Anal. Theory Appl.
Vol. 28, No. 1 (2012), 19–26

SOME NEW TYPE OF DIFFERENCE SEQUENCE SPACES DEFINED BY MODULUS FUNCTION AND STATISTICAL CONVERGENCE

Ayhan Esi

(*Adiyaman University, Turkey*)

Binod Chandra Tripathy

(*Institute of Advanced Study in Science and Technology, India)*

Received Jan. 13, 2011

Abstract. In this article we introduce the difference sequence spaces $W_0[f,\Delta_m]$, $W_1[f,\Delta_m]$, $W_\infty[f,\Delta_m]$ and $S[f,\Delta_m]$, defined by a modulus function f. We obtain a relation between $W_1[f,\Delta_m]\cap\ell_\infty[f,\Delta_m]$ and $S[f,\Delta_m]\cap\ell_\infty[f,\Delta_m]$ and prove some inclusion results.

Key words: *strongly Cesàro summable sequence, modulus function, statistical convergence*

AMS (2010) subject classification: 40A05, 40A35, 46A45

1 Introduction

Throughout the article w, ℓ_∞, c, c_0 denote the spaces of all, bounded, convergent and null sequences respectively. The zero sequence is denoted by $\theta=(0,0,0,\cdots)$.

The notion of difference sequence was introduced by Kizmaz[4] as follows:

$$Z(\Delta)=\{(x_k)\in w:(\Delta x_k)\in Z\},$$

for $Z=\ell_\infty$, c and c_0, where $\Delta x_k=x_k-x_{k+1}$, for all $k\in\mathbf{N}$.

For further investigation see the work [1],[11-15], [17-21].

The notion of modulus function was introduced by Nakano[6] and further investigated by Ruckle[8], Maddox[5], Tripathy and Chandra[16] and many others.

Definition 1.1. A function $f:[0,\infty)\to[0,\infty)$ is called a modulus if

(i) $f(x)=o$ if and only if $x=0$;

(ii) $f(x+y) \leq f(x)+f(y)$;

(iii) f is increasing;

(iv) f is continuous from the right at 0.

It is immediate from (ii) and (iv)that f is continuous everywhere on $[0,\infty)$.

The notion of statistical convergence was introduced by Fast[2] and Schoenberg[9] independently. Later on it was further investigated by Fridy [3], Rath and Tripathy[7], Tripathy[10],[11], Tripathy and Sarma[21], Tripathy and Sen[22] and many others from sequence space point of view and linked with the summability theory. The notion depends on certain density of subsets of N, the set of natural numbers.

Definition 1.2. A subset E of N is said to have density $\delta(E)$ if

$$\delta(E) = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \chi_E(k) \text{ exists,}$$

where χ_E is the *characteristic function* of E.

Definition 1.3. A sequence (x_n) is said to be *statistically convergent* to L if for every $\varepsilon > 0$, $\delta(\{k \in N : |x_k - L| \geq \varepsilon\}) = 0$. We write $stat - \lim x_k = L$.

2 Definitions and Preliminaries

Definition 2.1. A sequence space E is said to be solid (or normal) if $(x_k) \in E$ implies $(\alpha_k x_k) \in E$, for all sequences of scalars (α_k) with $|\alpha_k| \leq 1$, for all $k \in N$.

Definition 2.2. A sequence space E is said to be monotone if it contains the canonical preimages of all its step spaces.

Remark 2.1. It is clear from the above two definitions that “if a sequence space E is solid, then it is monotone".

Definition 2.3. A sequence space E is said to be convergence free if $(y_k) \in E$ whenever $(x_k) \in E$ and $y_k = 0$ whenever $x_k = 0$.

Definition 2.4. A sequence space E is said to be *symmetric* if $(x_{\pi(n)}) \in E$, whenever $(x_n) \in E$, where π is a permutation of N.

Definition 2.5. A sequence space E is said to be *convergence free* if $(y_n) \in E$, whenever $(x_n) \in E$ and $x_n = 0$ implies $y_n = 0$.

Let $m \in N$ be fixed, then the following new type of difference sequence spaces are introduced and studied by Tripathy and Esi[19].

$$Z(\Delta_m) = \{x = (x_k) \in w : (\Delta_m x_k) \in Z\},$$

for $Z=\ell_\infty$, c and c_0, where $\Delta_m x = (\Delta_m x_k) = (x_k - x_{k+m})$.

The above notion of difference sequence spaces generalize that of difference sequence spaces studied by Kizmaz[4].

