Anal. Theory Appl.
Vol. 27, No. 4 (2011), 365–376
DOI10.1007/s10049-011-0365-z

# BOUNDEDNESS OF COMMUTATORS FOR MARCINKIEWICZ INTEGRALS ON WEIGHTED HERZ-TYPE HARDY SPACES

Cuilan Wu

(*Xuzhou Normal University, China)*

Received Dec. 2, 2010



**Abstract.** In this paper, the authors study the boundedness of the operator $\mu_\Omega^b$, the commutator generated by a function $b \in \text{Lip}_\beta(\mathbf{R^n})(0<\beta<1)$ and the Marcinkiewicz integral $\mu_\Omega$ on weighted Herz-type Hardy spaces.

**Key words:** *Marcinkiewicz integral, commutator, weighted Herz space, Hardy space*

**AMS (2010) subject classification:** 42B20, 42B25

## 1 Introduction and Main Result

Let $S^{n-1}$ denote the unit sphere of $\mathbf{R^n}(n\geq 2)$ with Lebesgue measure $d\sigma = d\sigma(x')$. Let $\Omega \in L^1(S^{n-1})$ be homogeneous of degree zero on $\mathbf{R^n}$ and satisfy the cancelation condition

$$\int_{S^{n-1}} \Omega(x')\mathrm{d}\sigma(x') = 0,$$

where $x' = x/|x|$ for any $x \neq 0$. The higher-dimentional Marcinkiewicz integral $\mu_\Omega$ is defined by

$$\mu_\Omega(f)(x) = \Big(\int_0^\infty |F_{\Omega,t}(f)(x)|^2 \frac{\mathrm{d}t}{t^3}\Big)^{1/2},$$

where

$$F_{\Omega,t}(f)(x) = \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)\mathrm{d}y.$$

Supported by the Natural Science Foundation of Xuzhou Normal University (09XLB02).

The operator $\mu_\Omega$ is first defined by Stein[1]. Meanwhile, Stein has proved that if $\Omega$ is continuous and satisfies the Lip$\alpha(S^{n-1})(0<\alpha\le 1)$ condition

$$|\Omega(x')-\Omega(y')|\le C|x'-y'|^{\alpha}, \qquad \forall x',y'\in S^{n-1},$$

then $\mu_\Omega$ is an operator of strong type $(p,p)(1<p\le 2)$ and of weak type $(1,1)$. In [2], it is proved that if $\Omega\in C^1(S^{n-1})$, then $\mu_\Omega$ is bounded on $L^p(\mathbf{R^n})$ for $1<p<\infty$. The boundedness of $\mu_\Omega$ have been discussed by many authors(see [3-4] etc).

On the other hand, let $b\in L_{loc}(\mathbf{R^n})$, the commutator $\mu_\Omega^b$ is defined by

$$\mu_\Omega^b(f)(x)=\Big(\int_0^\infty |F_{\Omega,b,t}(f)(x)|^2\frac{dt}{t^3}\Big)^{1/2},$$

where

$$F_{\Omega,b,t}(f)(x)=\int_{|x-y|\le t}\frac{\Omega(x-y)}{|x-y|^{n-1}}(b(x)-b(y))f(y)dy.$$

In this paper $b\in$Lip$_\beta(\mathbf{R^n})\,(0<\beta<1)$, which is the homogeneous Lipschitz space consisting of all functions $f$ such that

$$||f||_{Lip_\beta}=\sup_{x,y\in\mathbf{R^n},x\neq y}\frac{|f(x)-f(y)|}{|x-y|^{\beta}}<\infty.$$

Obviously, if $b\in$Lip$_\beta(\mathbf{R^n})(0<\beta<1)$, then

$$|b(x)-b(y)|\le C||b||_{\mathrm{Lip}_\beta}|x-y|^{\beta}\ \ (\forall x,y\in\mathbf{R^n}).$$

Recently,Cheng and Shu[5] considered the commutator $\mu_\Omega^b$ on Herz-type Hardy spaces, and proved the following theorem.

**Theorem A.** *Suppose that* $\Omega\in\mathrm{Lip}\nu(S^{n-1})(0<\nu\le 1), b\in\mathrm{Lip}_\beta(\mathbf{R^n})(0<\beta<\min\{1/2,\nu\})$, $0<p<\infty, 1<q_1,q_2<\infty$ *and*

$$1/q_1-1/q_2=\beta/n,\ n(1-1/q_1)\le\alpha<n(1-1/q_1)+\beta,$$

*then* $\mu_\Omega^b$ *is bounded from* $H\dot{K}_{q_1}^{\alpha,p}(\mathbf{R^n})$ *to* $\dot{K}_{q_2}^{\alpha,p}(\mathbf{R^n})$.

Lu and Yang[6] introduced the weighted Herz-type Hardy space, and built the atomic decomposition theory. Motivated by [5-6], we consider the weighted boundedness of $\mu_\Omega^b$ and present our result as follows.

