Anal. Theory Appl.
Vol. 28, No. 1 (2012), 87–94

MÜNTZ RATIONAL APPROXIMATION FOR SPECIAL FUNCTION CLASSES IN $Ba[0,1]$ SPACES

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Received Oct. 14, 2008; Revised Feb. 9, 2012

Abstract. In this paper,we research the *Müntz* rational approximation of two kinds of special function classes, and give the corresponding estimates of approximation rates of these classes.

Key words: *Müntz rational approximation, bounded variation function class, Sobolev function class, approximation rate*

AMS (2010) subject classification: 41A25, 43A90

1 Introduction

The space Ba introduced by Ding Xiaqi is a new function space[1].

Definition . Let

$$B=\{L_{p_1}[0,1],L_{p_2}[0,1],\cdots,L_{p_m}[0,1],\cdots\}=:\{L_{p_1},L_{p_2},\cdots,L_{p_m},\cdots\}$$

be a sequence of Lebesgue spaces, $p_m>1(m=1,2,\cdots)$, $a=\{a_1,a_2,\cdots,a_m,\cdots\}$ be a nonnegative real number sequence, if for $f(x)\in\bigcap_{m=1}^{\infty}L_{p_m}$, there is a real number $\alpha>0$, such that

$$I(f,\alpha)=\sum_{m=1}^{\infty}a_m\alpha^m\|f\|_{L_{pm}}^m<+\infty,$$

then we say $f(x)\in Ba$, and the norm of Ba is defined by

$$\|f\|_{Ba}=\inf\{\alpha>0:I(f,\frac{1}{\alpha})\leq 1\}. \tag{1.1}$$

Supported by the National Natural Science Foundation of China(11161033) and Inner Mongolia Natural Science Foundation (2009MS0105).

Communication author: Garidi Wu

Ba is a Banach space under the norm defined by (1.1)[1].

If we choose $B = \{L_p, L_p, \cdots, L_p, \cdots\}$, $a = \{1, 0, \cdots, 0, \cdots\}$, then we get $I(f,\alpha) = \alpha\|f\|_{L_p}$ and

$$\|f\|_{Ba} = \inf\{\alpha > 0 : I(f, \frac{1}{\alpha}) = \frac{\|f\|_{L_p}}{\alpha} \le 1\} = \|f\|_{L_p}.$$

In this paper, we always suppose that $p_0 = \inf_m\{p_m\} > 1$, and denote

$$s = \inf_{m\ge 1}\{a_m^{\frac{1}{m}}\}, \quad q = \sup_{m\ge 1}\{a_m^{\frac{1}{m}}\}.$$

For convenience , we denote

$$\|f\|_p = \|f\|_{L_p}, \qquad 1 \le p < +\infty,$$

$$\|f\|_\infty = \|f\|_C = \max_{0\le t\le 1}|f(t)|, \qquad p = \infty.$$

C always denotes an absolutely positive constant, and $C(s, q, \cdots)$ denotes a positive constant depending on the letters in the brackets. Their values may be different in different place.

Let $L_p[0,1]$ be the space of all p-power integrable functions on [0,1], $1 \le p < +\infty$. when $p = +\infty$, it can be considered as $C[0,1]$, that is, the space of all continuous functions on [0,1]. Also, we denote by $AC[0,1]$ all the absolutely continuous functions on [0,1].

For any given real sequence $\{\lambda_n\}_{n=1}^{\infty}$, denote by $\Pi_n(\Lambda)$ the set of *Müntz* polynomials of degree n, that is, all linear combinations of $\{x^{\lambda_1}, x^{\lambda_2}, \cdots, x^{\lambda_n}\}$, and let $R_n(\Lambda)$ be the *Müntz* rational functions of degree n, that is,

$$R_n(\Lambda) = \{\frac{P(x)}{Q(x)} : P(x), Q(x) \in \Pi_n(\Lambda), Q(x) \ge 0, x \in [0,1]\},$$

if Q(0)=0, we assume that $\lim_{x\to 0^+}\frac{P(x)}{Q(x)}$ exists and is finite.