Let $m \in N$ be fixed and f be a modulus function. Then we introduce the following sequence spaces in this article.

$W_0[f,\Delta_m] = \left\{ (x_k) \in w : \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(|\Delta_m x_k|) = 0 \right\}.$

$W_1[f,\Delta_m] = \left\{ (x_k) \in w : \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(|\Delta_m x_k - L|) = 0, \text{ for some } L \in C \right\}.$

$W_\infty[f,\Delta_m] = \left\{ (x_k) \in w : \sup_{n} \frac{1}{n} \sum_{k=1}^{n} f(|\Delta_m x_k|) < \infty \right\}.$

$S[f,\Delta_m] = \left\{ (x_k) \in w : \operatorname{stat-lim}_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(|\Delta_m x_k|) = 0 \right\}.$

$\ell_\infty[f,\Delta_m] = \left\{ (x_k) \in w : \sup_{k} f(|\Delta_m x_k|) < \infty \right\}.$

3 Main Results

In this section we prove the results involving the classes of sequences $W_0[f,\Delta_m]$, $W_1[f,\Delta_m]$, $W_\infty[f,\Delta_m]$ and $S[f,\Delta_m]$. The proof of the following result is routine verification.

Proposition 3.1. *The classes of sequences* $W_0[f,\Delta_m]$, $W_1[f,\Delta_m]$,$W_\infty[f,\Delta_m]$ and $S[f,\Delta_m]$ *are linear spaces over* C *the field of complex numbers.*

Theorem 3.2. *For any modulus function* f, *we have*

$$W_0[f,\Delta_m] \subset W_1[f,\Delta_m] \subset W_\infty[f,\Delta_m].$$

Proof. The first inclusion is obvious. Now we prove the second inclusion *i.e.* $W_1[f,\Delta_m] \subset W_\infty[f,\Delta_m]$. Let $(x_k) \in W_1[f,\Delta_m]$, then there exists $L \in C$ such that

$$\frac{1}{n}\sum_{k=1}^{n} f(|\Delta_m x_k - L|) \to 0, \text{ as } n \to \infty.$$

The proof follows from the following inequality

$$\frac{1}{n}\sum_{k=1}^{n} f(|\Delta_m x_k|) \le \frac{1}{n}\sum_{k=1}^{n} f(|\Delta_m x_k - L|) + f(|L|).$$

Theorem 3.3. *The spaces* $W_0[f,\Delta_m]$, $W_1[f,\Delta_m]$ *and* $W_\infty[f,\Delta_m]$ *are linear topological spaces paranormed by*

$$G(x) = \sup_{r} 2^{-r} \sum_{k\in I_r} f(|\Delta_m x_k|),$$

where the summation is over $2^r \le k < 2^{r+1}$, for $r = 0,1,2,\cdots$.

Proof. We prove it for $W_\infty[f,\Delta_m]$ and for the other cases it will follow on applying similar technique. Let x, $y \in W_\infty[f,\Delta_m]$. Then clearly $G(x) \geq 0$, for all x, $G(\theta) = 0$, $G(-x) = G(x)$ and $G(x+y) \leq G(x)+G(y)$. We have for $\lambda \in C$,

$$G(\lambda x) \leq \{1+[|\lambda|]\}G(x).$$

Hence $x \to \theta$ and λ fixed implies $G(\lambda x) \to 0.$.

Next let $\lambda \to 0$ and (x_k) be fixed. Without loss of generality, let $|\lambda| < 1$. Then

$$G(x) = \sup_r 2^{-r} \sum_{k\in I_r} f(|\Delta_m x_k|) < \infty.$$

If the supremum is attained by a finite value of r, then it is clear that $G(x) \to 0$, as $\lambda \to 0$. Next let the supremum be attained for larger values of r. Then

$$2^{-r} \sum_{k\in I_r} f(|\Delta_m x_k|) < \infty.$$

implies that for a given $\varepsilon > 0$, there exists n_0 such that

$$2^{-r} \sum_{k\in I_r, k\geq n_0} f(|\Delta_m x_k|) < \frac{\varepsilon}{2}. \tag{1}$$

Next we have $2^{-r} \sum_{k\in I_r, k\leq n_0} f(|\Delta_m x_k|)$ is finite. Since f is continuous and $\lambda \to 0$, we can choose λ such that

$$2^{-r} \sum_{k\in I_r, k\leq n_0} f(|\lambda \Delta_m x_k|) < \frac{\varepsilon}{2}. \tag{2}$$

Since f is increasing, we have from (1) that

$$2^{-r} \sum_{k\in I_r, k\geq n_0} f(|\lambda \Delta_m x_k|) < \frac{\varepsilon}{2}. \tag{3}$$

Thus we have from (2) and (3) that $G(\lambda x) < \varepsilon$.

