Anal. Theory Appl.
Vol. 28, No. 2 (2012), 156–171
DOI : 10.3969/j.issn.1672-4070.2012.02.006

L^p INEQUALITIES AND ADMISSIBLE OPERATOR FOR POLYNOMIALS

M. Bidkham, H. A. Soleiman Mezerji
(*Semnan University, Iran*)
A. Mir
(*University of Kashmir, India)*

Received Aug. 5, 2011

Abstract. Let $p(z)$ be a polynomial of degree at most n. In this paper we obtain some new results about the dependence of

$$\left\| p(Rz) - \beta p(rz) + \alpha \left\{ \left(\frac{R+1}{r+1} \right)^n - |\beta| \right\} p(rz) \right\|_s$$

on $\|p(z)\|_s$ for every $\alpha,\ \beta \in \mathbf{C}$ with $|\alpha| \leq 1,\ |\beta| \leq 1,\ R > r \geqslant 1$, and $s > 0$. Our results not only generalize some well known inequalities, but also are variety of interesting results deduced from them by a fairly uniform procedure.

Key words: *L^p inequality polynomials, Rouche's theorem, admissible operator*

AMS (2010) subject classification: 39B82, 39B52, 46H25

1 Introduction and Statement of Results

Let P_n be the class of all complex polynomials

$$p(z) = \sum_{j=0}^{n} a_j z^j$$

of degree at most n and $p'(z)$ its derivative. For $p \in P_n$, define

$$\|p(z)\|_s := \left\{ \frac{1}{2\pi} \int_0^{2\pi} |p(e^{i\theta})|^s \right\}^{\frac{1}{s}}, \qquad 1 \leq s < \infty$$

and

$$\|p(z)\|_\infty := \max_{|z|=1} |p(z)|.$$

According to a famous result Known as Bernstein's inequality[4], we have

$$\|p'(z)\|_\infty \le n\|p(z)\|_\infty. \tag{1}$$

Also concerning the maximum modulus of $p(z)$ on $|z| = R > 1$, we have

$$\|p(Rz)\|_\infty \le R^n\|p(z)\|_\infty \tag{2}$$

(for reference see [11]). Zygmund[13] has shown

$$\|p'(z)\|_s \le n\|p(z)\|_s, \qquad s \geqslant 1. \tag{3}$$

whereas we can deduce the following inequality by applying a result of Hardy [9],

$$\|p(Rz)\|_s \le R^n\|p(z)\|_s, \quad R > 1, \quad s > 0. \tag{4}$$

Also Arestov[1] proved that (3) remains true for $0 < s < 1$ as well. It is clear that the inequalities (1) and (2) can be obtained by letting $s \longrightarrow \infty$ in the inequalities (3) and (4) respectively. If we restrict ourselves to the class of polynomials having no zeros in $|z| < 1$, the inequalities (3) and (4) can be improved. In fact, it was shown by De-Bruijn[6] for $s \geqslant 1$ and Rahman and Schmeisser[12] extended it for $0 < s < 1$ that if $p(z)$ is a polynomial of degree n having no zeros in $|z| < 1$, the inequality (3) can be replaced by

$$\|p'(z)\|_s \le n\frac{\|p(z)\|_s}{\|1+z\|_s}, \qquad s > 0. \tag{5}$$

Also Boas and Rahman[5] proved for $s \geqslant 1$ and Rahman and Schmeisser[12] extended it for $0 < s < 1$ that if $p(z)$ is a polynomial of degree n having no zeros in $|z| < 1$, the inequality (4) can be replaced by

$$\|p(Rz)\|_s \le \frac{\|R^n z+1\|_s}{\|1+z\|_s}\|p(z)\|_s, \quad R > 1, \quad s > 0. \tag{6}$$

Aziz and Rather[2] obtained a generalization of the inequalities (3) and (4). In fact, they have shown that if $p \in P_n$, then for every $R > 1$ and $s \geqslant 1$,

$$\|p(Rz) - p(z)\|_s \le (R^n - 1)\|p(z)\|_s. \tag{7}$$

Recently Aziz and Rather [3] considered a more general problem of investigating the dependence of

$$\|p(Rz) - \beta p(rz)\|_s \quad on \quad \|p(z)\|_s$$

for every $\beta \in \mathbf{C}$ with $|\beta| \le 1,\ R > r \geqslant 1,\ s > 0$ and extended the inequality (7) for $0 < s < 1$ as following.

Theorem A. *If $p \in P_n$, then for every $\beta \in \mathbf{C}$ with $|\beta| \le 1$ and $R > r \geqslant 1$, $s > 0$,*

$$\|p(Rz) - \beta p(rz)\|_s \le |R^n - \beta r^n| \, \|p(z)\|_s .$$

Also for the class of polynomials not vanishing in $|z| < 1$, they proved:

Theorem B. *If $p \in P_n$ and $p(z)$ does not vanish in $|z| < 1$, then for every $\beta \in \mathbf{C}$ with $|\beta| \le 1$ and $R > r \geqslant 1$, $s > 0$*

$$\left\|p(Re^{i\theta}) - \beta p(re^{i\theta})\right\|_s \le \frac{\|(R^n - \beta r^n)z + (1-\beta)\|_s}{\|1+z\|_s} \|p(z)\|_s. \tag{8}$$

For self-inversive polynomials, the following inequality was proved by Dewan and Govil[7].

$$\|p(Rz) - p(z)\|_s \le (R^n - 1)\|p(z)\|_s \,, \qquad s \geqslant 1. \tag{9}$$

Aziz and Rather[3] generalized (9) by proving the following interesting result.

Theorem C. *If $p \in P_n$ is self-inversive polynomial, then for every $\beta \in \mathbf{C}$ with $|\beta| \le 1$ and $R > r \geqslant 1$, $s > 0$*

$$\|p(Rz) - \beta p(rz)\|_s \le \frac{\|(R^n - \beta r^n)z + (1-\beta)\|_s}{\|1+z\|_s} \|p(z)\|_s. \tag{10}$$

In this paper, we first prove the following result which among other things includes Theorem A as a special case.

Theorem 1. *If $p \in P_n$, then for every α, $\beta \in \mathbf{C}$ with $|\alpha| \le 1$, $|\beta| \le 1$ and $R > r \geqslant 1$, $s > 0$,*

$$\left\|p(Rz) - \beta p(rz) + \alpha\left\{\left(\frac{R+1}{r+1}\right)^n - |\beta|\right\} p(rz)\right\|_s \le \left|R^n - \beta r^n + \alpha r^n \left\{\left(\frac{R+1}{r+1}\right)^n - |\beta|\right\}\right| \|p(z)\|_s .$$

The result is best possible and equality holds for $p(z) = \lambda z^n$, $\lambda \ne 0$.