For $f(x) \in Ba[0,1]$, define the best Müntz rational approximation as

$$R_n(\Lambda)_{Ba} = \inf_{r\in R_n(\Lambda)}\|f - r\|_{Ba}.$$

Our main results are following

Theorem 1. *Assume* $\frac{1}{2} \le \alpha < +\infty$, *given* $M > 0$, *if* $\lambda_{n+1} - \lambda_n \ge Mn^{\alpha}$ *for all* $n \ge 1$, *then for any* $f \in BV[0,1]$, *there is a positive constant* $C(s,q,M)$, *such that*

$$R_n(\Lambda)_{Ba[0,1]} \le C(s,q,M)n^{-\frac{1}{p_0}}V(f).$$

We denote by

$$W_{Ba}^1[0,1] = \{f : f \in AC[0,1], f' \in Ba[0,1]\}$$

the Sobolev function class in Ba space.

Theorem 2. *Assume* $\frac{1}{2} \le \alpha < +\infty$, *given* $M > 0$, *if* $\lambda_{n+1} - \lambda_n \ge Mn^{\alpha}$ *for all* $n \ge 1$, *then for any* $f \in W_{Ba}^1[0,1]$, *there is a positive constant* $C(s,q,M,p_0)$, *such that*

$$R_n(f,\Lambda)_{Ba[0,1]} \le C(s,q,M,p_0)n^{-1}\|f'\|_{Ba[0,1]}.$$

2 Auxiliary Lemmas

For any $x\in[0,1]$, let

$$x=1+\cos\theta,\qquad \frac{\pi}{2}\le\theta\le\pi,$$
$$x_j=1+\cos\theta_j,\qquad \theta_j=\frac{2n-2j+1}{2n}\pi,\qquad j=1,2,\cdots,[\frac{n}{2}].$$

For convenience, we denote $x_0=0$, $x_{[\frac{n}{2}]+1}=1$.

Furthermore, set

$$P_j(x)=x^{\lambda_j}\prod_{l=1}^{j}x_l^{-\Delta\lambda_l},\quad r_k(x)=\frac{P_k(x)}{\sum\limits_{l=1}^{[\frac{n}{2}]}P_l(x)},\ \ j=1,2,\cdots,[\frac{n}{2}],\ \ k=1,2,\cdots,[\frac{n}{2}],$$

where $\Delta\lambda_1=\lambda_1,\ \Delta\lambda_k=\lambda_k-\lambda_{k-1},\ k=2,3,\cdots$.

We construct the rational operator as follows:

$$L_n(f,x)=\sum_{k=1}^{[\frac{n}{2}]}f(x_k)r_k(x).$$

We have

Lemma 1 [12]**.** *Let* $x\in[x_{j-1},x_j],\ \ j=1,2,\cdots,[\frac{n}{2}]+1$*, If* $1\le k\le[\frac{n}{2}]$*, then*

$$r_k(x)\le C_M e^{-M|\sqrt{j}-\sqrt{k}|}.$$

Lemma 2 [12]**.** *If* $x\in[x_{j-1},x_j],\ \ j=1,2,\cdots,[\frac{n}{2}]+1$*, then*

$$|x-x_k|\le C(\frac{(|j-k|+1)^2}{n^2}+\frac{|j-k|+1}{n}\sqrt{x}),\ \ k=1,2,\cdots,[\frac{n}{2}].$$

Lemma 3 [5]**.** *Let* $a=(a_1,a_2,\cdots,a_m,\cdots)$ *be a nonnegative real number sequence,* $B=\{L_{p_1},L_{p_2},\cdots,L_{p_m},\cdots\}$ *be a sequence of Lebesgue spaces* $(p_m>1,m=1,2,\cdots)$*. If* $f\in Ba[0,1]$*, then*

$$\|f\|_{p_m}\le\frac{1}{s}\|f\|_{Ba},\ \ 1<p_m\le\infty.$$

Lemma 4. *For any* $f\in L_{p_m}[0,1]$*, we denote the Hardy-Littlewood maximum function of* f *by*

$$M(f)(x)=\sup_{t\ne x,0\le t\le 1}\frac{1}{t-x}\int_x^t|f(u)|\mathrm{d}u,$$

if $p_0=\inf\limits_m\{p_m\}>1$*, then* $M(f)\in L_{p_m}[0,1]$*, and*

$$\|M(f)\|_{p_m}\le C(p_0)\|f\|_{p_m}.$$

Proof. For $p_m > 1$, and $f \in L_{p_m}$, from the proof of Lemma 6 in [7], we have $M(f) \in L_{p_m}$, and