Hence the spaces are paranormed by G.

Theorem 3.4. *The spaces $W_0[f,\Delta_m]$, $W_1[f,\Delta_m]$, $W_\infty[f,\Delta_m]$ and $S[f,\Delta_m]$ are not monotone and as such are not solid.*

Proof. The proof follows from the following examples.

Example 3.1. Consider the space $W_\infty[f,\Delta_m]$. Let $f(x) = x$ for all $x \in [0,\infty)$ and let $m = 2$. Then the sequence (x_k) defined by $x_k = k$ for all $k \in N$ belongs to $W_\infty[f,\Delta_m]$.

Consider its canonical preimage space $[W_\infty[f,\Delta_m]]_J$ defined by $(y_k) \in [W_\infty[f,\Delta_m]]_J$ implies $y_k = x_k$ for $k = 4i+1$ and $k = 4i+2$, for all $i \in N$ and $y_k = 0$, otherwise. Then $(y_k) \notin W_\infty[f,\Delta_m]$. Hence the space $W_\infty[f,\Delta_m]$ is not monotone and as such is not solid.

Example 3.2. Let $f(x)=x$ for all $x\in[0,\infty)$ and let $m=2$. If one considers the sequence (x_k) defined by $x_k=1$ for all $k\in N$, then the sequence (x_k) belongs to the spaces $W_0[f,\Delta_m]$, $W_1[f,\Delta_m]$ and $S[f,\Delta_m]$. Consider the canonical preimage spaces of these spaces defined as in Example 3.1. Then (x_k) neither belongs to $W_0[f,\Delta_m]$ nor to $W_1[f,\Delta_m]$ nor to $S[f,\Delta_m]$. Hence the spaces $W_0[f,\Delta_m]$, $W_1[f,\Delta_m]$ and $S[f,\Delta_m]$ are not monotone and hence are not solid.

Theorem 3.5. *The spaces $W_0[f,\Delta_m]$, $W_1[f,\Delta_m]$, $W_\infty[f,\Delta_m]$ and $S[f,\Delta_m]$ are not convergence free.*

Proof. The result follows from the following example.

Example 3.3. Let $f(x)=x$ for all $x\in[0,\infty)$ and let $m=3$. Then the sequence (x_k) defined by $x_k=2$ for all $k\in N$ belongs to the spaces $W_0[f,\Delta_m]$, $W_1[f,\Delta_m]$, $W_\infty[f,\Delta_m]$. Consider the sequence (y_k) defined by $y_k=k^2$ for all $k\in N$. Then (y_k) neither belongs neither to $W_0[f,\Delta_m]$ nor to $W_1[f,\Delta_m]$ nor to $W_\infty[f,\Delta_m]$.

Hence the spaces are not convergence free.

Theorem 3.6. *The spaces $W_0[f,\Delta_m]$, $W_1[f,\Delta_m]$, $W_\infty[f,\Delta_m]$ and $S[f,\Delta_m]$ are not symmetric.*

Proof. The result follows from the following example.

Example 3.4. Let $f(x)=x$, for all $x\in[0,\infty)$ and $m=2$. Consider the sequence (x_n) defined by $x_n=k$, for all $n=k^4$, $k\in N$ and $x_n=0$, otherwise. Then (x_n) belongs to $W_0[f,\Delta_m]$, $W_1[f,\Delta_m]$, $W_\infty[f,\Delta_m]$ and $S[f,\Delta_m]$. Consider its rearrangement defined by

$$(y_n)=(1,0,0,2,0,0,3,0,0,4,0,0,5,0,0,\cdots).$$

Then (y_n) neither belongs to $W_0[f,\Delta_m]$, $W_1[f,\Delta_m]$nor to $W_\infty[f,\Delta_m]$ and nor to $S[f,\Delta_m]$. Hence the spaces are not symmetric.

Theorem 3.7. *Let f be a modulus, then*

(i) $W_Z[\Delta_m]\subset W_0[f,\Delta_m]$, *for* $Z=1,0,\infty$.

(ii) *If* $\beta=\lim\limits_{i\to\infty}\frac{f(t)}{t}>0$, *then* $W_1[\Delta_m]=W_1[f,\Delta_m]$.

Proof. (i) It can be proved by using the techniques applied for establishing Theorem 4 of Maddox[5].