If we take $\alpha = 0$, then Theorem 1 reduces to Theorem A due to Aziz and Rather[3].

If we assume $\alpha = \beta = 1$ in Theorem 1, then we get the following result.

Corollary 1. *If $p \in P_n$, then for $R > r \geqslant 1$, $s > 0$,*

$$\left\|p(Rz) - p(rz) + \left\{\left(\frac{R+1}{r+1}\right)^n - 1\right\} p(rz)\right\|_s \le \left|R^n - r^n + r^n \left\{\left(\frac{R+1}{r+1}\right)^n - 1\right\}\right| \|p(z)\|_s . \tag{11}$$

The result is best possible and equality holds for $p(z) = \lambda z^n$, $\lambda \ne 0$.

If we take $\beta = 1$ in Theorem 1, we conclude the following result.

Corollary 2. *If $p \in P_n$, then for every $\alpha \in \mathbf{C}$ with $|\alpha| \le 1$ and $R > r \geqslant 1$, $s > 0$,*

$$\left\|p(Rz) - p(rz) + \alpha\left\{\left(\frac{R+1}{r+1}\right)^n - 1\right\} p(rz)\right\|_s \le \left|R^n - r^n + \alpha r^n \left\{\left(\frac{R+1}{r+1}\right)^n - 1\right\}\right| \|p(z)\|_s . \tag{12}$$

The result is best possible and equality holds for $p(z) = \lambda z^n$, $\lambda \neq 0$.

If we divide two sides of (12) by $(R-r)$ and let $R \longrightarrow r$, we get:

Corollary 3. *If* $p \in P_n$ *in , then for every* $\alpha \in \mathbf{C}$ *with* $|\alpha| \leq 1$ *and* $r \geqslant 1$, $s > 0$,

$$\left\| zp'(rz) + \frac{n\alpha}{r+1} p(rz) \right\|_s \leq nr^{n-1} \left| 1 + \frac{r\alpha}{r+1} \right| \|p(z)\|_s. \tag{13}$$

The result is best possible and equality holds for $p(z) = \lambda z^n$, $\lambda \neq 0$.

Remark 1. If we let $s \longrightarrow \infty$ in (13), then it reduces to the following interesting inequality.

$$\left| zp'(rz) + \frac{n\alpha}{r+1} p(rz) \right| \leq nr^{n-1} \left| 1 + \frac{r\alpha}{r+1} \right| \max_{|z|=1} |p(z)|, \quad |\alpha| \leq 1, \quad r \geq 1, \quad |z| = 1. \tag{14}$$

For $r = 1$, (14) reduces to the following inequality which is due to Jain[8].

$$\left| zp'(z) + \frac{n\alpha}{2} p(z) \right| \leq n \left| 1 + \frac{\alpha}{2} \right| \max_{|z|=1} |p(z)|, \quad |\alpha| \leq 1, \quad |z| = 1. \tag{15}$$

Therefore, for $r = 1$ in (13), we get the following interesting result which is a generalization of (15).

Corollary 4. *If* $p \in P_n$, *then for every* $\alpha \in \mathbf{C}$ *with* $|\alpha| \leq 1$, $s > 0$,

$$\left\| zp'(z) + \frac{n\alpha}{2} p(z) \right\|_s \leq n \left| 1 + \frac{\alpha}{2} \right| \|p(z)\|_s. \tag{16}$$

The result is best possible and equality holds for $p(z) = \lambda z^n$, $\lambda \neq 0$.

For $\alpha = 0$, (16) reduces to (3).

For $p \in P_n$ and $p(z)$ does not vanish in $|z| < 1$, we prove the following generalization of (8) and improvement of (16).

Theorem 2. *If* $p \in P_n$ *and* $p(z)$ *does not vanish in* $|z| < 1$, *then for every* α, $\beta \in \mathbf{C}$ *with* $|\alpha| \leq 1$, $|\beta| \leq 1$ *and* $R > r \geqslant 1$, $s > 0$,

$$\begin{aligned} &\left\| \left[p(Rz) - \beta p(rz) + \alpha \left\{ \left(\frac{R+1}{r+1} \right)^n - |\beta| \right\} p(rz) \right] \right\|_s \\ &\leq \frac{\left\| \left[R^n - \beta r^n + \alpha r^n \left\{ \left(\frac{R+1}{r+1}\right)^n - |\beta| \right\} \right] z + \left[1 - \beta + \alpha \left\{ \left(\frac{R+1}{r+1}\right)^n - |\beta| \right\} \right] \right\|_s}{\|1+z\|_s} \|p(z)\|_s. \end{aligned} \tag{17}$$

The result is best possible and equality holds for $p(z) = \lambda z^n + \gamma$, $|\lambda| = |\gamma| = 1$.

For $\alpha = 0$, Theorem 2 reduces to Theorem B. Also if we take $\alpha = \beta = 0$, then (17) reduces to (6).

The following consequence is concluded by applying Minkowski's inequality in right hand side of (17).

Corollary 5. *If $p \in P_n$ and $p(z)$ does not vanish in $|z| < 1$, then for every α, $\beta \in \mathbf{C}$ with $|\alpha| \leq 1$, $|\beta| \leq 1$, and $R > r \geqslant 1$, $s \geqslant 1$,*

$$\left\| \left[p(Rz) - \beta p(rz) + \alpha \left\{ \left(\frac{R+1}{r+1} \right)^n - |\beta| \right\} p(rz) \right] \right\|_s \leq \frac{\left| R^n - \beta r^n + \alpha r^n \left\{ \left(\frac{R+1}{r+1} \right)^n - |\beta| \right\} \right| + \left| 1 - \beta + \alpha \left\{ \left(\frac{R+1}{r+1} \right)^n - |\beta| \right\} \right|}{\|1+z\|_s} \|p(z)\|_s. \tag{18}$$

The result is best possible and equality holds for $p(z) = \lambda z^n + \gamma$, $|\lambda| = |\gamma| = 1$.

Remark 2. If we take $\beta = 1$, and divide both sides of (18) by $R - r$ and let $R \longrightarrow r$, we get the following result.

Corollary 6. *If $p \in P_n$ and $p(z)$ does not vanish in $|z| < 1$, then for every $\alpha \in \mathbf{C}$ with $|\alpha| \leq 1$ and $r \geqslant 1$, $s \geqslant 1$,*

$$\left\| zp'(rz) + \frac{n\alpha}{r+1} p(rz) \right\|_s \leq \frac{n \left\{ r^{n-1} \left| 1 + \frac{r\alpha}{r+1} \right| + \left| \frac{\alpha}{r+1} \right| \right\}}{\|1+z\|_s} \|p(z)\|_s. \tag{19}$$

The result is best possible and equality holds for $p(z) = \lambda z^n + \gamma$, $|\lambda| = |\gamma| = 1$.