$$\|M(f)\|_{p_m} \le 2(\frac{5p_m}{p_m-1})^{\frac{1}{p_m}} \|f\|_{p_m} \le 2(\frac{5p_0}{p_0-1})^{\frac{1}{p_0}} \|f\|_{p_m},$$

this means

$$\|M(f)\|_{p_m} \le C(p_0)\|f\|_{p_m},$$

where

$$C(p_0) = 2(\frac{5p_0}{p_0-1})^{\frac{1}{p_0}}.$$

By the definition of norm of the space Ba and Lemma 4, it is easy to see the following conclusion

Corollary *If $f \in Ba[0,1]$, then $M(f) \in Ba[0,1]$, and*

$$\|M(f)\|_{Ba} \le C(s,q,p_0)\|f\|_{Ba}.$$

3 Proof of Theorem

Proof of Theorem 1. We need only to prove

$$\|f - L_n(f)\|_{Ba[0,1]} \le C(s,q,M)n^{-\frac{1}{p_0}}V(f).$$

Applying Jordan decomposition, for any $f \in BV[0,1]$, there exist two monotonically increasing functions $g(x), h(x)$, such that

$$f = g - h, \ \ V(f) = V(g) + V(h).$$

Furthermore, suppose $g(0) = h(0) = 0$, define the monotonic function $g_m(x)$ as follows

$$g_m(x) = \begin{cases} 0, & 0 \le x < x_m, \\ g(x) - g(x_m), & x_m \le x < x_{m+1}, \\ d_m, & x_{m+1} \le x \le 1, \end{cases}$$

where

$$d_m = g(x_{m+1}) - g(x_m), \ \ x_m = 1 + \cos\theta_m, \ \ \theta_m = \frac{2n-2m+1}{2n}\pi, \ \ m = 0, 1, \cdots, [\frac{n}{2}] + 1.$$

Set $x_0 = 0, \ \ x_{[\frac{n}{2}]+1} = 1$, then

$$g(x) = \sum_{m=0}^{[\frac{n}{2}]+1} g_m(x), \ \ V(g) = \sum_{m=0}^{[\frac{n}{2}]+1} V(g_m) = \sum_{m=0}^{[\frac{n}{2}]+1} d_m.$$

First of all, we will show

$$\|g-L_n(g)\|_{Ba}\le C(s,q,M)n^{-\frac{1}{p_0}}V(g).$$

We have

$$\begin{aligned}\|g-L_n(g)\|_{L_{p_m}[0,1]} &= \|\sum_{m=0}^{[\frac{n}{2}]+1}(g_m(x)-L_n(g_m,x))\|_{L_{p_m}[0,1]}\\ &\le \sum_{m=0}^{[\frac{n}{2}]+1}\|g_m(x)-L_n(g_m,x)\|_{L_{p_m}[0,1]},\end{aligned}$$

hence we will estimate $\|g_m(x)-L_n(g_m,x)\|_{L_{p_m}[0,1]}$.

$$\begin{aligned}\|g_m(x)-L_n(g_m,x)\|_{L_{p_m}[0,1]} &= (\int_0^1|g_m(x)-L_n(g_m,x)|^{p_m}\mathrm{d}x)^{\frac{1}{p_m}}\\ &= (\sum_{j=1}^{[\frac{n}{2}]+1}\int_{x_{j-1}}^{x_j}|\sum_{k=1}^{[\frac{n}{2}]}(g_m(x)-g_m(x_k))r_k(x)|^{p_m}\mathrm{d}x)^{\frac{1}{p_m}}\\ &= ([\sum_{j=1}^{m}+\sum_{j=m+2}^{[\frac{n}{2}]+1}]\int_{x_{j-1}}^{x_j}|\sum_{k=1}^{[\frac{n}{2}]}(g_m(x)-g_m(x_k))r_k(x)|^{p_m}\mathrm{d}x\\ &+ \int_{x_m}^{x_{m+1}}|\sum_{k=1}^{[\frac{n}{2}]}(g_m(x)-g_m(x_k))r_k(x)|^{p_m}\mathrm{d}x)^{\frac{1}{p_m}}.\end{aligned}$$