(ii) For any modulus function f, let

$$\beta=\lim_{i\to\infty}\frac{f(t)}{t}>0.$$

Then we can find a $\eta>0$ such that $f(t)\geq\eta t$, for all $t>0$. Then clearly $(x_k)\in W_1[f,\Delta_m]$ will imply $(x_k)\in W_1[\Delta_m]$. The equality follows from (i).

The proof of the following result is a routine verification.

Theorem 3.8. *Let f be a modulus function. Then*

(i) $W_Z[f,\Delta] \subset W_Z[f,\Delta_m]$, *for* $Z = 1,0,\infty$.

(ii) $W_1[f,\Delta_m] \subset W_0[f,\Delta_2 m]$.

(iii) *If m is even, then* $W_1[f,\Delta] \subset W_0[f,\Delta_m]$.

Theorem 3.9. (i) *If* $(x_k) \in W_1[f,\Delta_m]$, *then* $(x_k) \in S[f,\Delta_m]$.

(ii) $W_1[f,\Delta_m] \cap \ell_\infty[f,\Delta_m] = S[f,\Delta_m] \cap \ell_\infty[f,\Delta_m]$.

Proof. (i) Suppose that $(x_k) \in W_1[f,\Delta_m]$ and let $\varepsilon > 0$ be given. Let

$$\frac{1}{n}\sum_{k=1}^{n} f(|\Delta_m x_k - L|) = \sum_1 + \sum_2,$$

where $\sum_1$ and $\sum_2$ denote the sums taken over all values of k for which $f(|\Delta_m x_k - L|) < \varepsilon$ and the sums taken over all those values of k for which $f(|\Delta_m x_k - L|) \geq \varepsilon$ respectively. Then we have

$$\frac{1}{n}\sum_{k=1}^{n} f(|\Delta_m x_k - L|) \geq \frac{1}{n} \text{ Card } \{k \leq n : |\Delta_m x_k - L| \geq \varepsilon\} .\varepsilon.$$

Since $(x_k) \in W_1[f,\Delta_m]$, so we have $\delta(\{k \leq n : |\Delta_m x_k - L| \geq \varepsilon\}) = 0$.

Hence $(x_k) \in S[f,\Delta_m]$.

(ii) Suppose that $(x_k) \in W_1[f,\Delta_m] \cap \ell_\infty[f,\Delta_m]$ and let $\varepsilon > 0$ be given. Let

$$\frac{1}{n}\sum_{k=1}^{n} f(|\Delta_m x_k - L|) = \sum_1 + \sum_2,$$

where $\sum_1$ and $\sum_2$ denote the sums taken over all values of k for which $f(|\Delta_m x_k - L|) < \varepsilon$ and the sums taken over all those values of k for which $f(|\Delta_m x_k - L|) \geq \varepsilon$ respectively. Since $(x_k) \in \ell_\infty[f,\Delta_m]$, so we can find $M > 0$ such that

$$\sup_k f(|\Delta_m x_k - L|) \leq M.$$

Now we have

$$\frac{1}{n}\sum_{k=1}^{n} f(|\Delta_m x_k - L|) \leq \frac{1}{n} \text{ Card } \{k \leq n : |\Delta_m x_k - L| \geq \varepsilon\}.M + \varepsilon.$$

Since ε is arbitrarily small, we have

$$\frac{1}{n}\sum_{k=1}^{n} f(|\Delta_m x_k - L|) \to 0, \text{ as } n \to \infty.$$

Hence we have

$$W_1[f,\Delta_m] \cap \ell_\infty[f,\Delta_m] \supseteq S[f,\Delta_m] \cap \ell_\infty[f,\Delta_m].$$

The reverse inclusion follows from (i). Hence the equality follows.

Conclusion. In this article we have investigated different properties of the classes of sequences $W_0[f,\Delta_m]$, $W_1[f,\Delta_m]$, $W_\infty[f,\Delta_m]$ and $S[f,\Delta_m]$. We have defined these spaces by using a new type of difference operator and a modulus function. The technique used in this article can be applied for further investigating some other classes of sequences.

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Ayhan Esi
Adiyaman University
Science and Art Faculty
02040, Adiyaman
Turkey

E-mail: aesi23@hotmail.com

Binod Chandra Tripathy
Mathematical Sciences Division
Institute of Advanced Study in Science and Technology
Paschim Boragaon Garchuk
Guwahati-781035
India

E-mail: tripathybc@yahoo.com; tripathybc@rediffmail.com