By letting $s \longrightarrow \infty$ in (19), we get the following result.

Corollary 7. *If $p \in P_n$ and $p(z)$ does not vanish in $|z| < 1$, then for every $\alpha \in \mathbf{C}$ with $|\alpha| \leq 1$, and $r \geqslant 1$,*

$$\max_{|z|=1} \left| zp'(rz) + \frac{n\alpha}{r+1} p(rz) \right| \leq \frac{n}{2} \left\{ r^{n-1} \left| 1 + \frac{r\alpha}{r+1} \right| + \left| \frac{\alpha}{r+1} \right| \right\} \max_{|z|=1} |p(z)|. \tag{20}$$

The result is best possible and equality holds for $p(z) = \lambda z^n + \gamma$, $|\lambda| = |\gamma| = 1$.

For $r = 1$, (20) reduces to the following inequality which is due to Jain[8].

$$\left| zp'(z) + \frac{n\alpha}{2} p(z) \right| \leq \frac{n}{2} \left\{ \left| 1 + \frac{\alpha}{2} \right| + \left| \frac{\alpha}{2} \right| \right\} \max_{|z|=1} |p(z)|, \quad |\alpha| \leq 1, \quad |z| = 1. \tag{21}$$

Therefore, for $r = 1$ in (19), we get the following interesting result which is a generalization of (21) and improvement of (16).

Corollary 8. *If $p \in P_n$ and $p(z)$ does not vanish in $|z| < 1$, then for every $\alpha \in \mathbf{C}$ with $|\alpha| \leq 1$, $s \geqslant 1$,*

$$\left\| zp'(z) + \frac{n\alpha}{2} p(z) \right\|_s \leq \frac{n \left\{ \left| 1 + \frac{\alpha}{2} \right| + \left| \frac{\alpha}{2} \right| \right\}}{\|1+z\|_s} \|p(z)\|_s. \tag{22}$$

The result is best possible and equality holds for $p(z) = \lambda z^n + \gamma$, $|\lambda| = |\gamma| = 1$.

If we take $\alpha = 0$, then (22) reduces to (5) for $s \geqslant 1$.

Finally, we prove the following result for self inversive polynomials which is an improvement as well a generalization of some well known results.

Theorem 3. *If $p \in P_n$ is self-inversive polynomial, then for every $\alpha,\ \beta \in \mathbf{C}$ with $|\alpha| \leq 1,\ |\beta| \leq 1$, and $R > r \geqslant 1,\ s > 0$,*

$$\begin{aligned} &\left\| \left[p(Rz) - \beta p(rz) + \alpha \left\{ \left(\frac{R+1}{r+1}\right)^n - |\beta| \right\} p(rz) \right] \right\|_s \\ &\leq \frac{\left\| \left[R^n - \beta r^n + \alpha r^n \left\{ \left(\frac{R+1}{r+1}\right)^n - |\beta| \right\} \right] z + \left[1 - \beta + \alpha \left\{ \left(\frac{R+1}{r+1}\right)^n - |\beta| \right\} \right] \right\|_s}{\|1+z\|_s} \|p(z)\|_s. \end{aligned} \tag{23}$$

The result is best possible and equality holds for $p(z) = z^n + 1$.

For $\alpha = 0$, Theorem 3 reduces to Theorem C. If we take $\alpha = 0,\ \beta = r = 1$ in Theorem 3, then the inequality (23) reduces to (9). Also from this theorem, we can deduce so many interesting results in a similar manner as the previous one.

2 Lemmas

For the proof of the theorems, we require the following lemmas.

Lemma 1. *If $p(z)$ is a polynomial of degree n having all its zeros in $|z| \leq 1$, then for every $R \geqslant r \geqslant 1$, and $|z| = 1$,*

$$|p(Rz)| \geqslant \left(\frac{R+1}{r+1}\right)^n |p(rz)|. \tag{24}$$

Lemma 2. *If $F(z)$ be a polynomial of degree n having all its zeros in $|z| \leq 1$ and $f(z)$ be a polynomial of degree at most n such that $|f(z)| \leq |F(z)|$ for $|z| = 1$, then for all $\alpha,\ \beta \in \mathbf{C}$ with $|\alpha| \leq 1,\ |\beta| \leq 1$, and $R > r \geqslant 1,\ |z| \geqslant 1$*

$$\begin{aligned} &\left| f(Rz) - \beta f(rz) + \alpha \left\{ \left(\frac{R+1}{r+1}\right)^n - |\beta| \right\} f(rz) \right| \\ &\quad \leq \left| F(Rz) - \beta F(rz) + \alpha \left\{ \left(\frac{R+1}{r+1}\right)^n - |\beta| \right\} F(rz) \right|. \end{aligned} \tag{25}$$

Lemmas 1 and 2 are due to Liman[10].

For $\gamma = (\gamma_0, \gamma_1, \cdots, \gamma_n)$ and $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$, let

$$\Lambda_\gamma p(z) = \sum_{j=0}^{n} \gamma_j a_j z^j.$$

The operator Λ_γ is said to be admissible if it preserves one of the following properties:

1) $p(z)$ has all its zeros in $|z| \leq 1$.

2) $p(z)$ has all its zeros in $|z| \geq 1$.

Now we state a result of Arestov[1].

Lemma 3. *Let $\phi(x)=\psi(\log x)$ where ψ is a convex nondecreasing function on $\mathbf{R}$. Then for all $p\in P_n$ and each admissible operator Λ_γ,*

$$\int_0^{2\pi}\phi\left(|\Lambda_\gamma p(e^{i\theta})|\right)\mathrm{d}\theta\le\int_0^{2\pi}\phi\left(C(\gamma,n)|p(e^{i\theta})|\right)\mathrm{d}\theta \tag{26}$$

where $C(\gamma,n)=\mathrm{Max}\,(|\gamma_0|,\ |\gamma_n|)$.

