Abstract. Let $p(z)$ be a polynomial of degree at most $n$. In this paper we obtain some new results about the dependence of

$$\left\| p(Rz) - \beta p(rz) + \alpha \left\{ \left( \frac{R+1}{r+1} \right)^n - |\beta| \right\} p(rz) \right\|_s$$

on $\|p(z)\|_s$ for every $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1$, $|\beta| \leq 1$, $R > r \geq 1$, and $s > 0$. Our results not only generalize some well known inequalities, but also are a variety of interesting results deduced from them by a fairly uniform procedure.

Key words: $L^p$ inequality polynomials, Rouche’s theorem, admissible operator

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1 Introduction and Statement of Results

Let $P_n$ be the class of all complex polynomials

$$p(z) = \sum_{j=0}^{n} a_j z^j$$

of degree at most $n$ and $p'(z)$ its derivative. For $p \in P_n$, define

$$\|p(z)\|_s := \left\{ \frac{1}{2\pi} \int_0^{2\pi} |p(e^{i\theta})|^s \right\}^{\frac{1}{s}}, \quad 1 \leq s < \infty$$

and

$$\|p(z)\|_\infty := \max_{|z|=1} |p(z)|.$$
According to a famous result known as Bernstein’s inequality\[^4\], we have
\[ \|p'(z)\|_\infty \leq n\|p(z)\|_\infty. \] (1)

Also concerning the maximum modulus of \(p(z)\) on \(|z| = R > 1\), we have
\[ \|p(Rz)\|_\infty \leq R^n\|p(z)\|_\infty \] (2)
(for reference see [11]). Zygmund\[^{13}\] has shown
\[ \|p'(z)\|_s \leq n\|p(z)\|_s, \quad s \geq 1. \] (3)
whereas we can deduce the following inequality by applying a result of Hardy\[^{9}\],
\[ \|p(Rz)\|_s \leq R^n\|p(z)\|_s, \quad R > 1, \quad s > 0. \] (4)

Also Arestov\[^{1}\] proved that (3) remains true for \(0 < s < 1\) as well. It is clear that the inequalities (1) and (2) can be obtained by letting \(s \to \infty\) in the inequalities (3) and (4) respectively. If we restrict ourselves to the class of polynomials having no zeros in \(|z| < 1\), the inequalities (3) and (4) can be improved. In fact, it was shown by De-Bruijn\[^{6}\] for \(s \geq 1\) and Rahman and Schmeisser\[^{12}\] extended it for \(0 < s < 1\) that if \(p(z)\) is a polynomial of degree \(n\) having no zeros in \(|z| < 1\), the inequality (3) can be replaced by
\[ \|p'(z)\|_s \leq n\|p(z)\|_s / (1 + \|z\|^s), \quad s > 0. \] (5)

Also Boas and Rahman\[^{5}\] proved for \(s \geq 1\) and Rahman and Schmeisser\[^{12}\] extended it for \(0 < s < 1\) that if \(p(z)\) is a polynomial of degree \(n\) having no zeros in \(|z| < 1\), the inequality (4) can be replaced by
\[ \|p(Rz)\|_s \leq \frac{R^n\|z + 1\|^s}{1 + \|z\|^s}\|p(z)\|_s, \quad R > 1, \quad s > 0. \] (6)

Aziz and Rather\[^{2}\] obtained a generalization of the inequalities (3) and (4). In fact, they have shown that if \(p \in P_n\), then for every \(R > 1\) and \(s \geq 1\),
\[ \|p(Rz) - p(z)\|_s \leq (R^n - 1)\|p(z)\|_s. \] (7)

Recently Aziz and Rather\[^{3}\] considered a more general problem of investigating the dependence of
\[ \|p(Rz) - \beta p(rz)\|_s \quad \text{on} \quad \|p(z)\|_s \]
for every \(\beta \in \mathbb{C}\) with \(|\beta| \leq 1\), \(R > r \geq 1\), \(s > 0\) and extended the inequality (7) for \(0 < s < 1\) as following.
Theorem A. If $p \in P_n$, then for every $\beta \in \mathbb{C}$ with $|\beta| \leq 1$ and $R > r \geq 1$, $s > 0,$
\[ \|p(Rz) - \beta p(rz)\|_s \leq |R^n - \beta r^n| \|p(z)\|_s. \]

Also for the class of polynomials not vanishing in $|z| < 1$, they proved:

Theorem B. If $p \in P_n$ and $p(z)$ does not vanish in $|z| < 1$, then for every $\beta \in \mathbb{C}$ with $|\beta| \leq 1$ and $R > r \geq 1$, $s > 0$
\[ \|p(Re^{i\theta}) - \beta p(re^{i\theta})\|_s \leq \frac{\|(R^n - \beta r^n)z + (1 - \beta)\|_s}{\|1 + z\|_s} \|p(z)\|_s. \] (8)

For self-inversive polynomials, the following inequality was proved by Dewan and Govil\cite{Dewan}.
\[ \|p(Rz) - p(z)\|_s \leq (R^n - 1) \|p(z)\|_s, \quad s \geq 1. \] (9)

Aziz and Rather\cite{Aziz} generalized (9) by proving the following interesting result.

Theorem C. If $p \in P_n$ is self-inversive polynomial, then for every $\beta \in \mathbb{C}$ with $|\beta| \leq 1$ and $R > r \geq 1$, $s > 0$
\[ \|p(Rz) - \beta p(rz)\|_s \leq \frac{\|(R^n - \beta r^n)z + (1 - \beta)\|_s}{\|1 + z\|_s} \|p(z)\|_s. \] (10)

In this paper, we first prove the following result which among other things includes Theorem A as a special case.

Theorem 1. If $p \in P_n$, then for every $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1, \ |\beta| \leq 1$ and $R > r \geq 1$, $s > 0,$
\[ \|p(Rz) - \beta p(rz) + \alpha \left\{ \left( \frac{R + 1}{r + 1} \right)^n - |\beta| \right\} p(rz)\|_s \leq R^n - \beta r^n + \alpha r^n \left\{ \left( \frac{R + 1}{r + 1} \right)^n - |\beta| \right\} \|p(z)\|_s. \]
The result is best possible and equality holds for $p(z) = \lambda z^n, \lambda \neq 0$.

If we take $\alpha = 0$, then Theorem 1 reduces to Theorem A due to Aziz and Rather\cite{Aziz}.

If we assume $\alpha = \beta = 1$ in Theorem 1, then we get the following result.

Corollary 1. If $p \in P_n$, then for $R > r \geq 1$, $s > 0,$
\[ \|p(Rz) - p(rz) + \left\{ \left( \frac{R + 1}{r + 1} \right)^n - 1 \right\} p(rz)\|_s \leq R^n - r^n + \left\{ \left( \frac{R + 1}{r + 1} \right)^n - 1 \right\} \|p(z)\|_s. \] (11)

The result is best possible and equality holds for $p(z) = \lambda z^n, \lambda \neq 0$.