If $0\le p\le 1$ and $a,b\ge 0$, from [11] we have $(a+b)^p\le a^p+b^p$. So it is easy to verify that

$$\begin{aligned}\|g_m-L_n(g_m,x)\|_{L_{p_m}[0,1]} &\le (\sum_{j=1}^{m}\int_{x_{j-1}}^{x_j}|\sum_{k=1}^{[\frac{n}{2}]}(g_m(x)-g_m(x_k))r_k(x)|^{p_m}\mathrm{d}x)^{\frac{1}{p_m}}\\ &+ (\sum_{j=m+2}^{[\frac{n}{2}]+1}\int_{x_{j-1}}^{x_j}|\sum_{k=1}^{[\frac{n}{2}]}|(g_m(x)-g_m(x_k))r_k(x)|^{p_m}\mathrm{d}x)^{\frac{1}{p_m}}\\ &+ (\int_{x_m}^{x_{m+1}}|\sum_{k=1}^{[\frac{n}{2}]}|(g_m(x)-g_m(x_k))r_k(x)|^{p_m}\mathrm{d}x)^{\frac{1}{p_m}} =: K_1+K_2+K_3.\end{aligned}$$

In the following, we use the method of [10]. From the construction of $r_k(x)$, we seen that $r_k(x)\ge 0$, and $\sum_{k=1}^{[\frac{n}{2}]}r_k(x)=1$. On the other hand,

$$\begin{aligned}|x_j-x_{j-1}| &= |\cos\frac{2n-2j+1}{2n}\pi-\cos\frac{2n-2j+3}{2n}\pi|\\ &= |2\sin\frac{n-j+1}{n}\pi\sin\frac{1}{2n}\pi|\\ &= O(\frac{n-j+1}{n}\frac{1}{2n})=O(n^{-1}).\end{aligned}$$

Hence, by using Lemma 1 and Lemma 2, completely similar to the proof of Theorem 3 in [10], we can get

$$K_1 \le C_M d_m n^{-\frac{1}{p_m}},$$
$$K_2 \le C_M d_m n^{-\frac{1}{p_m}},$$

$$\begin{aligned} K_3 &= \Big(\int_{x_m}^{x_{m+1}} \Big|\sum_{k=1}^{[\frac{n}{2}]}(g_m(x)-g_m(x_k))r_k(x)\Big|^{p_m}\mathrm{d}x\Big)^{\frac{1}{p_m}} \\ &= \Big(\int_{x_m}^{x_{m+1}} \Big|g_m(x)-\sum_{k=m+1}^{[\frac{n}{2}]} d_m r_k(x)\Big|^{p_m}\mathrm{d}x\Big)^{\frac{1}{p_m}} \\ &\le d_m\Big(\int_{x_m}^{x_{m+1}} \Big(1+\sum_{k=m+1}^{[\frac{n}{2}]} r_k(x)\Big)^{p_m}\mathrm{d}x\Big)^{\frac{1}{p_m}} \qquad \Big(\sum_{k=1}^{[\frac{n}{2}]} r_k(x)=1\Big) \\ &\le 2d_m n^{-\frac{1}{p_m}}, \end{aligned}$$

therefore,

$$\|g_m - L_n(g_m)\|_{L_{p_m}[0,1]} \le C_M d_m n^{-\frac{1}{p_m}},$$

furthermore,

$$\|g - L_n(g)\|_{L_{p_m}[0,1]} \le C_M n^{-\frac{1}{p_m}} \sum_{m=0}^{[\frac{n}{2}]+1} d_m = C_M n^{-\frac{1}{p_m}} V(g),$$

$$\|g - L_n(g)\|_{Ba[0,1]} \le C(s,q,M) n^{-\frac{1}{p_m}} V(g).$$

As $h(x)$ is also a monotonically increasing function, so we also have

$$\|h - L_n(h)\|_{Ba[0,1]} \le C(s,q,M) n^{-\frac{1}{p_m}} V(h),$$

where $p_0 = \inf_m \{p_m\} > 1$. Hence

$$\|f - L_n(f)\|_{Ba[0,1]} \le C(s,q,M) n^{-\frac{1}{p_m}} (V(h)+V(g)) \le C(s,q,M) n^{-\frac{1}{p_0}} V(f).$$

Theorem 1 is proved.