**Theorem 1.** *Suppose that* $\Omega\in\mathrm{Lip}\nu(S^{n-1})(0<\nu\le 1), b\in\mathrm{Lip}_\beta(\mathbf{R^n})(0<\beta<\min\{1/2,\nu\})$, $0<p_1\le p_2<\infty, 1<q_1,q_2<\infty$ *and*

$$1/q_1-1/q_2=\beta/n,\ n(1-1/q_1)\le\alpha<n(1-1/q_1)+\beta,$$

*and* $\omega_1\in A_1$, $\omega_2^{q_2}\in A_1$,*then* $\mu_\Omega^b$ *is bounded from* $H\dot{K}_{q_1}^{\alpha,p_1}(\omega_1,\omega_2^{q_1})$ *to* $\dot{K}_{q_2}^{\alpha,p_2}(\omega_1,\omega_2^{q_2})$.

## 2 Preliminaries

To prove our result, let us recall some definitions. In the following definitions, the function $\omega$ is a locally integrable nonnegative function on $\mathbf{R^n}$. Moreover, $C>0$, $Q$ denotes a cube in $\mathbf{R^n}$ with sides parallel to the coordinate axes, and $|Q|$ denotes the Lebesgue measure of $Q$.

*Definition* 1[7,8]. (1) A function $\omega$ is said to belong to $A_p(1<p<\infty)$ if

$$\sup_Q\Big(\frac{1}{|Q|}\int_Q\omega(x)\mathrm{d}x\Big)\Big(\frac{1}{|Q|}\int_Q\omega(x)^{-1/(p-1)}\mathrm{d}x\Big)^{p-1}\leq C.$$

For the case $p=1,\omega\in A_1$ if

$$\frac{1}{|Q|}\int_Q\omega(x)\mathrm{d}x\leq C\operatorname*{essinf}_Q\{\omega(x)\}.$$

(2) A function $\omega$ is said to belong to $A(p,q)(1<p,q<\infty)$ if

$$\sup_Q\Big(\frac{1}{|Q|}\int_Q\omega(x)^q\mathrm{d}x\Big)^{1/q}\Big(\frac{1}{|Q|}\int_Q\omega(x)^{-p'}\mathrm{d}x\Big)^{1/p'}\leq C,$$

where $p'=p/(p-1)$.

(3) If there exist $C,\delta>0$, such that for any $E\subset Q$

$$\frac{\omega(E)}{\omega(Q)}\leq C\Big(\frac{|E|}{|Q|}\Big)^{\delta},$$

then we call $\omega\in A_\infty$.

**Elementary properties of** $A_p$ **(see [7])**.

(a) $A_1\subset A_p\subset A_q$ if $1<p<q<\infty$.

(b) If $\omega(x)\in A_p$, then for any $0<\varepsilon<1,\omega(x)^\varepsilon\in A_p$.

(c) If $\omega(x)\in A_p$, then there are $C>0$ and $\varepsilon>0$, such that, for any $Q\subset\mathbf{R^n}$,

$$\Big(\frac{1}{|Q|}\int_Q\omega(x)^{1+\varepsilon}\mathrm{d}x\Big)^{1/(1+\varepsilon)}\leq C\Big(\frac{1}{|Q|}\int_Q\omega(x)\mathrm{d}x\Big).$$

**The relations between** $A_p$ **and** $A(p,q)$ **(see [7])**. Suppose that $0<\alpha<n,1<p<n/\alpha$ and $1/q=1/p-\alpha/n$. Then we have the following conclusions:

$$\omega(x)\in A(p,q)\Longleftrightarrow\omega(x)^q\in A_{q(n-\alpha)/n}\Longleftrightarrow\omega(x)^q\in A_{1+q/p'}\Longleftrightarrow\omega(x)^{-p'}\in A_{1+p'/q}.$$

**The definition of reverse Hölder condition.** If there exists $r>1$ such that

$$\Big(\frac{1}{|Q|}\int_Q\omega(x)^r\mathrm{d}x\Big)^{1/r}\leq C\Big(\frac{1}{|Q|}\int_Q\omega(x)\mathrm{d}x\Big),$$

then $\omega$ is said to satisfy the reverse Hölder condition of order $r$ and is written by $\omega \in RH_r$. It follows from Hölder inequality that $\omega \in RH_r$ implies $\omega \in RH_s$ for $s < r$. It is known that if $\omega \in RH_r (r > 1)$ then $\omega \in RH_{r+\varepsilon}$ for some $\varepsilon > 0$.

Set $B_k = \{x \in \mathbf{R^n} : |x| \leq 2^k\}, C_k = B_k \setminus B_{k-1}$ and $\chi_k = \chi_{C_k}$ denotes the characteristic function of $C_k$ for $k \in Z$. Moreover, for any nonnegative weight function $\omega$ and Lebesgue measurable function $f$,we write

$$||f||_{L^q(\omega)} = \Big( \int_{\mathbf{R^n}} |f(x)|^q \omega(x) \mathrm{d}x \Big)^{1/q}$$

*Definition 2*[9]. Let $0 < \alpha < \infty$, $0 < p < \infty$, $1 < q < \infty$, $\omega_1$ and $\omega_2$ be nonnegative weight functions. The homogeneous weight Herz space $\dot{K}_q^{\alpha,p}(\omega_1, \omega_2)$ is defined by

$$\dot{K}_q^{\alpha,p}(\omega_1, \omega_2) = \{f \in L^q_{loc}(\mathbf{R^n} \setminus \{0\}, \omega_2) : ||f||_{\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)} < \infty\},$$

where

$$||f||_{\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)} = \Big( \sum_{k=-\infty}^{\infty} [\omega_1(B_k)]^{\alpha p/n} ||f\chi_k||^p_{L^q(\omega_2)} \Big)^{1/p}.$$