If we take $\beta = 1$ in Theorem 1, we conclude the following result.

Corollary 2. If $p \in P_n$, then for every $\alpha \in \mathbb{C}$ with $|\alpha| \leq 1$ and $R > r \geq 1$, $s > 0,$
\[ \|p(Rz) - p(rz) + \alpha \left\{ \left( \frac{R + 1}{r + 1} \right)^n - 1 \right\} p(rz)\|_s \leq R^n - r^n + \alpha r^n \left\{ \left( \frac{R + 1}{r + 1} \right)^n - 1 \right\} \|p(z)\|_s. \] (12)
The result is best possible and equality holds for \( p(z) = \lambda z^n, \lambda \neq 0 \).

If we divide two sides of (12) by \( (R - r) \) and let \( R \to r \), we get:

**Corollary 3.** If \( p \in P_n \) in , then for every \( \alpha \in \mathbb{C} \) with \( |\alpha| \leq 1 \) and \( r \geq 1, s > 0, \)

\[
\left\| z p'(rz) + \frac{n \alpha}{r + 1} p(rz) \right\|_s \leq n r^{n-1} \left| 1 + \frac{r \alpha}{r + 1} \right| \| p(z) \|_s. \tag{13}
\]

The result is best possible and equality holds for \( p(z) = \lambda z^n, \lambda \neq 0 \).

**Remark 1.** If we let \( s \to \infty \) in (13), then it reduces to the following interesting inequality.

\[
\left| z p'(rz) + \frac{n \alpha}{r + 1} p(rz) \right| \leq n r^{n-1} \left| 1 + \frac{r \alpha}{r + 1} \right| \max_{|z|=1} |p(z)|, \quad |\alpha| \leq 1, \quad s > 1, \quad |z| = 1. \tag{14}
\]

For \( r = 1 \), (14) reduces to the following inequality which is due to Jain\(^8\).

\[
\left| z p'(z) + \frac{n \alpha}{2} p(z) \right| \leq n \left| 1 + \frac{\alpha}{2} \right| \max_{|z|=1} |p(z)|, \quad |\alpha| \leq 1, \quad |z| = 1. \tag{15}
\]

Therefore, for \( r = 1 \) in (13), we get the following interesting result which is a generalization of (15).

**Corollary 4.** If \( p \in P_n \), then for every \( \alpha \in \mathbb{C} \) with \( |\alpha| \leq 1, s > 0, \)

\[
\left\| z p'(z) + \frac{n \alpha}{2} p(z) \right\|_s \leq n \left| 1 + \frac{\alpha}{2} \right| \| p(z) \|_s. \tag{16}
\]

The result is best possible and equality holds for \( p(z) = \lambda z^n, \lambda \neq 0 \).

For \( \alpha = 0 \), (16) reduces to (3).

For \( p \in P_n \) and \( p(z) \) does not vanish in \( |z| < 1 \), we prove the following generalization of (8) and improvement of (16).

**Theorem 2.** If \( p \in P_n \) and \( p(z) \) does not vanish in \( |z| < 1 \), then for every \( \alpha, \beta \in \mathbb{C} \) with \( |\alpha| \leq 1, |\beta| \leq 1 \) and \( R > r \geq 1, s > 0, \)

\[
\left\| p(Rz) - \beta p(rz) + \alpha \left\{ \left( \frac{R + 1}{r + 1} \right)^n - |\beta| \right\} p(rz) \right\|_s
\leq \left\| \frac{R^n - \beta r^n + \alpha n \left( \frac{R + 1}{r + 1} \right)^n - |\beta| \right\} z + \left[ 1 - \beta + \alpha \left\{ \left( \frac{R + 1}{r + 1} \right)^n - |\beta| \right\} \right] z \| p(z) \|_s, \tag{17}
\]

The result is best possible and equality holds for \( p(z) = \lambda z^n + \gamma, |\lambda| = |\gamma| = 1 \).

For \( \alpha = 0 \), Theorem 2 reduces to Theorem B. Also if we take \( \alpha = \beta = 0 \), then (17) reduces to (6).

The following consequence is concluded by applying Minkowski’s inequality in right hand side of (17).
Corollary 5. If \( p \in P_n \) and \( p(z) \) does not vanish in \( |z| < 1 \), then for every \( \alpha, \beta \in \mathbb{C} \) with 
\[ |\alpha| \leq 1, \quad |\beta| \leq 1, \quad \text{and} \quad R > r \geq 1, \quad s \geq 1, \]
\[ \left\| \left[ p(Rz) - \beta p(rz) + \alpha \left\{ \frac{R+1}{r+1} \right\}^n p(rz) \right] \right\|_s \leq \left\| R^n - \beta r^n + \alpha r^n \left\{ \frac{R+1}{r+1} \right\}^n - |\beta| \right\| p(z) \right\|_s. \]  
(18)
The result is best possible and equality holds for \( p(z) = \lambda z^n + \gamma \), \( |\lambda| = |\gamma| = 1 \).

Remark 2. If we take \( \beta = 1 \), and divide both sides of (18) by \( R - r \) and let \( R \to r \), we get the following result.

Corollary 6. If \( p \in P_n \) and \( p(z) \) does not vanish in \( |z| < 1 \), then for every \( \alpha \in \mathbb{C} \) with 
\[ |\alpha| \leq 1 \] and \( r \geq 1, \quad s \geq 1 \),
\[ \left\| zp'(rz) + \frac{n\alpha}{r+1} p(rz) \right\|_s \leq \frac{n}{\|1+z\|_s} \left\{ r^{n-1} \left| 1 + \frac{r\alpha}{r+1} \right| + \left| \frac{r\alpha}{r+1} \right| \right\} \|p(z)\|_s. \]  
(19)
The result is best possible and equality holds for \( p(z) = \lambda z^n + \gamma \), \( |\lambda| = |\gamma| = 1 \).

By letting \( s \to \infty \) in (19), we get the following result.

Corollary 7. If \( p \in P_n \) and \( p(z) \) does not vanish in \( |z| < 1 \), then for every \( \alpha \in \mathbb{C} \) with 
\[ |\alpha| \leq 1, \quad \text{and} \quad r \geq 1, \]
\[ \max_{|z|=1} \left| zp'(rz) + \frac{n\alpha}{r+1} p(rz) \right| \leq \frac{n}{2} \left\{ r^{n-1} \left| 1 + \frac{\alpha}{r+1} \right| + \left| \frac{\alpha}{r+1} \right| \right\} \max_{|z|=1} |p(z)|. \]  
(20)
The result is best possible and equality holds for \( p(z) = \lambda z^n + \gamma \), \( |\lambda| = |\gamma| = 1 \).