Proof of Theorem 2. Using Lemma 4, choosing $\zeta_j \in [x_{j-1}, x_j] =: \Delta_j$, such that

$$M(f')(\zeta_j) = \inf\{M(f')(x) : x \in \Delta_j\}.$$

Applying Jensen inequality, we can get

$$\begin{aligned} |f(x)-f(\zeta_j)| &\le |x-\zeta_j| \frac{1}{|x-\zeta_j|}\Big|\int_{\zeta_j}^{x} f'(t)\mathrm{d}t\Big| \\ &\le |x-\zeta_j| M(f')(\zeta_j) \\ &\le |x-\zeta_j| \Big(\frac{1}{|\Delta_j|}\int_{\Delta_j} M(f')^p(t)\mathrm{d}t\Big)^{\frac{1}{p}} \\ &\le C|x-\zeta_j| n^{\frac{1}{p}} \Big(\int_{\Delta_j} M(f')^p(t)\mathrm{d}t\Big)^{\frac{1}{p}}. \end{aligned}$$

Modifying the construction of $L_n(f,x)$ suitably,

$$L_n^*(f,x)=\sum_{j=1}^{[\frac{n}{2}]} f(\zeta_j)r_j(x),$$

then we need only to show

$$\|f-L_n^*(f,x)\|_{Ba[0,1]}\le C(s,q,M,p_0)n^{-1}\|f'\|_{Ba[0,1]}.$$

Using the results mentioned above, and applying Lemma 4, Jensen inequality, we get

$$\begin{aligned}\|f-L_n^*(f,x)\|_{L_{p_m}[0,1]}^{p_m} &= \int_0^1 |\sum_{j=1}^{[\frac{n}{2}]}(f(x)-f(\zeta_j))r_j(x)|^{p_m}\mathrm{d}x\\ &\le C^{p_m}\int_0^1 |\sum_{j=1}^{[\frac{n}{2}]} n^{\frac{1}{p_m}}|x-\zeta_j|r_j(x)(\int_{\Delta_j}|M(f')(t)|^{p_m}\mathrm{d}t)^{\frac{1}{p_m}}|^{p_m}\mathrm{d}x\\ &\le C^{p_m}n\int_0^1\sum_{j=1}^{[\frac{n}{2}]}|r_j(x)|^{p_m}|x-\zeta_j|^{p_m}\int_{\Delta_j}|M(f')(t)|^{p_m}\mathrm{d}t\mathrm{d}x\\ &\le (C(p_0))^{p_m}n\int_0^1\sum_{j=1}^{[\frac{n}{2}]}|r_j(x)|^{p_m}|x-\zeta_j|^{p_m}\int_{\Delta_j}|f'(t)|^{p_m}\mathrm{d}t\mathrm{d}x\\ &\le (C(p_0))^{p_m}n\sum_{j=1}^{[\frac{n}{2}]}\int_{\Delta_j}|f'(t)|^{p_m}\mathrm{d}t\int_0^1|r_j(x)|^{p_m}|x-\zeta_j|^{p_m}\mathrm{d}x.\end{aligned}$$

Considered $0\le r_j(x)\le 1$, by using Lemma 1 and Lemma 2, completely similar to the proof of Theorem 2 in [10], we have

$$\int_0^1 |r_j(x)|^{p_m}|x-\zeta_j|^{p_m}\mathrm{d}x\le\sum_{k=1}^{[\frac{n}{2}]+1}\int_{\Delta_k} r_j(x)|x-\zeta_j|^{p_m}\mathrm{d}x\le C_MC^{p_m}\frac{1}{n^{p_m+1}}.$$

From the above, we have

$$\|f-L_n^*(f,x)\|_{L_{p_m}[0,1]}\le C(M,p_0)n^{-1}\|f'\|_{L_{p_m}[0,1]},$$

$$\|f-L_n^*(f,x)\|_{Ba[0,1]}\le C(s,q,M,p_0)n^{-1}\|f'\|_{Ba[0,1]}.$$

Theorem 2 is proved.

Acknowledgements. The authors thank the referee for giving valuable suggestions to this paper which make them revise this paper in a better form.

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