*Definition 3*[6]. Let $0 < \alpha < \infty$, $0 < p < \infty$, $1 < q < \infty$, $\omega_1$, $\omega_2 \in A_1$. The homogeneous weighted Herz-type Hardy space $H\dot{K}_q^{\alpha,p}(\omega_1, \omega_2)$ is defined by

$$H\dot{K}_q^{\alpha,p}(\omega_1, \omega_2) = \{f \in S'(\mathbf{R^n}) : G(f) \in \dot{K}_q^{\alpha,p}(\omega_1, \omega_2)\},$$

and

$$||f||_{H\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)} = ||G(f)||_{\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)},$$

where $S'(\mathbf{R^n})$ is the space of tempered distributions on $\mathbf{R^n}$ and $G(f)$ is the grand maximal function of $f$.

*Eefinition 4*[6]. Let $\omega_1$, $\omega_2 \in A_1, 1 < q < \infty, n(1-1/q) \leq \alpha < \infty$. A function $a(x)$ on $\mathbf{R^n}$ is called a central $(\alpha, q; \omega_1, \omega_2)$ atom if $a$ satisfies

1) Supp$a \subset B(0,r)$ for some $r > 0$;

2) $||a||_{L^q(\omega_2)} \leq [\omega_1(B(0,r))]^{-\alpha/n}$;

3) $\int_{\mathbf{R^n}} a(x) x^s \mathrm{d}x = 0$, when $|s| \leq [\alpha - n(1-1/q)]$.

To prove our result, we need the following lemmas.

**Lemma 1**[6]. *Let $\omega_1$, $\omega_2 \in A_1, 0 < p < \infty, 1 < q < \infty$ and $\alpha \geq n(1-1/q)$. A distribution $f$ on $\mathbf{R^n}$ belongs to $H\dot{K}_q^{\alpha,p}(\omega_1, \omega_2)$ if and only if it can be written as $f = \sum_{j=-\infty}^{\infty} \lambda_j a_j$ in the* *distributional sense, where each $a_j$ is a central $(\alpha,q;\omega_1,\omega_2)$ atom on $B_j$ and*

$$\sum_{j=-\infty}^{\infty}|\lambda_j|^p<\infty.$$

*Moreover,*

$$||f||_{H\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)}\sim\inf\{(\sum_{j=-\infty}^{\infty}|\lambda_j|^p)^{1/p}\}$$

*with the infimum taken over all the decomposition $f$ as above.*

**Lemma 2**[10]. *Let*

$$\Omega\in \mathrm{Lip}\nu(S^{n-1})(0<\nu\le 1), b\in \mathrm{Lip}_\beta(\mathbf{R^n})(0<\beta<1).$$

*If $1<p<n/\beta, 1/q=1/p-\beta/n$ and $\omega\in A(p,q)$, then there is $C>0$ such that*

$$||\mu_\Omega^b(f)||_{L^q(\omega^q)}\le C||b||_{Lip_\beta}||f||_{L^p(\omega^p)}.$$

**Lemma 3**[11]. *$\omega^r\in A_\infty(r>1)$ if and only if $\omega\in RH_r$.*

**Lemma 4**[12]. *If $\omega\in A_1$, then there are $C>0$ and $\delta>0(0<\delta<1)$ such that*

$$\frac{\omega(B_k)}{\omega(B_j)}\le C2^{(k-j)n}, k>j;$$

$$\frac{\omega(B_k)}{\omega(B_j)}\le C2^{(k-j)n\delta}, k\le j$$

.

## 3. Proof of Theorem 1

From $p_1\le P_2$, it follows that

$$\dot{K}_q^{\alpha,p_1}(\omega_1,\omega_2)\subseteq\dot{K}_q^{\alpha,p_2}(\omega_1,\omega_2),\ \ H\dot{K}_q^{\alpha,p_1}(\omega_1,\omega_2)\subseteq H\dot{K}_q^{\alpha,p_2}(\omega_1,\omega_2).$$

Hence we only prove Theorem 1 for $p_1=p_2=p$.

Let $f\in H\dot{K}_{q_1}^{\alpha,p}(\omega_1,\omega_2^{q_1})$,applying the atomic decomposition theory (see Lemma 1), we write $f=\sum_{j=-\infty}^{\infty}\lambda_ja_j$, where each $a_j$ is a central $(\alpha,q_1;\omega_1,\omega_2^{q_1})$ atom, supp $a_j\subset B_j=B(0,2^j)$ and

$$\|a_j\|_{L^{q_1}(\omega_2^{q_1})}\le[\omega_1(B_j)]^{-\alpha/n},\ \sum_{j=-\infty}^{\infty}|\lambda_j|^p<\infty.$$

Then we have

$$\begin{aligned}
||\mu_\Omega^b(f)||^p_{\dot K^{\alpha,p}_{q_2}(\omega_1,\omega_2^{q_2})} &= \sum_{k=-\infty}^{\infty}[\omega_1(B_k)]^{\alpha p/n}||\mu_\Omega^b(f)\chi_k||^p_{L^{q_2}(\omega_2^{q_2})}\\
&\le C\sum_{k=-\infty}^{\infty}[\omega_1(B_k)]^{\alpha p/n}\Big(\sum_{j=-\infty}^{k-3}|\lambda_j|||\mu_\Omega^b(a_j)\chi_k||_{L^{q_2}(\omega_2^{q_2})}\Big)^p\\
&\quad +C\sum_{k=-\infty}^{\infty}[\omega_1(B_k)]^{\alpha p/n}\Big(\sum_{j=k-2}^{\infty}|\lambda_j|||\mu_\Omega^b(a_j)\chi_k||_{L^{q_2}(\omega_2^{q_2})}\Big)^p\\
&= C(I+II).
\end{aligned}$$