For \( r = 1 \), (20) reduces to the following inequality which is due to Jain\[^8\].
\[ \left| zp'(z) + \frac{n\alpha}{2} p(z) \right| \leq \frac{n}{2} \left\{ \left| 1 + \frac{\alpha}{2} \right| + \left| \frac{\alpha}{2} \right| \right\} \max_{|z|=1} |p(z)|, \quad |\alpha| \leq 1, \quad |z| = 1. \]  
(21)
Therefore, for \( r = 1 \) in (19), we get the following interesting result which is a generalization of (21) and improvement of (16).

Corollary 8. If \( p \in P_n \) and \( p(z) \) does not vanish in \( |z| < 1 \), then for every \( \alpha \in \mathbb{C} \) with 
\[ |\alpha| \leq 1, \quad s \geq 1, \]
\[ \left\| zp'(z) + \frac{n\alpha}{2} p(z) \right\|_s \leq \frac{n}{\|1+z\|_s} \left\{ \left| 1 + \frac{\alpha}{2} \right| + \left| \frac{\alpha}{2} \right| \right\} \|p(z)\|_s. \]  
(22)
The result is best possible and equality holds for \( p(z) = \lambda z^n + \gamma \), \( |\lambda| = |\gamma| = 1 \).

If we take \( \alpha = 0 \), then (22) reduces to (5) for \( s \geq 1 \).

Finally, we prove the following result for self inverse polynomials which is an improvement as well a generalization of some well known results.
Theorem 3. If \( p \in P_n \) is self-inversive polynomial, then for every \( \alpha, \beta \in \mathbb{C} \) with \( |\alpha| \leq 1, |\beta| \leq 1, \) and \( R > r \geq 1, s > 0, \)

\[
\| p(Rz) - \beta p(rz) + \alpha \left\{ \left( \frac{R+1}{r+1} \right)^n - \left| \beta \right| \right\} p(rz) \|_s \\
\leq \frac{\|R^n - \beta r^n + \alpha r^n \left\{ \left( \frac{R+1}{r+1} \right)^n - \left| \beta \right| \right\} z + [1 - \beta + \alpha \left\{ \left( \frac{R+1}{r+1} \right)^n - \left| \beta \right| \right\}] \|_s \|p(z)\|_s.
\]

(23)

The result is best possible and equality holds for \( p(z) = z^n + 1. \)

For \( \alpha = 0, \) Theorem 3 reduces to Theorem C. If we take \( \alpha = 0, \beta = r = 1 \) in Theorem 3, then the inequality (23) reduces to (9). Also from this theorem, we can deduce so many interesting results in a similar manner as the previous one.

2 Lemmas

For the proof of the theorems, we require the following lemmas.

**Lemma 1.** If \( p(z) \) is a polynomial of degree \( n \) having all its zeros in \( |z| \leq 1, \) then for every \( R \geq r \geq 1, \) and \( |z| = 1, \)

\[
|p(Rz)| \geq \left( \frac{R+1}{r+1} \right)^n |p(rz)|.
\]

(24)

**Lemma 2.** If \( F(z) \) be a polynomial of degree \( n \) having all its zeros in \( |z| \leq 1 \) and \( f(z) \) be a polynomial of degree at most \( n \) such that \( |f(z)| \leq |F(z)| \) for \( |z| = 1, \) then for all \( \alpha, \beta \in \mathbb{C} \) with \( |\alpha| \leq 1, |\beta| \leq 1, \) and \( R > r \geq 1, |z| \geq 1, \)

\[
\left| f(Rz) - \beta f(rz) + \alpha \left\{ \left( \frac{R+1}{r+1} \right)^n - \left| \beta \right| \right\} f(rz) \right| \\
\leq \left| F(Rz) - \beta F(rz) + \alpha \left\{ \left( \frac{R+1}{r+1} \right)^n - \left| \beta \right| \right\} F(rz) \right|.
\]

(25)

Lemmas 1 and 2 are due to Liman\(^{[10]}\).

For \( \gamma = (\gamma_0, \gamma_1, \cdots, \gamma_n) \) and \( p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0, \) let

\[
\Lambda_\gamma p(z) = \sum_{j=0}^{n} \gamma_j a_j z^j.
\]

The operator \( \Lambda_\gamma \) is said to be admissible if it preserves one of the following properties:

1) \( p(z) \) has all its zeros in \( |z| \leq 1. \)

2) \( p(z) \) has all its zeros in \( |z| \geq 1. \)
Now we state a result of Arestov\[1\].

**Lemma 3.** Let \( \phi(x) = \psi(\log x) \) where \( \psi \) is a convex nondecreasing function on \( \mathbb{R} \). Then for all \( p \in \mathcal{P}_n \) and each admissible operator \( \Lambda_\gamma \),

\[
\int_{0}^{2\pi} \phi \left( |\Lambda_\gamma p(e^{i\theta})| \right) d\theta \leq \int_{0}^{2\pi} \phi \left( C(\gamma,n)|p(e^{i\theta})| \right) d\theta
\]

(26)

where \( C(\gamma,n) = \max \left( |\gamma_0|, |\gamma_n| \right) \).

By applying Lemma 3 to the function \( \phi(x) = x^s \) for every \( s > 0 \), we get

\[
\int_{0}^{2\pi} \phi \left( |\Lambda_\gamma p(e^{i\theta})|^s \right) d\theta \leq (C(\gamma,n))^s \int_{0}^{2\pi} |p(e^{i\theta})|^s d\theta.
\]

(27)

**Lemma 4.** If \( p \in \mathcal{P}_n \) and \( p(z) \) does not vanish in \( |z| < 1 \), then for every \( \alpha, \beta \in \mathbb{C} \) with \( |\alpha| \leq 1, |\beta| \leq 1 \) and \( R > r \geq 1, s > 0 \) and \( \gamma \) real,

\[
\int_{0}^{2\pi} \left[ p(Re^{i\theta}) - \beta p(re^{i\theta}) + \alpha \left\{ \left( \frac{R+1}{r+1} \right)^n - |\beta| \right\} p(re^{i\theta}) \right. \\
\left. + e^{\gamma} \left( R^np(e^{i\theta}/R) - \overline{\beta}r^n p(e^{i\theta}/r) + \alpha \theta^n \left\{ \left( \frac{R+1}{r+1} \right)^n - |\beta| \right\} p(e^{i\theta}/r) \right\}^s d\theta \right. \\
\leq \left. \left| R^n - \beta r^n + \alpha \theta^n \left\{ \left( \frac{R+1}{r+1} \right)^n - |\beta| \right\} \right|^s \int_{0}^{2\pi} |p(e^{i\theta})|^s d\theta. \right)
\]

(28)