By applying Lemma 3 to the function $\phi(x)=x^s$ for every $s>0$, we get

$$\int_0^{2\pi}\left(|\Lambda_\gamma p(e^{i\theta})|^s\right)\mathrm{d}\theta\le(C(\gamma,n))^s\int_0^{2\pi}|p(e^{i\theta})|^s\mathrm{d}\theta. \tag{27}$$

Lemma 4. *If $p\in P_n$ and $p(z)$ does not vanish in $|z|<1$, then for every $\alpha,\ \beta\in\mathbf{C}$ with $|\alpha|\le1,\ |\beta|\le1$ and $R>r\geqslant1$, $s>0$ and γ real,*

$$\begin{aligned}\int_0^{2\pi}&\left|\left[p(Re^{i\theta})-\beta p(re^{i\theta})+\alpha\left\{\left(\frac{R+1}{r+1}\right)^n-|\beta|\right\}p(re^{i\theta})\right]\right.\\&\left.+e^{i\gamma}\left[R^np(e^{i\theta}/R)-\overline{\beta}r^np(e^{i\theta}/r)+\overline{\alpha}r^n\left\{\left(\frac{R+1}{r+1}\right)^n-|\beta|\right\}p(e^{i\theta}/r)\right]\right|^s\mathrm{d}\theta\\&\le\left|R^n-\beta r^n+\alpha r^n\left\{\left(\frac{R+1}{r+1}\right)^n-|\beta|\right\}\right.\\&\left.+e^{i\gamma}\left[1-\overline{\beta}+\overline{\alpha}\left\{\left(\frac{R+1}{r+1}\right)^n-|\beta|\right\}\right]\right|^s\int_0^{2\pi}|p(e^{i\theta})|^s\mathrm{d}\theta.\end{aligned} \tag{28}$$

Proof of Lemma 4. Let $q(z)=z^n\overline{p(1/\overline{z})}$. Applying Lemma 2 to the polynomials $p(z)$ and $q(z)$, we get for any $\alpha,\ \beta\in\mathbf{C}$ with $|\alpha|\le1,\ |\beta|\le1$ and $R>r\geqslant1$,

$$\begin{aligned}&\left|p(Rz)-\beta p(rz)+\alpha\left\{\left(\frac{R+1}{r+1}\right)^n-|\beta|\right\}p(rz)\right|\\&\le\left|q(Rz)-\beta q(rz)+\alpha\left\{\left(\frac{R+1}{r+1}\right)^n-|\beta|\right\}q(rz)\right|\\&=\left|R^np(z/R)-\overline{\beta}r^np(z/r)+\overline{\alpha}r^n\left\{\left(\frac{R+1}{r+1}\right)^n-|\beta|\right\}p(z/r)\right|,\quad\text{for}\quad|z|=1.\end{aligned} \tag{29}$$

On the other hand, we have $|q(rz)|\le|q(Rz)|$ for $|z|=1,\ R>r\geqslant1$. Since $q(Rz)$ has all its zeros in $|z|\le1/R<1$, a direct application of Rouche's theorem shows that the polynomial $q(Rz)-\beta q(rz)$ has all its zeros in $|z|<1$ for every $\beta\in\mathbf{C}$ with $|\beta|\le1$. By Lemma 1, we have for $R>r\geqslant1$,

$$|q(Rz)-\beta q(rz)|\geqslant|q(Rz)|-|\beta||q(rz)|>\left\{\left(\frac{R+1}{r+1}\right)^n-|\beta|\right\}|q(rz)|,\quad\text{for}\quad|z|=1. \tag{30}$$

Therefore, again by applying Rouche's theorem, it follows that for any $\alpha \in \mathbf{C}$ with $|\alpha| \le 1$ and $R > r \geqslant 1$, all the zeros of the polynomial

$$H(z) := q(Rz) - \beta q(rz) + \alpha\left\{\left(\frac{R+1}{r+1}\right)^n - |\beta|\right\} q(rz)$$

lie in $|z| < 1$. This implies that the polynomial

$$z^n\overline{H(1/\overline{z})} = R^n p(z/R) - \overline{\beta} r^n p(z/r) + \overline{\alpha} r^n\left\{\left(\frac{R+1}{r+1}\right)^n - |\beta|\right\} p(z/r)$$

has all its zeros in $|z| > 1$. Hence the function

$$f(z) := \frac{p(Rz) - \beta p(rz) + \alpha\left\{\left(\frac{R+1}{r+1}\right)^n - |\beta|\right\} p(rz)}{R^n p(z/R) - \overline{\beta} r^n p(z/r) + \overline{\alpha} r^n\left\{\left(\frac{R+1}{r+1}\right)^n - |\beta|\right\} p(z/r)}$$

is analytic in $|z| \le 1$ and $|f(z)| \le 1$ for $|z| = 1$. By applying the Maximum Modulus Principle, we get

$$|f(z)| < 1, \qquad \text{for} \qquad |z| < 1.$$

Equivalently,

$$\begin{aligned}&\left|p(Rz) - \beta p(rz) + \alpha\left\{\left(\frac{R+1}{r+1}\right)^n - |\beta|\right\} p(rz)\right| \\ &< \left|R^n p(z/R) - \overline{\beta} r^n p(z/r) + \overline{\alpha} r^n\left\{\left(\frac{R+1}{r+1}\right)^n - |\beta|\right\} p(z/r)\right|, \qquad \text{for} \quad |z| < 1.\end{aligned} \tag{31}$$

A direct application of Rouche's theorem shows that

$$\begin{aligned}\Lambda_\gamma p(z) = {} & p(Rz) - \beta p(rz) + \alpha\left\{\left(\frac{R+1}{r+1}\right)^n - |\beta|\right\} p(rz) \\ & + e^{i\gamma}\left[R^n p(z/R) - \overline{\beta} r^n p(z/r) + \overline{\alpha} r^n\left\{\left(\frac{R+1}{r+1}\right)^n - |\beta|\right\} p(z/r)\right] \\ = {} & \left(R^n - \beta r^n + \alpha r^n\left\{\left(\frac{R+1}{r+1}\right)^n - |\beta|\right\} + e^{i\gamma}\left[1 - \overline{\beta} + \overline{\alpha}\left\{\left(\frac{R+1}{r+1}\right)^n - |\beta|\right\}\right]\right) a_n z^n \\ & + \cdots\cdots \\ & + \left(1 - \beta + \alpha\left\{\left(\frac{R+1}{r+1}\right)^n - |\beta|\right\} + e^{i\gamma}\left[R^n - \overline{\beta} r^n + \overline{\alpha} r^n\left\{\left(\frac{R+1}{r+1}\right)^n - |\beta|\right\}\right]\right) a_0\end{aligned} \tag{32}$$

does not vanish in $|z| < 1$ for every $\alpha,\ \beta \in \mathbf{C}$ with $|\alpha| \le 1,\ |\beta| \le 1$ and $R > r \geqslant 1$, and γ real. Therefore Λ_γ is an admissible operator. By applying (27), the desired result follows. This completes the proof of Lemma 4.