For $II$, since

$$\omega_2^{q_2}\in A_1$$

and

$$1/q_1-1/q_2=\beta/n,$$

then $\omega_2\in A(q_1,q_2)$. By lemma 2, we know $\mu_\Omega^b$ is bounded from $L^{q_1}(\omega_2^{q_1})$ to $L^{q_2}(\omega_2^{q_2})$, it is easy to verify that

$$II\le C\sum_{k=-\infty}^{\infty}[\omega_1(B_k)]^{\alpha p/n}\Big(\sum_{j=k-2}^{\infty}|\lambda_j|||a_j||_{L^{q_1}(\omega_2^{q_1})}\Big)^p.$$

When $0<p\le 1$,

$$\begin{aligned}
II &\le C\sum_{k=-\infty}^{\infty}[\omega_1(B_k)]^{\alpha p/n}\sum_{j=k-2}^{\infty}|\lambda_j|^p||a_j||^p_{L^{q_1}(\omega_2^{q_1})}\\
&\le C\sum_{k=-\infty}^{\infty}[\omega_1(B_k)]^{\alpha p/n}\sum_{j=k-2}^{\infty}|\lambda_j|^p[\omega_1(B_j)]^{-\alpha p/n}\\
&\le C\sum_{k=-\infty}^{\infty}\sum_{j=k-2}^{\infty}|\lambda_j|^p2^{(k-j)\alpha p}\\
&\le C\sum_{j=-\infty}^{\infty}|\lambda_j|^p\Big(\sum_{k=-\infty}^{j+2}2^{(k-j)\alpha p}\Big)\\
&\le C\sum_{j=-\infty}^{\infty}|\lambda_j|^p.
\end{aligned}$$

If $p>1$, by Hölder's inequality, we get

$$
\begin{aligned}
II &\le C\sum_{k=-\infty}^{\infty}[\omega_1(B_k)]^{\alpha p/n}\Big(\sum_{j=k-2}^{\infty}|\lambda_j|[\omega_1(B_j)]^{-\alpha/n}\Big)^p\\
&\le C\sum_{k=-\infty}^{\infty}\Big(\sum_{j=k-2}^{\infty}|\lambda_j|2^{(k-j)\alpha}\Big)^p\\
&\le C\sum_{k=-\infty}^{\infty}\Big(\sum_{j=k-2}^{\infty}|\lambda_j|2^{(k-j)\alpha/2}\cdot 2^{(k-j)\alpha/2}\Big)^p\\
&\le C\sum_{k=-\infty}^{\infty}\Big[\Big(\sum_{j=k-2}^{\infty}|\lambda_j|^p 2^{(k-j)\alpha p/2}\Big)\Big(\sum_{j=k-2}^{\infty}2^{(k-j)\alpha p'/2}\Big)^{p/p'}\Big]\\
&\le C\sum_{j=-\infty}^{\infty}|\lambda_j|^p\sum_{k=-\infty}^{j+2}2^{(k-j)\alpha p/2}\\
&\le C\sum_{j=-\infty}^{\infty}|\lambda_j|^p.
\end{aligned}
$$

Let us now estimate $I$. By the definition of $\mu_\Omega^b$, we have

$$||\mu_\Omega^b(a_j)\chi_k||_{L^{q_2}(\omega_2^{q_2})}=\Big(\int_{C_k}|\mu_\Omega^b(a_j)(x)|^{q_2}\omega_2(x)^{q_2}dx\Big)^{1/q_2}$$

$$
\begin{aligned}
&= \Big(\int_{C_k}\Big|\int_0^\infty\Big|\int_{|x-y|\le t}\frac{\Omega(x-y)}{|x-y|^{n-1}}(b(x)-b(y))a_j(y)dy\Big|^2\frac{dt}{t^3}\Big|^{q_2/2}\omega_2(x)^{q_2}dx\Big)^{1/q_2}\\
&\le C\Big(\int_{C_k}\Big|\int_0^{|x|}\Big|\int_{|x-y|\le t}\frac{\Omega(x-y)}{|x-y|^{n-1}}(b(x)-b(y))a_j(y)dy\Big|^2\frac{dt}{t^3}\Big|^{q_2/2}\omega_2(x)^{q_2}dx\Big)^{1/q_2}\\
&\quad +C\Big(\int_{C_k}\Big|\int_{|x|}^\infty\Big|\int_{|x-y|\le t}\frac{\Omega(x-y)}{|x-y|^{n-1}}(b(x)-b(y))a_j(y)dy\Big|^2\frac{dt}{t^3}\Big|^{q_2/2}\omega_2(x)^{q_2}dx\Big)^{1/q_2}\\
&= C(I_1+I_2).
\end{aligned}
$$