**Proof of Lemma 4.** Let \( q(z) = z^s \overline{p(1/z)} \). Applying Lemma 2 to the polynomials \( p(z) \) and \( q(z) \), we get for any \( \alpha, \beta \in \mathbb{C} \) with \( |\alpha| \leq 1, |\beta| \leq 1 \) and \( R > r \geq 1, \)

\[
\left| p(Rz) - \beta p(rz) + \alpha \left\{ \left( \frac{R+1}{r+1} \right)^n - |\beta| \right\} p(rz) \right| \\
\leq \left| q(Rz) - \beta q(rz) + \alpha \left\{ \left( \frac{R+1}{r+1} \right)^n - |\beta| \right\} q(rz) \right| \\
= R^n p(z/R) - \beta r^n p(z/r) + \alpha \theta^n \left\{ \left( \frac{R+1}{r+1} \right)^n - |\beta| \right\} p(z/r), \quad \text{for } |z| = 1.
\]

(29)

On the other hand, we have \( |q(rz)| \leq |q(Rz)| \) for \( |z| = 1, R > r \geq 1 \). Since \( q(Rz) \) has all its zeros in \( |z| \leq 1/R < 1 \), a direct application of Rouche’s theorem shows that the polynomial \( q(Rz) - \beta q(rz) \) has all its zeros in \( |z| < 1 \) for every \( \beta \in \mathbb{C} \) with \( |\beta| \leq 1 \). By Lemma 1, we have for \( R > r \geq 1, \)

\[
|q(Rz) - \beta q(rz)| \geq |q(Rz)| - |\beta||q(rz)| > \left\{ \left( \frac{R+1}{r+1} \right)^n - |\beta| \right\} |q(rz)|, \quad \text{for } |z| = 1.
\]

(30)
Therefore, again by applying Rouche’s theorem, it follows that for any $\alpha \in \mathbb{C}$ with $|\alpha| \leq 1$ and $R > r \geq 1$, all the zeros of the polynomial

$$H(z) := q(Rz) - \beta q(rz) + \alpha \left\{ \left( \frac{R+1}{r+1} \right)^n - |\beta| \right\} q(rz)$$

lie in $|z| < 1$. This implies that the polynomial

$$z^n H(1/z) = R^n p(z/R) - \overline{\beta} r^n p(z/r) + \overline{\alpha} r^n \left\{ \left( \frac{R+1}{r+1} \right)^n - |\beta| \right\} p(z/r)$$

has all its zeros in $|z| > 1$. Hence the function

$$f(z) := \frac{p(Rz) - \beta p(rz) + \alpha \left\{ \left( \frac{R+1}{r+1} \right)^n - |\beta| \right\} p(rz)}{R^n p(z/R) - \overline{\beta} r^n p(z/r) + \overline{\alpha} r^n \left\{ \left( \frac{R+1}{r+1} \right)^n - |\beta| \right\} p(z/r)}$$

is analytic in $|z| \leq 1$ and $|f(z)| \leq 1$ for $|z| = 1$. By applying the Maximum Modulus Principle, we get

$$|f(z)| < 1, \quad \text{for } |z| < 1.$$ 

Equivalently,

$$\left| p(Rz) - \beta p(rz) + \alpha \left\{ \left( \frac{R+1}{r+1} \right)^n - |\beta| \right\} p(rz) \right| < \left| R^n p(z/R) - \overline{\beta} r^n p(z/r) + \overline{\alpha} r^n \left\{ \left( \frac{R+1}{r+1} \right)^n - |\beta| \right\} p(z/r) \right|,$$

for $|z| < 1$. 

(31)

A direct application of Rouche’s theorem shows that

$$\Lambda_{\gamma} p(z) = p(Rz) - \beta p(rz) + \alpha \left\{ \left( \frac{R+1}{r+1} \right)^n - |\beta| \right\} p(rz)$$

$$+ e^{\gamma} \left[ R^n p(z/R) - \overline{\beta} r^n p(z/r) + \overline{\alpha} r^n \left\{ \left( \frac{R+1}{r+1} \right)^n - |\beta| \right\} p(z/r) \right]$$

$$= \left( R^n - \beta r^n + \alpha r^n \left\{ \left( \frac{R+1}{r+1} \right)^n - |\beta| \right\} + e^{\gamma} \left[ 1 - \overline{\beta} + \overline{\alpha} \left\{ \left( \frac{R+1}{r+1} \right)^n - |\beta| \right\} \right] \right) a_n z^n$$

$$+ \ldots$$

$$+ \left( 1 - \beta + \alpha \left\{ \left( \frac{R+1}{r+1} \right)^n - |\beta| \right\} + e^{\gamma} \left[ R^n - \overline{\beta} r^n + \overline{\alpha} r^n \left\{ \left( \frac{R+1}{r+1} \right)^n - |\beta| \right\} \right] \right) a_0 z^0$$

(32)

does not vanish in $|z| < 1$ for every $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1$, $|\beta| \leq 1$ and $R > r \geq 1$, and $\gamma$ real. Therefore $\Lambda_{\gamma}$ is an admissible operator. By applying (27), the desired result follows. This completes the proof of Lemma 4.
Lemma 5. If \( p \in P_n \), then for every \( \alpha, \beta \in \mathbb{C} \) with \(|\alpha| \leq 1, |\beta| \leq 1 \) and \( R > r \geq 1, s > 0 \) and \( \gamma \) real,

\[
\int_{0}^{2\pi} \left[ p(Re^{i\theta}) - \beta p(re^{i\theta}) + \alpha \left( \frac{R+1}{r+1} \right)^n - |\beta| \right] p(re^{i\theta}) \, d\theta \\
+ e^{i\gamma} \left[ R^n p(z/R) - \beta r^n p(z/r) + \overline{\alpha} r^n \left( \frac{R+1}{r+1} \right)^n - |\beta| \right] p(z/r) \right] ^{s} d\theta \\
\leq \left| R^n - \beta r^n + \alpha r^n \left( \frac{R+1}{r+1} \right)^n - |\beta| \right| \\
+ e^{i\gamma} \left[ 1 - \beta + \overline{\alpha} \left( \frac{R+1}{r+1} \right)^n - |\beta| \right] \right] ^{s} \int_{0}^{2\pi} |p(e^{i\theta})|^s \, d\theta. \tag{33}
\]

Proof of Lemma 5. Let \( p(z) \) be a polynomial of degree at most \( n \), we can write \( p(z) = p_1(z)p_2(z) \) such that \( p_1(z) \) is a polynomial of degree \( k \geq 1 \) having all its zeros in \(|z| > 1 \) and \( p_2(z) \) is a polynomial of degree \( n - k \) having all its zeros in \(|z| < 1 \). First we suppose that \( p_1(z) \) does not vanish on \(|z| = 1 \) and hence all the zeros of \( p_1(z) \) lie in \(|z| > 1 \). Let \( q_2(z) = z^{n-k}p_2(1/z) \), then all the zeros of \( q_2(z) \) lie in \(|z| > 1 \) and \(|q_2(z)| = |p_2(z)| \) for \(|z| = 1 \). Therefore the polynomial \( g(z) = p_1(z)q_2(z) \) is a polynomial of degree \( n \) not vanishing in \(|z| < 1 \) and for \(|z| = 1 \),

\[
|g(z)| = |p_1(z)||q_2(z)| = |p_1(z)||p_2(z)| = |p(z)|. \tag{34}
\]