Lemma 5. *If $p \in P_n$, then for every $\alpha,\ \beta \in \mathbf{C}$ with $|\alpha| \le 1,\ |\beta| \le 1$ and $R > r \geqslant 1$, $s > 0$ and γ real,*

$$\begin{aligned}
&\int_0^{2\pi}\left|\left[p(Re^{i\theta}) - \beta p(re^{i\theta}) + \alpha\left\{\left(\frac{R+1}{r+1}\right)^n - |\beta|\right\}p(re^{i\theta})\right]\right.\\
&\qquad\left.+e^{i\gamma}\left[R^n p(z/R) - \overline{\beta} r^n p(z/r) + \overline{\alpha} r^n\left\{\left(\frac{R+1}{r+1}\right)^n - |\beta|\right\}p(z/r)\right]\right|^s d\theta\\
&\qquad\le\left|R^n - \beta r^n + \alpha r^n\left\{\left(\frac{R+1}{r+1}\right)^n - |\beta|\right\}\right.\\
&\qquad\left.+e^{i\gamma}\left[1-\overline{\beta}+\overline{\alpha}\left\{\left(\frac{R+1}{r+1}\right)^n - |\beta|\right\}\right]\right|^s\int_0^{2\pi}|p(e^{i\theta})|^s d\theta.
\end{aligned} \tag{33}$$

Proof of Lemma 5. Let $p(z)$ be a polynomial of degree at most n, we can write $p(z) = p_1(z)p_2(z)$ such that $p_1(z)$ is a polynomial of degree $k \geqslant 1$ having all its zeros in $|z| \geqslant 1$ and $p_2(z)$ is a polynomial of degree $n-k$ having all its zeros in $|z| < 1$. First we suppose that $p_1(z)$ does not vanish on $|z| = 1$ and hence all the zeros of $p_1(z)$ lie in $|z| > 1$. Let $q_2(z) = z^{n-k}\overline{p_2(1/\overline{z})}$, then all the zeros of $q_2(z)$ lie in $|z| > 1$ and $|q_2(z)| = |p_2(z)|$ for $|z| = 1$. Therefore the polynomial $g(z) = p_1(z)q_2(z)$ is a polynomial of degree n not vanishing in $|z| \le 1$ and for $|z| = 1$,

$$|g(z)| = |p_1(z)||q_2(z)| = |p_1(z)||p_2(z)| = |p(z)|. \tag{34}$$

A direct application of Rouche's theorem show that $h(z) := p(z) + \lambda g(z)$ does not vanish in $|z| < 1$, for every $\lambda \in \mathbf{C}$ with $|\lambda| > 1$. Also $h(z)$ does not vanish on $|z| = 1$, because if this is not true then it would contradict with (34). Thus $h(z)$ does not vanish in $|z| \le 1$ for any λ with $|\lambda| > 1$, so that all the zeros of $h(z)$ lie in $|z| \geqslant \rho$ for some $\rho > 1$ and hence all the zeros of $h(\rho z)$ lie in $|z| \geqslant 1$. Applying (31) to the polynomial $h(\rho z)$, we get

$$\begin{aligned}
&\left|h(R\rho z) - \beta h(r\rho z) + \alpha\left\{\left(\frac{R+1}{r+1}\right)^n - |\beta|\right\}h(r\rho z)\right|\\
&<\left|R^n h(\rho z/R) - \overline{\beta} r^n h(\rho z/r) + \overline{\alpha} r^n\left\{\left(\frac{R+1}{r+1}\right)^n - |\beta|\right\}h(\rho z/r)\right|, \qquad \text{for} \quad |z| < 1,\ R > r \geqslant 1.
\end{aligned} \tag{35}$$

Taking $z = e^{i\theta}/\rho,\ 0 \le \theta < 2\pi$, then $|z| = (1/\rho) < 1$ as $\rho > 1$, and we get

$$\begin{aligned}
&\left|h(Re^{i\theta}) - \beta h(re^{i\theta}) + \alpha\left\{\left(\frac{R+1}{r+1}\right)^n - |\beta|\right\}h(re^{i\theta})\right|\\
&<\left|R^n h(e^{i\theta}/R) - \overline{\beta} r^n h(e^{i\theta}/r) + \overline{\alpha} r^n\left\{\left(\frac{R+1}{r+1}\right)^n - |\beta|\right\}h(e^{i\theta}/r)\right|,\ \ 0 \le \theta < 2\pi\ \ ,\ R > r \geqslant 1.
\end{aligned} \tag{36}$$

Or

$$\begin{aligned}&\left|h(Rz)-\beta h(rz)+\alpha\left\{\left(\frac{R+1}{r+1}\right)^n-|\beta|\right\}h(rz)\right|\\&<\left|R^nh(z/R)-\overline{\beta}r^nh(z/r)+\overline{\alpha}r^n\left\{\left(\frac{R+1}{r+1}\right)^n-|\beta|\right\}h(z/r)\right|,\qquad \text{for}\quad |z|=1.\end{aligned}\tag{37}$$

By Rouche's theorem, it follows that the polynomial

$$\begin{aligned}T(z):=&\left(h(Rz)-\beta h(rz)+\alpha\left\{\left(\frac{R+1}{r+1}\right)^n-|\beta|\right\}h(rz)\right)\\&+e^{i\gamma}\left(R^nh(z/R)-\overline{\beta}r^nh(z/r)+\overline{\alpha}r^n\left\{\left(\frac{R+1}{r+1}\right)^n-|\beta|\right\}h(z/r)\right),\end{aligned}$$

does not vanish in $|z|\le 1$ for every $\alpha,\ \beta\in\mathbf{C}$ with $|\alpha|\le 1,\ |\beta|\le 1, R>r\geqslant 1$, and γ real. If we replace $h(z)$ by $p(z)+\lambda g(z)$, then the polynomial

$$\begin{aligned}T(z)=&\left\{p(Rz)-\beta p(rz)+\alpha\left\{\left(\frac{R+1}{r+1}\right)^n-|\beta|\right\}p(rz)\right.\\&\left.+e^{i\gamma}\left[R^np(z/R)-\overline{\beta}r^np(z/r)+\overline{\alpha}r^n\left\{\left(\frac{R+1}{r+1}\right)^n-|\beta|\right\}p(z/r)\right]\right\}\\&+\lambda\left\{g(Rz)-\beta g(rz)+\alpha\left\{\left(\frac{R+1}{r+1}\right)^n-|\beta|\right\}g(rz)\right.\\&\left.+e^{i\gamma}\left[R^ng(z/R)-\overline{\beta}r^ng(z/r)+\overline{\alpha}r^n\left\{\left(\frac{R+1}{r+1}\right)^n-|\beta|\right\}g(z/r)\right]\right\}\end{aligned}\tag{38}$$

does not vanish in $|z|\le 1$ for every $\alpha,\ \lambda,\ \beta\in\mathbf{C}$ with $|\alpha|\le 1,\ |\beta|\le 1,\ |\lambda|>1, R>r\geqslant 1$, and γ real. This implies