For $I_1$, when $x\in C_k, y\in B_j, j\le k-3$ we get $|x|\sim|x-y|\approx 2^k$ and

$$\Big|\frac{1}{|x|^2}-\frac{1}{|x-y|^2}\Big|\le C\frac{|y|}{|x|^3}.$$

By Minkowski inequality and $\Omega\in\mathrm{Lip}_\nu(S^{n-1})\subset L^\infty(S^{n-1})$,

$$
\begin{aligned}
I_1 &\le C\Big(\int_{C_k}\Big(\int_{\mathbf{R^n}}\frac{|\Omega(x-y)|}{|x-y|^{n-1}}|b(x)-b(y)||a_j(y)|\Big(\int_{|x-y|}^{|x|}\frac{dt}{t^3}\Big)^{1/2}\mathrm{d}y\Big)^{q_2}\omega_2(x)^{q_2}\mathrm{d}x\Big)^{1/q_2}\\
&\le C||\Omega||_\infty||b||_{Lip_\beta}\Big(\int_{C_k}\Big(\int_{B_j}\frac{|x-y|^\beta}{|x-y|^{n-1}}|a_j(y)|\frac{|y|^{1/2}}{|x|^{3/2}}\mathrm{d}y\Big)^{q_2}\omega_2(x)^{q_2}\mathrm{d}x\Big)^{1/q_2}\\
&\le C||b||_{Lip_\beta}2^{j/2}\Big(\int_{C_k}\frac{1}{|x|^{(n+1/2-\beta)q_2}}\Big(\int_{B_j}|a_j(y)|\mathrm{d}y\Big)^{q_2}\omega_2(x)^{q_2}\mathrm{d}x\Big)^{1/q_2} \qquad (1)\\
&\le C||b||_{Lip_\beta}2^{j/2}2^{-k(n+1/2-\beta)}\int_{B_j}|a_j(y)|\mathrm{d}y\Big(\int_{C_k}\omega_2(x)^{q_2}\mathrm{d}x\Big)^{1/q_2}.
\end{aligned}
$$

Since $\omega_2^{q_2} \in A_1$, we have $\omega_2^{q_2} \in A_{1+q_2/q_1'} \subset A_\infty$, by Lemma 3, it follows that $\omega_2 \in RH_{q_2}$, i.e.

$$\Big(\frac{1}{|B_k|}\int_{B_k}\omega_2(x)^{q_2}\mathrm{d}x\Big)^{1/q_2} \le C\Big(\frac{1}{|B_k|}\int_{B_k}\omega_2(x)\mathrm{d}x\Big),$$

$$\Big(\int_{B_k}\omega_2(x)^{q_2}\mathrm{d}x\Big)^{1/q_2} \le C|B_k|^{1/q_2-1}\int_{B_k}\omega_2(x)\mathrm{d}x \le C2^{kn(1/q_2-1)}\omega_2(B_k). \tag{2}$$

By $(\omega_2^{q_2})^{1/q_2} = \omega_2 \in A_1$ and the definition of $A_1$, we get

$$\frac{1}{|B_j|}\int_{B_j}\omega_2(y)\mathrm{d}y \le C\operatorname*{essinf}_{y\in B_j}\{\omega_2(y)\} \le C\omega_2(y), \quad \text{a. e.} \quad y \in B_j,$$

i.e.

$$\omega_2(B_j) \le C|B_j|\omega_2(y),\ a.e.\ y \in B_j. \tag{3}$$

By Hölder's inequality and (3), we obtain

$$\begin{aligned}
\int_{B_j}|a_j(y)|\mathrm{d}y &= \int_{B_j}|a_j(y)|\omega_2(y)\omega_2(y)^{-1}\mathrm{d}y \\
&\le C\frac{|B_j|}{\omega_2(B_j)}\int_{B_j}|a_j(y)|\omega_2(y)\mathrm{d}y \\
&\le C\frac{|B_j|}{\omega_2(B_j)}\Big(\int_{B_j}|a_j(y)|^{q_1}\omega_2(y)^{q_1}\mathrm{d}y\Big)^{1/q_1}\Big(\int_{B_j}1\mathrm{d}y\Big)^{1/q_1'} \\
&\le C\frac{|B_j|}{\omega_2(B_j)}|B_j|^{1/q_1'}||a_j||_{L^{q_1}(\omega_2^{q_1})}.
\end{aligned}$$

Using (2),(3),$||a_j||_{L^{q_1}(\omega_2^{q_1})} \le [\omega_1(B_j)]^{-\alpha/n}$ and lemma 4,we get

$$\begin{aligned}
\int_{B_j}|a_j(y)|dy\Big(\int_{C_k}\omega_2(x)^{q_2}dx\Big)^{1/q_2} &\le C2^{jn(1+1/q_1')}2^{kn(1/q_2-1)}[\omega_1(B_j)]^{-\alpha/n}\frac{\omega_2(B_k)}{\omega_2(B_j)} \\
&\le C2^{jn(1+1/q_1')}2^{kn(1/q_2-1)}[\omega_1(B_j)]^{-\alpha/n}2^{(k-j)n} \\
&= C[\omega_1(B_j)]^{-\alpha/n}2^{jn/q_1'}2^{kn/q_2}.
\end{aligned} \tag{4}$$