A direct application of Rouche’s theorem show that \( h(z) := p(z) + \lambda g(z) \) does not vanish in \(|z| < 1, \lambda \in \mathbb{C} \) with \(|\lambda| > 1 \). Also \( h(z) \) does not vanish on \(|z| = 1 \), because if this is not true then it would contradict with (34). Thus \( h(z) \) does not vanish in \(|z| \leq 1 \) for any \( \lambda \) with \(|\lambda| > 1 \), so that all the zeros of \( h(z) \) lie in \(|z| > \rho \) for some \( \rho > 1 \) and hence all the zeros of \( h(\rho z) \) lie in \(|z| \geq 1 \). Applying (31) to the polynomial \( h(\rho z) \), we get

\[
\left| h(\rho z) - \beta h(\rho z) + \alpha \left( \frac{R+1}{r+1} \right)^n - |\beta| \right| h(\rho z) \\
< \left| R^n h(\rho z/R) - \beta r^n h(\rho z/r) + \overline{\alpha} r^n \left( \frac{R+1}{r+1} \right)^n - |\beta| \right| h(\rho z/r), \quad \text{for } |z| < 1, R > r \geq 1. \tag{35}
\]

Taking \( z = e^{i\theta}/\rho, 0 \leq \theta < 2\pi \), then \(|z| = (1/\rho) < 1 \) as \( \rho > 1 \), and we get

\[
\left| h(Re^{i\theta}) - \beta h(re^{i\theta}) + \alpha \left( \frac{R+1}{r+1} \right)^n - |\beta| \right| h(re^{i\theta}) \\
< \left| R^n h(e^{i\theta}/R) - \beta r^n h(e^{i\theta}/r) + \overline{\alpha} r^n \left( \frac{R+1}{r+1} \right)^n - |\beta| \right| h(e^{i\theta}/r), \quad 0 \leq \theta < 2\pi, R > r \geq 1. \tag{36}
\]
Or
\[
|h(Rz) - \beta h(rz) + \alpha \left\{ \left( \frac{R + 1}{r + 1} \right)^n - |\beta| \right\} h(rz)| < |R^n h(z/R) - \overline{\beta} r^n h(z/r) + \overline{\alpha} r^n \left\{ \left( \frac{R + 1}{r + 1} \right)^n - |\beta| \right\} h(z/r)|, \quad \text{for } |z| = 1. \tag{37}
\]

By Rouche’s theorem, it follows that the polynomial
\[
T(z) := \left( h(Rz) - \beta h(rz) + \alpha \left\{ \left( \frac{R + 1}{r + 1} \right)^n - |\beta| \right\} h(rz) \right)
+ e^{i\gamma} \left( R^n h(z/R) - \overline{\beta} r^n h(z/r) + \overline{\alpha} r^n \left\{ \left( \frac{R + 1}{r + 1} \right)^n - |\beta| \right\} h(z/r) \right)
\]
does not vanish in $|z| \leq 1$ for every $\alpha$, $\beta \in \mathbb{C}$ with $|\alpha| \leq 1$, $|\beta| \leq 1$, $R > r \geq 1$, and $\gamma$ real. If we replace $h(z)$ by $p(z) + \lambda g(z)$, then the polynomial
\[
T(z) = \left\{ p(Rz) - \beta p(rz) + \alpha \left\{ \left( \frac{R + 1}{r + 1} \right)^n - |\beta| \right\} p(rz) \right.
+ e^{i\gamma} \left[ R^n p(z/R) - \overline{\beta} r^n p(z/r) + \overline{\alpha} r^n \left\{ \left( \frac{R + 1}{r + 1} \right)^n - |\beta| \right\} p(z/r) \right]
\]
\[
+ \lambda \left\{ g(Rz) - \beta g(rz) + \alpha \left\{ \left( \frac{R + 1}{r + 1} \right)^n - |\beta| \right\} g(rz) \right.
+ e^{i\gamma} \left[ R^n g(z/R) - \overline{\beta} r^n g(z/r) + \overline{\alpha} r^n \left\{ \left( \frac{R + 1}{r + 1} \right)^n - |\beta| \right\} g(z/r) \right] \}
\tag{38}
\]
does not vanish in $|z| \leq 1$ for every $\alpha$, $\lambda$, $\beta \in \mathbb{C}$ with $|\alpha| \leq 1$, $|\beta| \leq 1$, $|\lambda| > 1$, $R > r \geq 1$, and $\gamma$ real. This implies
\[
|p(Rz) - \beta p(rz) + \alpha \left\{ \left( \frac{R + 1}{r + 1} \right)^n - |\beta| \right\} p(rz) \]
\[
+ e^{i\gamma} \left[ R^n p(z/R) - \overline{\beta} r^n p(z/r) + \overline{\alpha} r^n \left\{ \left( \frac{R + 1}{r + 1} \right)^n - |\beta| \right\} p(z/r) \right|
\leq |g(Rz) - \beta g(rz) + \alpha \left\{ \left( \frac{R + 1}{r + 1} \right)^n - |\beta| \right\} g(rz) \]
\[
+ e^{i\gamma} \left[ R^n g(z/R) - \overline{\beta} r^n g(z/r) + \overline{\alpha} r^n \left\{ \left( \frac{R + 1}{r + 1} \right)^n - |\beta| \right\} g(z/r) \right| \}
\tag{39}
\]
for $|z| \leq 1$, $|\alpha| \leq 1$, $|\beta| \leq 1$, $R > r \geq 1$, and $\gamma$ real. If the inequality (39) is not true, then we
would have

\[
p(Rz_0) - \beta p(rz_0) + \alpha \left\{ \left( \frac{R+1}{r+1} \right)^n - |\beta| \right\} p(rz_0) \\
+ e^{iT} \left[ R^\alpha p(z_0/R) - \overline{\beta} r^n p(z_0/r) + \overline{\alpha} r^n \left\{ \left( \frac{R+1}{r+1} \right)^n - |\beta| \right\} p(z_0/r) \right] \\
> \left| g(Rz) - \beta g(rz) + \alpha \left\{ \left( \frac{R+1}{r+1} \right)^n - |\beta| \right\} g(rz) \\
+ e^{iT} \left[ R^\alpha g(z_0/R) - \overline{\beta} r^n g(z_0/r) + \overline{\alpha} r^n \left\{ \left( \frac{R+1}{r+1} \right)^n - |\beta| \right\} g(z_0/r) \right] \\
+ e^{iT} \left[ R^\alpha g(z_0/R) - \overline{\beta} r^n g(z_0/r) + \overline{\alpha} r^n \left\{ \left( \frac{R+1}{r+1} \right)^n - |\beta| \right\} g(z_0/r) \right],
\]