$$\begin{aligned}&\left|p(Rz)-\beta p(rz)+\alpha\left\{\left(\frac{R+1}{r+1}\right)^n-|\beta|\right\}p(rz)\right.\\&\qquad\left.+e^{i\gamma}\left[R^np(z/R)-\overline{\beta}r^np(z/r)+\overline{\alpha}r^n\left\{\left(\frac{R+1}{r+1}\right)^n-|\beta|\right\}p(z/r)\right]\right|\\&\le\left|g(Rz)-\beta g(rz)+\alpha\left\{\left(\frac{R+1}{r+1}\right)^n-|\beta|\right\}g(rz)\right.\\&\qquad\left.+e^{i\gamma}\left[R^ng(z/R)-\overline{\beta}r^ng(z/r)+\overline{\alpha}r^n\left\{\left(\frac{R+1}{r+1}\right)^n-|\beta|\right\}g(z/r)\right]\right|\end{aligned}\tag{39}$$

for $|z|\le 1,\ |\alpha|\le 1,\ |\beta|\le 1, R>r\geqslant 1$, and γ real. If the inequality (39) is not true, then we would have

$$\begin{aligned}&\Bigg|p(Rz_0)-\beta p(rz_0)+\alpha\left\{\left(\frac{R+1}{r+1}\right)^n-|\beta|\right\}p(rz_0)\\&\quad+e^{i\gamma}\left[R^np(z_0/R)-\overline{\beta}r^np(z_0/r)+\overline{\alpha}r^n\left\{\left(\frac{R+1}{r+1}\right)^n-|\beta|\right\}p(z_0/r)\right]\Bigg|\\&>\Bigg|g(Rz_0)-\beta g(rz_0)+\alpha\left\{\left(\frac{R+1}{r+1}\right)^n-|\beta|\right\}g(rz_0)\\&\quad+e^{i\gamma}\left[R^ng(z_0/R)-\overline{\beta}r^ng(z_0/r)+\overline{\alpha}r^n\left\{\left(\frac{R+1}{r+1}\right)^n-|\beta|\right\}g(z_0/r)\right]\Bigg|,\end{aligned}$$

for some z_0 with $|z_0|\le 1$. Since all the zeros of polynomial$g(z)$ lie in $|z|>1$, it follows (as before) that all the zeros of polynomial

$$\begin{aligned}&g(Rz)-\beta g(rz)+\alpha\left\{\left(\frac{R+1}{r+1}\right)^n-|\beta|\right\}g(rz)\\&\quad+e^{i\gamma}\left[R^ng(z/R)-\overline{\beta}r^ng(z/r)+\overline{\alpha}r^n\left\{\left(\frac{R+1}{r+1}\right)^n-|\beta|\right\}g(z/r)\right]\end{aligned}$$

also lie in $|z|>1$. Hence

$$\begin{aligned}&g(Rz_0)-\beta g(rz_0)+\alpha\left\{\left(\frac{R+1}{r+1}\right)^n-|\beta|\right\}g(rz_0)\\&\quad+e^{i\gamma}\left[R^ng(z_0/R)-\overline{\beta}r^ng(z_0/r)+\overline{\alpha}r^n\left\{\left(\frac{R+1}{r+1}\right)^n-|\beta|\right\}g(z_0/r)\right]\neq 0\end{aligned}$$

for any $|z_0|\le 1$. So we can take a suitable value for λ such that $|\lambda|>1$ and $T(z_0)=0$ with $|z_0|\le 1$. This clearly is a contradiction to the fact that $T(z)$ does not vanish in $|z|\le 1$. The inequality (39) gives for each $s>0$ and $0\le\theta<2\pi$,

$$\begin{aligned}&\int_0^{2\pi}\Bigg|p(Re^{i\theta})-\beta p(re^{i\theta})+\alpha\left\{\left(\frac{R+1}{r+1}\right)^n-|\beta|\right\}p(re^{i\theta})\\&\quad+e^{i\gamma}\left[R^np(e^{i\theta}/R)-\overline{\beta}r^np(e^{i\theta}/r)+\overline{\alpha}r^n\left\{\left(\frac{R+1}{r+1}\right)^n-|\beta|\right\}p(e^{i\theta}/r)\right]\Bigg|^s d\theta\\&\le\int_0^{2\pi}\Bigg|g(Re^{i\theta})-\beta g(re^{i\theta})+\alpha\left\{\left(\frac{R+1}{r+1}\right)^n-|\beta|\right\}g(re^{i\theta})\\&\quad+e^{i\gamma}\left[R^ng(e^{i\theta}/R)-\overline{\beta}r^ng(e^{i\theta}/r)+\overline{\alpha}r^n\left\{\left(\frac{R+1}{r+1}\right)^n-|\beta|\right\}g(e^{i\theta}/r)\right]\Bigg|^s d\theta.\end{aligned}\tag{40}$$

By applying Lemma 4 to $g(z)$ and using (34), we get for any $\alpha,\ \beta\in\mathbf{C}$ with $|\alpha|\le 1,\ |\beta|\le 1,$

$R > r \geqslant 1$, $s > 0$ and γ real,

$$\begin{aligned}
&\int_0^{2\pi} \Big| p(Re^{i\theta}) - \beta p(re^{i\theta}) + \alpha \left\{ \left(\frac{R+1}{r+1} \right)^n - |\beta| \right\} p(re^{i\theta}) \\
&\quad + e^{i\gamma} \left[R^n p(e^{i\theta}/R) - \overline{\beta} r^n p(e^{i\theta}/r) + \overline{\alpha} r^n \left\{ \left(\frac{R+1}{r+1} \right)^n - |\beta| \right\} p(e^{i\theta}/r) \right] \Big|^s \mathrm{d}\theta \\
&\leq \left| R^n - \beta r^n + \alpha r^n \left\{ \left(\frac{R+1}{r+1} \right)^n - |\beta| \right\} + e^{i\gamma} \left[1 - \overline{\beta} + \overline{\alpha} \left\{ \left(\frac{R+1}{r+1} \right)^n - |\beta| \right\} \right] \right|^s \int_0^{2\pi} |g(e^{i\theta})|^s \mathrm{d}\theta \\
&= \left| R^n - \beta r^n + \alpha r^n \left\{ \left(\frac{R+1}{r+1} \right)^n - |\beta| \right\} + e^{i\gamma} \left[1 - \overline{\beta} + \overline{\alpha} \left\{ \left(\frac{R+1}{r+1} \right)^n - |\beta| \right\} \right] \right|^s \int_0^{2\pi} |p(e^{i\theta})|^s \mathrm{d}\theta.
\end{aligned} \tag{41}$$