Combining (1),(4) with $1/q_1 - 1/q_2 = \beta/n$, we obtain

$$\begin{aligned}
I_1 &\le C||b||_{Lip_\beta}2^{j/2}2^{-k(n+1/2-\beta)}[\omega_1(B_j)]^{-\alpha/n}2^{jn/q_1'}2^{kn/q_2} \\
&\le C||b||_{Lip_\beta}[\omega_1(B_j)]^{-\alpha/n}2^{-(k-j)(1/2+n(1-1/q_1))}.
\end{aligned}$$

For $I_2$, applying the vanishing condition of $a_j$, we obtain

$$\begin{aligned}
I_2 &\le C\Big(\int_{C_k}\Big|\int_{|x|}^{\infty}\Big|\int_{|x-y|\le t}\frac{\Omega(x-y)}{|x-y|^{n-1}}(b(x)-b(y))a_j(y)\mathrm{d}y \\
&\quad -\int_{|x-y|\le t}\frac{\Omega(x)}{|x|^{n-1}}(b(x)-b(0))a_j(y)\mathrm{d}y\Big|^2\frac{dt}{t^3}\Big|^{q_2/2}\omega_2(x)^{q_2}\mathrm{d}x\Big)^{1/q_2} \\
&\le C\Big(\int_{C_k}\Big(\int_{\mathbf{R}^n}\Big|\frac{\Omega(x-y)}{|x-y|^{n-1}}-\frac{\Omega(x)}{|x|^{n-1}}\Big||b(x)-b(0)||a_j(y)|\Big(\int_{|x|}^{\infty}\frac{\mathrm{d}t}{t^3}\Big)^{1/2}\mathrm{d}y\Big)^{q_2}\omega_2(x)^{q_2}\mathrm{d}x\Big)^{1/q_2} \\
&\quad +C\Big(\int_{C_k}\Big(\int_{\mathbf{R}^n}\frac{|\Omega(x-y)|}{|x-y|^{n-1}}|b(y)-b(0)||a_j(y)|\Big(\int_{|x|}^{\infty}\frac{\mathrm{d}t}{t^3}\Big)^{1/2}\mathrm{d}y\Big)^{q_2}\omega_2(x)^{q_2}\mathrm{d}x\Big)^{1/q_2} \\
&= C(J_1+J_2).
\end{aligned}$$

Let us estimate $J_1$ and $J_2$ respectively. For $J_2$, because $\Omega\in\mathrm{Lip}\nu(S^{n-1})\subset L^\infty(S^{n-1})$, and $|x|\sim|x-y|$, it is easy to see that

$$\begin{aligned}
J_2 &\le C||\Omega||_\infty||b||_{Lip_\beta}\Big(\int_{C_k}\Big(\int_{B_j}\frac{|y|^\beta}{|x-y|^{n-1}}|a_j(y)|\frac{1}{|x|}\mathrm{d}y\Big)^{q_2}\omega_2(x)^{q_2}\mathrm{d}x\Big)^{1/q_2} \\
&\le C||b||_{Lip_\beta}2^{j\beta}\Big(\int_{C_k}\frac{1}{|x|^{nq_2}}\Big(\int_{B_j}|a_j(y)|\mathrm{d}y\Big)^{q_2}\omega_2(x)^{q_2}\mathrm{d}x\Big)^{1/q_2} \\
&\le C||b||_{Lip_\beta}2^{j\beta}2^{-kn}\int_{B_j}|a_j(y)|\mathrm{d}y\Big(\int_{C_k}\omega_2(x)^{q_2}\mathrm{d}x\Big)^{1/q_2}
\end{aligned}$$

Using (4) again,

$$\begin{aligned}
J_2 &\le C||b||_{Lip_\beta}2^{j\beta}2^{-kn}[\omega_1(B_j)]^{-\alpha/n}2^{jn/q_1'}2^{kn/q_2} \\
&\le C||b||_{Lip_\beta}[\omega_1(B_j)]^{-\alpha/n}2^{-(k-j)(\beta+n(1-1/q_1))}.
\end{aligned}$$

For $J_1$, we have

$$\begin{aligned}
J_1 &\le C\Big(\int_{C_k}\Big(\int_{\mathbf{R}^n}\Big|\frac{\Omega(x-y)}{|x-y|^{n-1}}-\frac{\Omega(x)}{|x-y|^{n-1}}+\frac{\Omega(x)}{|x-y|^{n-1}}-\frac{\Omega(x)}{|x|^{n-1}}\Big| \\
&\quad \cdot|b(x)-b(0)||a_j(y)|\frac{1}{|x|}\mathrm{d}y\Big)^{q_2}\omega_2(x)^{q_2}\mathrm{d}x\Big)^{1/q_2} \\
&\le C\Big(\int_{C_k}\Big(\int_{\mathbf{R}^n}|\Omega(x-y)-\Omega(x)|\frac{1}{|x-y|^{n-1}}|b(x)-b(0)||a_j(y)|\frac{1}{|x|}\mathrm{d}y\Big)^{q_2}\omega_2(x)^{q_2}\mathrm{d}x\Big)^{1/q_2} \\
&\quad +C\Big(\int_{C_k}\Big(\int_{\mathbf{R}^n}|\Omega(x)|\Big|\frac{1}{|x-y|^{n-1}}-\frac{1}{|x|^{n-1}}\Big||b(x)-b(0)||a_j(y)|\frac{1}{|x|}\mathrm{d}y\Big)^{q_2}\omega_2(x)^{q_2}\mathrm{d}x\Big)^{1/q_2} \\
&= C(J_{11}+J_{12})
\end{aligned}$$