for some \( z_0 \) with \( |z_0| \leq 1 \). Since all the zeros of polynomial \( g(z) \) lie in \( |z| > 1 \), it follows (as before) that all the zeros of polynomial

\[
g(Rz) - \beta g(rz) + \alpha \left\{ \left( \frac{R+1}{r+1} \right)^n - |\beta| \right\} g(rz) \\
+ e^{iT} \left[ R^\alpha g(z_0/R) - \overline{\beta} r^n g(z_0/r) + \overline{\alpha} r^n \left\{ \left( \frac{R+1}{r+1} \right)^n - |\beta| \right\} g(z_0/r) \right]
\]

also lie in \( |z| > 1 \). Hence

\[
g(Rz_0) - \beta g(rz_0) + \alpha \left\{ \left( \frac{R+1}{r+1} \right)^n - |\beta| \right\} g(rz_0) \\
+ e^{iT} \left[ R^\alpha g(z_0/R) - \overline{\beta} r^n g(z_0/r) + \overline{\alpha} r^n \left\{ \left( \frac{R+1}{r+1} \right)^n - |\beta| \right\} g(z_0/r) \right] \neq 0
\]

for any \( |z| \leq 1 \). So we can take a suitable value for \( \lambda \) such that \( |\lambda| > 1 \) and \( T(z_0) = 0 \) with \( |z_0| \leq 1 \). This clearly is a contradiction to the fact that \( T(z) \) does not vanish in \( |z| \leq 1 \). The inequality (39) gives for each \( s > 0 \) and \( 0 \leq \theta < 2\pi \),

\[
\int_0^{2\pi} \left| p(R e^{i\theta}) - \beta p(r e^{i\theta}) + \alpha \left\{ \left( \frac{R+1}{r+1} \right)^n - |\beta| \right\} p(r e^{i\theta}) \\
+ e^{iT} \left[ R^\alpha p(e^{i\theta}/R) - \overline{\beta} r^n p(e^{i\theta}/r) + \overline{\alpha} r^n \left\{ \left( \frac{R+1}{r+1} \right)^n - |\beta| \right\} p(e^{i\theta}/r) \right] \right|^s d\theta \\
\leq \int_0^{2\pi} \left| g(R e^{i\theta}) - \beta g(r e^{i\theta}) + \alpha \left\{ \left( \frac{R+1}{r+1} \right)^n - |\beta| \right\} g(r e^{i\theta}) \\
+ e^{iT} \left[ R^\alpha g(e^{i\theta}/R) - \overline{\beta} r^n g(e^{i\theta}/r) + \overline{\alpha} r^n \left\{ \left( \frac{R+1}{r+1} \right)^n - |\beta| \right\} g(e^{i\theta}/r) \right] \right|^s d\theta.
\]

By applying Lemma 4 to \( g(z) \) and using (34), we get for any \( \alpha, \beta \in \mathbb{C} \) with \( |\alpha| \leq 1, |\beta| \leq 1 \),
$R > r \geq 1$, $s > 0$ and $\gamma$ real,

$$
\int_0^{2\pi} \left| p(Re^{i\theta}) - \beta p(re^{i\theta}) + \alpha \left\{ \left( \frac{R+1}{r+1} \right)^n - |\beta| \right\} p(re^{i\theta}) \right|^s d\theta
+ e^{i\gamma} \left[ R^n p(e^{i\theta}/R) - \overline{\beta} r^n p(e^{i\theta}/r) + \overline{\alpha} r^n \left\{ \left( \frac{R+1}{r+1} \right)^n - |\beta| \right\} p(e^{i\theta}/r) \right]^s d\theta
\leq \left| R^n - \beta r^n + \alpha r^n \left\{ \left( \frac{R+1}{r+1} \right)^n - |\beta| \right\} \right|^s d\theta
+ e^{i\gamma} \left[ 1 - \overline{\beta} + \overline{\alpha} \left\{ \left( \frac{R+1}{r+1} \right)^n - |\beta| \right\} \right]^s \int_0^{2\pi} |g(e^{i\theta})|^s d\theta
= \left| R^n - \beta r^n + \alpha r^n \left\{ \left( \frac{R+1}{r+1} \right)^n - |\beta| \right\} \right|^s d\theta
+ e^{i\gamma} \left[ 1 - \overline{\beta} + \overline{\alpha} \left\{ \left( \frac{R+1}{r+1} \right)^n - |\beta| \right\} \right]^s \int_0^{2\pi} |p(e^{i\theta})|^s d\theta.
$$

(41)

Now if $p_1(z)$ has a zero on $|z| = 1$, then applying (41) to the polynomial $p^*(z) = p_1(z)p_2(z)$ where $t < 1$, we get for any $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1$, $|\beta| \leq 1$, $R > r \geq 1$, $s > 0$ and $\gamma$ real,

$$
\int_0^{2\pi} \left| p^*(Re^{i\theta}) - \beta p^*(re^{i\theta}) + \alpha \left\{ \left( \frac{R+1}{r+1} \right)^n - |\beta| \right\} p^*(re^{i\theta}) \right|^s d\theta
+ e^{i\gamma} \left[ R^n p^*(e^{i\theta}/R) - \overline{\beta} r^n p^*(e^{i\theta}/r) + \overline{\alpha} r^n \left\{ \left( \frac{R+1}{r+1} \right)^n - |\beta| \right\} p^*(e^{i\theta}/r) \right]^s d\theta
\leq \left| R^n - \beta r^n + \alpha r^n \left\{ \left( \frac{R+1}{r+1} \right)^n - |\beta| \right\} \right|^s d\theta
+ e^{i\gamma} \left[ 1 - \overline{\beta} + \overline{\alpha} \left\{ \left( \frac{R+1}{r+1} \right)^n - |\beta| \right\} \right]^s \int_0^{2\pi} |p^*(e^{i\theta})|^s d\theta.
$$

(42)

Letting $t \longrightarrow 1$ in (42) and using continuity, the desired result follows.

