)\sin ux(\cos hx-1)\sin vy(\cos ky-1)\mathrm{d}x\mathrm{d}y\end{aligned}$$

is uniform in $u,v\ge 0$, we may change the order of integration with respect to x and u, and from (7.2) we conclude that

$$\begin{aligned}&\Big|\int_0^\infty\int_0^\infty f(x,y)\frac{(1-\cos hx)^2}{x}\sin vy(\cos ky-1)\mathrm{d}x\mathrm{d}y\Big| \\ &\le Ch^{\alpha+1}k^\beta, \qquad \text{for all} \quad v\ge 0 \quad \text{and} \quad h,k>0.\end{aligned} \tag{7.3}$$

Next, we will integrate the double integral in (7.3) between the absolute value bars with respect to v over the interval $(0,k)$. By the same token as above, we may change the order of integration with respect to y and v, and from (7.3) we conclude that

$$\Big|\int_0^\infty \int_0^\infty f(x,y)\frac{(1-\cos hx)^2}{x}\frac{(1-\cos ky)^2}{y}\mathrm{d}x\mathrm{d}y$$

$$\le Ch^{\alpha+1}k^{\beta+1} \quad \text{for all} \quad h,k>0,$$

whence it follows that

$$\int_0^\infty \int_0^\infty \frac{f(x,y)}{xy}\Big(\sin\frac{hx}{2}\Big)^4\Big(\sin\frac{ky}{2}\Big)^4\mathrm{d}x\mathrm{d}y \le \frac{C}{16}h^{\alpha+1}k^{\beta+1} \quad \text{for all} \quad h,k>0,$$

where we have taken into account that $f\ge 0$. Making use of inequality (4.10), we even have

$$\frac{h^4k^4}{\pi^8}\int_0^{1/h}\int_0^{1/k} x^3y^3f(x,y)\mathrm{d}x\mathrm{d}y \le \frac{C}{16}h^{\alpha+1}k^{\beta+1},$$

or equivalently,

$$\int_0^{1/h}\int_0^{1/k} x^3y^3f(x,y)\mathrm{d}x\mathrm{d}y \le \frac{C\pi^8}{16}h^{\alpha-3}k^{\beta-3}. \tag{7.4}$$

First, applying Part (i) in Lemma 1 with $\gamma=\delta=3$ and $\mu=3-\alpha$, $\nu=3-\beta$, it follows from (7.4) that

$$\int_{1/h}^\infty\int_{1/k}^\infty f(x,y)\mathrm{d}x\mathrm{d}y = O\Big(\Big(\frac{1}{h}\Big)^{-\alpha}\Big(\frac{1}{k}\Big)^{-\beta}\Big). \tag{7.5}$$

Second, applying Part (ii) in Lemma 1 with $\gamma=\delta=2$ and $\mu=2-\alpha$, $\nu=2-\beta$ (we must have $\mu>0$, $\nu>0$, but this is the case since $0<\alpha,\ \beta<2$), it follows from (7.5) that

$$\int_0^{1/h}\int_0^{1/k} x^2y^2f(x,y)\mathrm{d}x\mathrm{d}y = O\Big(\Big(\frac{1}{h}\Big)^{2-\alpha}\Big(\frac{1}{k}\Big)^{2-\beta}\Big).$$

This proves (2.7) with $s=1/h$ and $t=1/k$, $h,k>0$.

The proof of Theorem 4 is complete.

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