Now If $p_1(z)$ has a zero on $|z| = 1$, then applying (41) to the polynomial $p^*(z) = p_1(tz)p_2(z)$ where $t < 1$, we get for any $\alpha,\ \beta \in \mathbf{C}$ with $|\alpha| \leq 1$, $|\beta| \leq 1$, $R > r \geqslant 1$, $s > 0$ and γ real,

$$\begin{aligned}
&\int_0^{2\pi} \Big| p^*(Re^{i\theta}) - \beta p^*(re^{i\theta}) + \alpha \left\{ \left(\frac{R+1}{r+1} \right)^n - |\beta| \right\} p^*(re^{i\theta}) \\
&+ e^{i\gamma} \left[R^n p^*(e^{i\theta}/R) - \overline{\beta} r^n p^*(e^{i\theta}/r) + \overline{\alpha} r^n \left\{ \left(\frac{R+1}{r+1} \right)^n - |\beta| \right\} p^*(e^{i\theta}/r) \right] \Big|^s \mathrm{d}\theta \\
&\leq \left| R^n - \beta r^n + \alpha r^n \left\{ \left(\frac{R+1}{r+1} \right)^n - |\beta| \right\} + e^{i\gamma} \left[1 - \overline{\beta} + \overline{\alpha} \left\{ \left(\frac{R+1}{r+1} \right)^n - |\beta| \right\} \right] \right|^s \int_0^{2\pi} |p^*(e^{i\theta})|^s \mathrm{d}\theta.
\end{aligned} \tag{42}$$

Letting $t \longrightarrow 1$ in (42) and using continuity, the desired result follows.

3 Proofs of the Theorems

Proof of Theorem 1. Since $p(z)$ is a polynomial of degree at most n, we can write $p(z) = p_1(z)p_2(z)$ such that $p_1(z)$ is a polynomial of degree $k \geqslant 1$ having all its zeros in $|z| \leq 1$ and $p_2(z)$ is a polynomial of degree $n-k$ having all its zeros in $|z| > 1$. Let $q_2(z) = z^{n-k}\overline{p_2(1/\overline{z})}$, then all the zeros of $q_2(z)$ lie in $|z| < 1$ and $|q_2(z)| = |p_2(z)|$ for $|z| = 1$. Now if we consider the polynomial $F(z) = p_1(z)q_2(z)$, then all the zeros of $F(z)$ lie in $|z| \leq 1$ and $|F(z)| = |p(z)|$ for $|z| = 1$. By applying Lemma 2 to the polynomials $F(z)$ and $p(z)$, we get for all $\alpha,\ \beta \in \mathbf{C}$ with $|\alpha| \leq 1$, $|\beta| \leq 1$, $R > r \geqslant 1$, *and* $|z| \geqslant 1$

$$\begin{aligned}
&\left| p(Rz) - \beta p(rz) + \alpha \left\{ \left(\frac{R+1}{r+1} \right)^n - |\beta| \right\} p(rz) \right| \\
&\quad \leq \left| F(Rz) - \beta F(rz) + \alpha \left\{ \left(\frac{R+1}{r+1} \right)^n - |\beta| \right\} F(rz) \right|.
\end{aligned} \tag{43}$$

Hence it gives for $s>0$

$$\begin{aligned}&\int_0^{2\pi}\left|p(Re^{i\theta})-\beta p(re^{i\theta})+\alpha\left\{\left(\frac{R+1}{r+1}\right)^n-|\beta|\right\}p(re^{i\theta})\right|^s d\theta\\&\quad\le\int_0^{2\pi}\left|F(Re^{i\theta})-\beta F(re^{i\theta})+\alpha\left\{\left(\frac{R+1}{r+1}\right)^n-|\beta|\right\}F(re^{i\theta})\right|^s d\theta.\end{aligned}\tag{44}$$

On the other hand, as in the proof of Lemma 4 for $H(z)$, we conclude that the polynomial

$$G(z):=F(Rz)-\beta F(rz)+\alpha\left\{\left(\frac{R+1}{r+1}\right)^n-|\beta|\right\}F(rz)$$

has all its zeros in $|z|\le 1$. Therefore, the operator Λ_γ defined by

$$\begin{aligned}\Lambda_\gamma F(z)&=F(Rz)-\beta F(rz)+\alpha\left\{\left(\frac{R+1}{r+1}\right)^n-|\beta|\right\}F(rz)\\&=\left(R^n-\beta r^n+\alpha r^n\left\{\left(\frac{R+1}{r+1}\right)^n-|\beta|\right\}\right)b_nz^n+\ldots+\left(1-\beta+\alpha\left\{\left(\frac{R+1}{r+1}\right)^n-|\beta|\right\}\right)b_0\end{aligned}$$

is admissible. Hence by (27), we get for each $s>0$

$$\begin{aligned}&\int_0^{2\pi}\left|F(Re^{i\theta})-\beta F(re^{i\theta})+\alpha\left\{\left(\frac{R+1}{r+1}\right)^n-|\beta|\right\}F(re^{i\theta})\right|^s d\theta\\&\quad\le\left|R^n-\beta r^n+\alpha r^n\left\{\left(\frac{R+1}{r+1}\right)^n-|\beta|\right\}\right|^s\int_0^{2\pi}|F(e^{i\theta})|^s d\theta.\end{aligned}\tag{45}$$

Combining (44) and (45) and using $|F(e^{i\theta})|=|p(e^{i\theta})|$, we get for every $\alpha,\ \beta\in\mathbf{C}$ with $|\alpha|\le 1,\ |\beta|\le 1,\ R>r\geqslant 1,$ *and* $s>0,$

$$\begin{aligned}&\int_0^{2\pi}\left|p(Re^{i\theta})-\beta p(re^{i\theta})+\alpha\left\{\left(\frac{R+1}{r+1}\right)^n-|\beta|\right\}p(re^{i\theta})\right|^s d\theta\\&\quad\le\left|R^n-\beta r^n+\alpha r^n\left\{\left(\frac{R+1}{r+1}\right)^n-|\beta|\right\}\right|^s\int_0^{2\pi}|p(e^{i\theta})|^s d\theta.\end{aligned}\tag{46}$$

This completes the proof of Theorem 1.