### 3 Proofs of the Theorems

**Proof of Theorem 1.** Since $p(z)$ is a polynomial of degree at most $n$, we can write $p(z) = p_1(z)p_2(z)$ such that $p_1(z)$ is a polynomial of degree $k \geq 1$ having all its zeros in $|z| \leq 1$ and $p_2(z)$ is a polynomial of degree $n - k$ having all its zeros in $|z| > 1$. Let $q_2(z) = z^{n-k}p_2(1/z)$, then all the zeros of $q_2(z)$ lie in $|z| < 1$ and $|q_2(z)| = |p_2(z)|$ for $|z| = 1$. Now if we consider the polynomial $F(z) = p_1(z)q_2(z)$, then all the zeros of $F(z)$ lie in $|z| \leq 1$ and $|F(z)| = |p(z)|$ for $|z| = 1$. By applying Lemma 2 to the polynomials $F(z)$ and $p(z)$, we get for all $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1$, $|\beta| \leq 1$, $R > r \geq 1$, and $|z| \geq 1$

$$
\left| p(Rz) - \beta p(rz) + \alpha \left\{ \left( \frac{R+1}{r+1} \right)^n - |\beta| \right\} p(rz) \right|
\leq \left| F(Rz) - \beta F(rz) + \alpha \left\{ \left( \frac{R+1}{r+1} \right)^n - |\beta| \right\} F(rz) \right|.
$$

(43)
Hence it gives for $s > 0$

$$
\int_0^{2\pi} |p(\text{Re}^i\theta) - \beta p(\text{re}^i\theta) + \alpha \left\{ \left( \frac{R + 1}{r + 1} \right)^n - |\beta| \right\} p(\text{re}^i\theta) |^s d\theta \\
\leq \int_0^{2\pi} |F(\text{Re}^i\theta) - \beta F(\text{re}^i\theta) + \alpha \left\{ \left( \frac{R + 1}{r + 1} \right)^n - |\beta| \right\} F(\text{re}^i\theta) |^s d\theta. \quad (44)
$$

On the other hand, as in the proof of Lemma 4 for $H(z)$, we conclude that the polynomial

$$
G(z) := F(Rz) - \beta F(rz) + \alpha \left\{ \left( \frac{R + 1}{r + 1} \right)^n - |\beta| \right\} F(rz)
$$

has all its zeros in $|z| \leq 1$. Therefore, the operator $\Lambda_\gamma$ defined by

$$
\Lambda_\gamma F(z) = F(Rz) - \beta F(rz) + \alpha \left\{ \left( \frac{R + 1}{r + 1} \right)^n - |\beta| \right\} F(rz)
$$

$$
= \left( R^n - \beta r^n + \alpha r^n \left\{ \left( \frac{R + 1}{r + 1} \right)^n - |\beta| \right\} \right) b_n z^n + \ldots + \left( 1 - \beta + \alpha \left\{ \left( \frac{R + 1}{r + 1} \right)^n - |\beta| \right\} \right) b_0
$$

is admissible. Hence by (27), we get for each $s > 0$

$$
\int_0^{2\pi} \left| F(\text{Re}^i\theta) - \beta F(\text{re}^i\theta) + \alpha \left\{ \left( \frac{R + 1}{r + 1} \right)^n - |\beta| \right\} F(\text{re}^i\theta) \right|^s d\theta \\
\leq \left| R^n - \beta r^n + \alpha r^n \left\{ \left( \frac{R + 1}{r + 1} \right)^n - |\beta| \right\} \right|^s \int_0^{2\pi} |F(\text{re}^i\theta)|^s d\theta. \quad (45)
$$

Combining (44) and (45) and using $|F(\text{e}^i\theta)| = |p(\text{e}^i\theta)|$, we get for every $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1, |\beta| \leq 1, R > r \geq 1, and s > 0$

$$
\int_0^{2\pi} \left| p(\text{Re}^i\theta) - \beta p(\text{re}^i\theta) + \alpha \left\{ \left( \frac{R + 1}{r + 1} \right)^n - |\beta| \right\} p(\text{re}^i\theta) \right|^s d\theta \\
\leq \left| R^n - \beta r^n + \alpha r^n \left\{ \left( \frac{R + 1}{r + 1} \right)^n - |\beta| \right\} \right|^s \int_0^{2\pi} |p(\text{e}^i\theta)|^s d\theta. \quad (46)
$$

This completes the proof of Theorem 1.

**Proof of Theorem 2.** Since $p \in P_4$ and $P(z) \neq 0$ in $|z| < 1$, then by using (29), we have for every $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1, |\beta| \leq 1, R > r \geq 1, and s > 0$

$$
|F(\theta)| \leq |G(\theta)|, \quad 0 \leq \theta < 2\pi, \quad (47)
$$

where

$$
F(\theta) = p(\text{Re}^i\theta) - \beta p(\text{re}^i\theta) + \alpha \left\{ \left( \frac{R + 1}{r + 1} \right)^n - |\beta| \right\} p(\text{re}^i\theta),
$$

$$
G(\theta) = R^n p(\text{e}^i\theta / R) - \beta r^n p(\text{e}^i\theta / r) + \alpha r^n \left\{ \left( \frac{R + 1}{r + 1} \right)^n - |\beta| \right\} p(\text{e}^i\theta / r).
$$
Using (28), we get
\[
\int_0^{2\pi} |F(\theta) + e^{i\gamma}G(\theta)|^s d\theta \leq \left| R^n - \beta r^n + \alpha r^n \left\{ \left( \frac{R + 1}{r + 1} \right)^n - |\beta| \right\} \right|
+ e^{i\gamma} \left[ 1 - \overline{\beta} + \overline{\alpha} \left\{ \left( \frac{R + 1}{r + 1} \right)^n - |\beta| \right\} \right] \int_0^{2\pi} |p(e^{i\theta})|^s d\theta.
\] (48)

By integrating both sides of (48) with respect to $\gamma$ in $[0, 2\pi]$, we get
\[
\int_0^{2\pi} \int_0^{2\pi} |F(\theta) + e^{i\gamma}G(\theta)|^s d\gamma d\theta \leq \left\{ \int_0^{2\pi} \left| R^n - \beta r^n + \alpha r^n \left\{ \left( \frac{R + 1}{r + 1} \right)^n - |\beta| \right\} \right|^s d\gamma \right\} \int_0^{2\pi} |p(e^{i\theta})|^s d\theta
+ e^{i\gamma} \left[ 1 - \overline{\beta} + \overline{\alpha} \left\{ \left( \frac{R + 1}{r + 1} \right)^n - |\beta| \right\} \right] \int_0^{2\pi} |p(e^{i\theta})|^s d\theta.
\] (49)

Now we use the fact that $|t + e^{i\gamma}|$ is an increasing function of $t$ for $t \geq 1$ which implies
\[
\int_0^{2\pi} |t + e^{i\gamma}|^s d\gamma \geq \int_0^{2\pi} |1 + e^{i\gamma}|^s d\gamma, \quad \gamma \in \mathbb{R}, \ s > 0, \ t \geq 1.
\] (50)