Note that from $C_k=B_k\setminus B_{k-1}$ and $x\in C_k, y\in B_j, j\le k-3$, it follows $|x|\sim|x-y|\approx 2^k, |C_k|\approx 2^{kn}$, and

$$|\Omega(x-y)-\Omega(x)|\le C\Big|\frac{x-y}{|x-y|}-\frac{x}{|x|}\Big|^\nu\le C\frac{|y|^\nu}{|x|^\nu},$$

Thus,

$$\begin{aligned} J_{11} &\leq C||b||_{Lip_\beta}\Big(\int_{C_k}\Big(\int_{B_j}\frac{|y|^\nu}{|x|^\nu}\frac{1}{|x|^{n-1}}\frac{|x|^\beta}{|x|}|a_j(y)|\mathrm{d}y\Big)^{q_2}\omega_2(x)^{q_2}\mathrm{d}x\Big)^{1/q_2}\\ &\leq C||b||_{Lip_\beta}2^{j\nu}\Big(\int_{C_k}|x|^{-(n+\nu-\beta)q_2}\Big(\int_{B_j}|a_j(y)\mathrm{d}y|\Big)^{q_2}\omega_2(x)^{q_2}\mathrm{d}x\Big)^{1/q_2}\\ &\leq C||b||_{Lip_\beta}2^{j\nu}2^{-k(n+\nu-\beta)}\int_{Bj}|a_j(y)|\mathrm{d}y\Big(\int_{C_k}\omega_2(x)^{q_2}\mathrm{d}x\Big)^{1/q_2} \end{aligned}$$

From (4),

$$\begin{aligned} J_{11} &\leq C||b||_{\mathrm{Lip}_\beta}2^{j\nu}2^{-k(n+\nu-\beta)}[\omega_1(B_j)]^{-\alpha/n}2^{jn/q_1'}2^{kn/q_2}\\ &\leq C||b||_{\mathrm{Lip}_\beta}[\omega_1(B_j)]^{-\alpha/n}2^{-(k-j)(\nu+n(1-1/q_1))}. \end{aligned}$$

For $J_{12}$, since $|x|\sim|x-y|$ and

$$\Big|\frac{1}{|x-y|^{n-1}}-\frac{1}{|x|^{n-1}}\Big|\leq C\frac{|y|}{|x|^n},$$

we have

$$\begin{aligned} J_{12} &\leq C||\Omega||_\infty||b||_{\mathrm{Lip}_\beta}\Big(\int_{C_k}\Big(\int_{B_j}\frac{|y|}{|x|^n}\frac{|x|^\beta}{|x|}|a_j(y)|\mathrm{d}y\Big)^{q_2}\omega_2(x)^{q_2}\mathrm{d}x\Big)^{1/q_2}\\ &\leq C||b||_{\mathrm{Lip}_\beta}2^j\Big(\int_{C_k}|x|^{-(n+1-\beta)q_2}\Big(\int_{B_j}|a_j(y)\mathrm{d}y|\Big)^{q_2}\omega_2(x)^{q_2}\mathrm{d}x\Big)^{1/q_2}\\ &\leq C||b||_{\mathrm{Lip}_\beta}2^j2^{-k(n+1-\beta)}\int_{Bj}|a_j(y)|\mathrm{d}y\Big(\int_{C_k}\omega_2(x)^{q_2}\mathrm{d}x\Big)^{1/q_2}\\ &\leq C||b||_{\mathrm{Lip}_\beta}2^j2^{-k(n+1-\beta)}[\omega_1(B_j)]^{-\alpha/n}2^{jn/q_1'}2^{kn/q_2}\\ &\leq C||b||_{\mathrm{Lip}_\beta}[\omega_1(B_j)]^{-\alpha/n}2^{-(k-j)(1+n(1-1/q_1))}. \end{aligned}$$

Set $s_1=1/2+n(1-1/q_1), s_2=\beta+n(1-1/q_1), s_3=\nu+n(1-1/q_1), s_4=1+n(1-1/q_1)$. By $0<\beta<\min\{1/2,\nu\}$ and

$$n(1-1/q_1)<\alpha<n(1-1/q_1)+\beta,$$

we have

$$\begin{aligned} I &\leq \sum_{k=-\infty}^{\infty}\Big(\sum_{j=-\infty}^{k-3}|\lambda_j|[\omega_1(B_k)]^{\alpha/n}||\mu_\Omega^b(a_j)\chi_k||_{L^{q_2}(\omega_2^{q_2})}\Big)^p\\ &\leq C\sum_{k=-\infty}^{\infty}\Big\{\sum_{j=-\infty}^{k-3}|\lambda_j|[\omega_1(B_k)]^{\alpha/n}(I_1+J_2+J_{11}+J_{12})\Big\}^p\\ &\leq C||b||_{\mathrm{Lip}_\beta}^p\sum_{k=-\infty}^{\infty}\Big\{\sum_{j=-\infty}^{k-3}|\lambda_j|\Big[2^{(k-j)(\alpha-s_1)}+2^{(k-j)(\alpha-s_2)}+2^{(k-j)(\alpha-s_3)}+2^{(k-j)(\alpha-s_4)}\Big]\Big\}^p. \end{aligned}$$