If we suppose that $F(\theta) \neq 0$, then by taking $t = |G(\theta)|/|F(\theta)|$, we have $t \geq 1$ by (47) and we get
\[
\int_0^{2\pi} |F(\theta) + e^{i\gamma}G(\theta)|^s d\gamma = |F(\theta)|^s \int_0^{2\pi} \left| 1 + e^{i\gamma} \frac{G(\theta)}{F(\theta)} \right|^s d\gamma
= |F(\theta)|^s \int_0^{2\pi} \left( \frac{G(\theta)}{F(\theta)} + e^{i\gamma} \right)^s d\gamma
= |F(\theta)|^s \int_0^{2\pi} \left( \frac{G(\theta)}{F(\theta)} \right)^s d\gamma + e^{i\gamma} |F(\theta)|^s \int_0^{2\pi} |1 + e^{i\gamma}|^s d\gamma \quad \text{(by (50)).}
\] (51)

It is clear that the inequality (51) holds for $F(\theta) = 0$ also. By using (51) in (49), we get for every $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1$, $|\beta| \leq 1$, $R > r \geq 1$ and $s > 0$,
\[
\left\{ \int_0^{2\pi} |1 + e^{i\gamma}|^s d\gamma \right\} \left\{ \int_0^{2\pi} |p(Re^{i\theta}) - \beta p(re^{i\theta}) + \alpha \left\{ \left( \frac{R + 1}{r + 1} \right)^n - |\beta| \right\} p(re^{i\theta})|^s d\theta \right\}
\leq \left\{ \int_0^{2\pi} \left| R^n - \beta r^n + \alpha r^n \left\{ \left( \frac{R + 1}{r + 1} \right)^n - |\beta| \right\} \right|^s d\gamma \right\} \int_0^{2\pi} |p(e^{i\theta})|^s d\theta
+ e^{i\gamma} \left[ 1 - \overline{\beta} + \overline{\alpha} \left\{ \left( \frac{R + 1}{r + 1} \right)^n - |\beta| \right\} \right] \int_0^{2\pi} |p(e^{i\theta})|^s d\theta.
\] (52)
But
\[
\left\{ \int_0^{2\pi} \left| R^n - \beta r^n + \alpha r^n \left\{ \left( \frac{R+1}{r+1} \right)^n - |\beta| \right\} \right| + e^{i\gamma} \left\{ 1 - \beta + |\alpha| \left\{ \left( \frac{R+1}{r+1} \right)^n - |\beta| \right\} \right\} ^s \right| \right| d\gamma \\
= \left\{ \int_0^{2\pi} \left| R^n - \beta r^n + \alpha r^n \left\{ \left( \frac{R+1}{r+1} \right)^n - |\beta| \right\} \right| + e^{i\gamma} \left\{ 1 - |\beta| + \alpha \left\{ \left( \frac{R+1}{r+1} \right)^n - |\beta| \right\} \right\} ^s \right| \right| d\gamma
\]
\[
= \left\{ \int_0^{2\pi} \left| \left( \frac{R+1}{r+1} \right)^n - |\beta| \right| + e^{i\gamma} \left\{ 1 - |\beta| + \alpha \left\{ \left( \frac{R+1}{r+1} \right)^n - |\beta| \right\} \right\} ^s \right| \right| d\gamma \]}
\[
= \left\{ \int_0^{2\pi} \left| \left( \frac{R+1}{r+1} \right)^n - |\beta| \right| + e^{i\gamma} \left\{ 1 - |\beta| + \alpha \left\{ \left( \frac{R+1}{r+1} \right)^n - |\beta| \right\} \right\} ^s \right| \right| d\gamma \]}
\[
= \left\{ \int_0^{2\pi} \left| \left( \frac{R+1}{r+1} \right)^n - |\beta| \right| + e^{i\gamma} \left\{ 1 - |\beta| + \alpha \left\{ \left( \frac{R+1}{r+1} \right)^n - |\beta| \right\} \right\} ^s \right| \right| d\gamma \]}
\[
= \left\{ \int_0^{2\pi} \left| \left( \frac{R+1}{r+1} \right)^n - |\beta| \right| + e^{i\gamma} \left\{ 1 - |\beta| + \alpha \left\{ \left( \frac{R+1}{r+1} \right)^n - |\beta| \right\} \right\} ^s \right| \right| d\gamma \]}
\[
= \left\{ \int_0^{2\pi} \left| \left( \frac{R+1}{r+1} \right)^n - |\beta| \right| + e^{i\gamma} \left\{ 1 - |\beta| + \alpha \left\{ \left( \frac{R+1}{r+1} \right)^n - |\beta| \right\} \right\} ^s \right| \right| d\gamma \]}
\[
\tag{53}
\]

Now by combining (52) and (53), we get the desired result.

**Proof of Theorem 3.** Since \( p(z) \) is a self-inversive polynomial, we have \( p(z) = aq(z) \), where \( |a| = 1 \) and \( q(z) = e^{i\theta}p(1/z) \). Therefore, we have for every \( \alpha, \beta \in \mathbb{C} \) with \( |\alpha| \leq 1, |\beta| \leq 1 \) and \( R > r \geq 1 \),
\[
|p(Rz) - \beta p(rz) + \alpha \left\{ \left( \frac{R+1}{r+1} \right)^n - |\beta| \right\} p(rz)| = |q(Rz) - \beta q(rz) + \alpha \left\{ \left( \frac{R+1}{r+1} \right)^n - |\beta| \right\} q(rz)|.
\]

Hence we can write
\[
|F(\theta)| = |G(\theta)|, \quad 0 \leq \theta < 2\pi,
\]
where
\[
F(\theta) = p(Re^{i\theta}) - \beta p(re^{i\theta}) + \alpha \left\{ \left( \frac{R+1}{r+1} \right)^n - |\beta| \right\} p(re^{i\theta}),
\]
\[
G(\theta) = R^n p(e^{i\theta}/R) - \beta r^n p(e^{i\theta}/r) + \alpha r^n \left\{ \left( \frac{R+1}{r+1} \right)^n - |\beta| \right\} p(e^{i\theta}/r).
\]

By applying Lemma 5, we have
\[
\int_0^{2\pi} |F(\theta) + e^{i\gamma}G(\theta)|^s d\theta \leq \int_0^{2\pi} \left| R^n - \beta r^n + \alpha r^n \left\{ \left( \frac{R+1}{r+1} \right)^n - |\beta| \right\} \right| + e^{i\gamma} \left\{ 1 - \beta + \alpha \left\{ \left( \frac{R+1}{r+1} \right)^n - |\beta| \right\} \right\} ^s \right| \right| d\gamma \]
\[
\leq \int_0^{2\pi} |p(e^{i\theta})|^s d\theta.
\]

By using the similar argument as in the proof of Theorem 2, we conclude the desired result. And this completes the proof of Theorem 3.
References


Department of Mathematics
Semnan University
Semnan
Iran

M. Bidkham
E-mail: mdbidkham@gmail.com

H. A. Soleiman Mezerji
E-mail: soleiman50@gmail.com

A. Mir
Department of Mathematics
University of Kashmir
Srinagar, 19006
India

E-mail: mabdullahmir@yahoo.com