When $0 < p \leq 1$,

$$\begin{aligned} I &\leq C||b||^p_{\mathrm{Lip}_\beta}\sum_{k=-\infty}^{\infty}\Big\{\sum_{j=-\infty}^{k-3}|\lambda_j|^p\Big[2^{(k-j)(\alpha-s_1)p}+2^{(k-j)(\alpha-s_2)p} \\ &\quad +2^{(k-j)(\alpha-s_3)p}+2^{(k-j)(\alpha-s_4)p}\Big]\Big\} \\ &\leq C||b||^p_{\mathrm{Lip}_\beta}\sum_{j=-\infty}^{\infty}|\lambda_j|^p\Big\{\sum_{k=j+3}^{\infty}\Big[2^{(k-j)(\alpha-s_1)p}+2^{(k-j)(\alpha-s_2)p} \\ &\quad +2^{(k-j)(\alpha-s_3)p}+2^{(k-j)(\alpha-s_4)p}\Big]\Big\} \\ &\leq C||b||^p_{\mathrm{Lip}_\beta}\sum_{j=-\infty}^{\infty}|\lambda_j|^p. \end{aligned}$$

If $p > 1$, applying Hölder's inequality, we obtain

$$\begin{aligned} I &\leq C||b||^p_{\mathrm{Lip}_\beta}\sum_{k=-\infty}^{\infty}\Big\{\Big[\sum_{j=-\infty}^{k-3}|\lambda_j|^p\Big(2^{(k-j)(\alpha-s_1)p/2}+2^{(k-j)(\alpha-s_2)p/2}+2^{(k-j)(\alpha-s_3)p/2} \\ &\quad +2^{(k-j)(\alpha-s_4)p/2}\Big)\Big]\cdot\Big[\sum_{j=-\infty}^{k-3}\Big(2^{(k-j)(\alpha-s_1)p'/2}+2^{(k-j)(\alpha-s_2)p'/2}+2^{(k-j)(\alpha-s_3)p'/2} \\ &\quad +2^{(k-j)(\alpha-s_4)p'/2}\Big)\Big]^{p/p'}\Big\} \\ &\leq C||b||^p_{\mathrm{Lip}_\beta}\sum_{j=-\infty}^{\infty}|\lambda_j|^p\Big(\sum_{k=j+3}^{\infty}\Big(2^{(k-j)(\alpha-s_1)p/2}+2^{(k-j)(\alpha-s_2)p/2}+2^{(k-j)(\alpha-s_3)p/2} \\ &\quad +2^{(k-j)(\alpha-s_4)p/2}\Big)\Big) \\ &\leq C||b||^p_{\mathrm{Lip}_\beta}\sum_{j=-\infty}^{\infty}|\lambda_j|^p. \end{aligned}$$

The estimates for $I$ and $II$ lead to

$$||\mu^b_\Omega(f)||_{\dot{K}^{\alpha,p}_{q_2}(\omega_1,\omega_2^{q_2})} \leq C||b||_{Lip\beta}(\sum_{j=-\infty}^{\infty}|\lambda_j|^p)^{1/p}.$$

## References

[1] Stein, E. M., On the Function of Littlewood-Paley, Lusin and Marcinkiewicz. Trans Amer Math Soc, 88(1958), 430-466.

[2] Benedek, A., Calderón, A. and Panzone, R., Convolution Operators on Banach Space Valued Functions, Proc Nat Acad Sci USA, 48(1962), 356-365.

[3] Ding, Y., Fan, D. and Pan, Y., $L_p$-boundedness of Marcinkiewicz Integrals with Hardy Functions Kernel, Acta Math Sinica(English Ser), 16:4(2000), 593-600.

[4] Chen, J., Fan, D. and Ying, Y., Rough Marcinkiewicz Integrals with $L(Log^{+}L)^{2}$ Kernels on Product Spaces, Advance in Math., 30:2(2002), 179-182.

[5] Cheng, M. and Shu, L., Boundedness of Commutators of Marcinkiewicz Integral on Hardy Space and Herz-type Hardy Space, Acta. Math. Scientia, 28:A(2008), 222-231.

[6] Lu, S. and Yang, D., The Weighted Herz-type Hardy Space and its Applications, Science in China(Ser A), 25:3(1995), 235-245.

[7] Ding, Y. and Lin, C., Two-weight Norm Inequalities for the Rough Fractional Integrals, IJMMS, 25:8(2001), 517-524.

[8] Zhou, M., Lecture of Harmonic Analysis, Beijing Univ Press,1999.

[9] Lu, S. and Yang, D., The Decomposition of the Weighted Herz Spaces on $R^n$ and its Applications, Science in China(Ser A), 38:2(1995), 147-158.

[10] Wang, X., Weighted Boundedness of Commutators of the Marcinkiewicz Integrals, Journal of Natural Science of Hei Longjiang Univ., 24:3(2007), 324-327.

[11] Stromberg, J. O. and Wheeden, R. L., Fractional Integrals on Weighted $H^p$ and $L^p$ Spaces, Trans Amer Math Soc, 287(1985), 293-321.

[12] Journé JL. Calderón-Zygmund Operators, Pesudo-differential Operators and the Chauchy Integral of Calderon. Lecture Notes in Math., 994(1983), 1-127.

School of Mathematical Science
Xuzhou Normal University
Xuzhou, 221116
P. R. China

E-mail: w-cuilan@126.com