Proof of Theorem 2. Since $p\in P_n$ and $P(z)\neq 0$ in $|z|<1$, then by using (29), we have for every $\alpha,\ \beta\in\mathbf{C}$ with $|\alpha|\le 1,\ |\beta|\le 1,\ R>r\geqslant 1$, and $s>0$,

$$|F(\theta)|\le|G(\theta)|,\quad 0\le\theta<2\pi,\tag{47}$$

where

$$\begin{aligned}F(\theta)&=p(Re^{i\theta})-\beta p(re^{i\theta})+\alpha\left\{\left(\frac{R+1}{r+1}\right)^n-|\beta|\right\}p(re^{i\theta}),\\G(\theta)&=R^np(e^{i\theta}/R)-\overline{\beta}r^np(e^{i\theta}/r)+\overline{\alpha}r^n\left\{\left(\frac{R+1}{r+1}\right)^n-|\beta|\right\}p(e^{i\theta}/r).\end{aligned}$$

Using (28), we get

$$\begin{aligned}\int_0^{2\pi}\left|F(\theta)+e^{i\gamma}G(\theta)\right|^s d\theta \le \Bigg|R^n-\beta r^n+\alpha r^n\left\{\left(\frac{R+1}{r+1}\right)^n-|\beta|\right\}\\ +e^{i\gamma}\left[1-\overline{\beta}+\overline{\alpha}\left\{\left(\frac{R+1}{r+1}\right)^n-|\beta|\right\}\right]\Bigg|^s\int_0^{2\pi}|p(e^{i\theta})|^s d\theta.\end{aligned}\tag{48}$$

By integrating both sides of (48) with respect to γ in $[0,\ 2\pi]$, we get

$$\begin{aligned}&\int_0^{2\pi}\int_0^{2\pi}\left|F(\theta)+e^{i\gamma}G(\theta)\right|^s d\gamma d\theta \le \Bigg\{\int_0^{2\pi}\Bigg|R^n-\beta r^n+\alpha r^n\left\{\left(\frac{R+1}{r+1}\right)^n-|\beta|\right\}\\ &+e^{i\gamma}\left[1-\overline{\beta}+\overline{\alpha}\left\{\left(\frac{R+1}{r+1}\right)^n-|\beta|\right\}\right]\Bigg|^s d\gamma\Bigg\}\left\{\int_0^{2\pi}|p(e^{i\theta})|^s d\theta\right\}.\end{aligned}\tag{49}$$

Now we use the fact that $|t+e^{i\gamma}|$ is an increasing function of t for $t\geqslant 1$ which implies

$$\int_0^{2\pi}|t+e^{i\gamma}|^s d\gamma \geqslant \int_0^{2\pi}|1+e^{i\gamma}|^s d\gamma\ , \quad \gamma\in\mathbf{R},\ \ s>0,\quad t\geqslant 1.\tag{50}$$

If we suppose that $F(\theta)\neq 0$, then by taking $t=|G(\theta)|/|F(\theta)|$, we have $t\geqslant 1$ by (47) and we get

$$\begin{aligned}\int_0^{2\pi}\left|F(\theta)+e^{i\gamma}G(\theta)\right|^s d\gamma &= |F(\theta)|^s\int_0^{2\pi}\left|1+e^{i\gamma}\frac{G(\theta)}{F(\theta)}\right|^s d\gamma\\ &= |F(\theta)|^s\int_0^{2\pi}\left|\frac{G(\theta)}{F(\theta)}+e^{i\gamma}\right|^s d\gamma\\ &= |F(\theta)|^s\int_0^{2\pi}\left|\left|\frac{G(\theta)}{F(\theta)}\right|+e^{i\gamma}\right|^s d\gamma\\ &\geqslant |F(\theta)|^s\int_0^{2\pi}\left|1+e^{i\gamma}\right|^s d\gamma \qquad (\text{by } (50)).\end{aligned}\tag{51}$$

It is clear that the inequality (51) holds for $F(\theta)=0$ also. By using (51) in (49), we get for every $\alpha,\ \beta\in\mathbf{C}$ with $|\alpha|\le 1,\ |\beta|\le 1, R>r\geqslant 1$ and $s>0$,

$$\begin{aligned}&\left\{\int_0^{2\pi}\left|1+e^{i\gamma}\right|^s d\gamma\right\}\left\{\int_0^{2\pi}\left|p(Re^{i\theta})-\beta p(re^{i\theta})+\alpha\left\{\left(\frac{R+1}{r+1}\right)^n-|\beta|\right\}p(re^{i\theta})\right|^s d\theta\right\}\\ &\le\Bigg\{\int_0^{2\pi}\Bigg|R^n-\beta r^n+\alpha r^n\left\{\left(\frac{R+1}{r+1}\right)^n-|\beta|\right\}\\ &\quad +e^{i\gamma}\left[1-\overline{\beta}+\overline{\alpha}\left\{\left(\frac{R+1}{r+1}\right)^n-|\beta|\right\}\right]\Bigg|^s d\gamma\Bigg\}\left\{\int_0^{2\pi}|p(e^{i\theta})|^s d\theta\right\}.\end{aligned}\tag{52}$$

But

$$\begin{aligned}
&\left\{\int_0^{2\pi}\left|\left[R^n-\beta r^n+\alpha r^n\left\{\left(\frac{R+1}{r+1}\right)^n-|\beta|\right\}\right]+e^{i\gamma}\left[1-\overline{\beta}+\overline{\alpha}\left\{\left(\frac{R+1}{r+1}\right)^n-|\beta|\right\}\right]\right|^s d\gamma\right\}\\
&=\left\{\int_0^{2\pi}\left|\left|R^n-\beta r^n+\alpha r^n\left\{\left(\frac{R+1}{r+1}\right)^n-|\beta|\right\}\right|+e^{i\gamma}\left|\left[1-\overline{\beta}+\overline{\alpha}\left\{\left(\frac{R+1}{r+1}\right)^n-|\beta|\right\}\right]\right|\right|^s d\gamma\right\}\\
&=\left\{\int_0^{2\pi}\left|\left|R^n-\beta r^n+\alpha r^n\left\{\left(\frac{R+1}{r+1}\right)^n-|\beta|\right\}\right|+e^{i\gamma}\left|\left[1-\beta+\alpha\left\{\left(\frac{R+1}{r+1}\right)^n-|\beta|\right\}\right]\right|\right|^s d\gamma\right\}\\
&=\left\{\int_0^{2\pi}\left|\left|R^n-\beta r^n+\alpha r^n\left\{\left(\frac{R+1}{r+1}\right)^n-|\beta|\right\}\right|e^{i\gamma}+\left|\left[1-\beta+\alpha\left\{\left(\frac{R+1}{r+1}\right)^n-|\beta|\right\}\right]\right|\right|^s d\gamma\right\}\\
&=\left\{\int_0^{2\pi}\left|\left[R^n-\beta r^n+\alpha r^n\left\{\left(\frac{R+1}{r+1}\right)^n-|\beta|\right\}\right]e^{i\gamma}+\left[1-\beta+\alpha\left\{\left(\frac{R+1}{r+1}\right)^n-|\beta|\right\}\right]\right|^s d\gamma\right\}.
\end{aligned} \tag{53}$$

Now by combining (52) and (53), we get the desired result.

Proof of Theorem 3. Since $p(z)$ is a self-inversive polynomial, we have $p(z)=aq(z)$, where $|a|=1$ and $q(z)=z^n\overline{p(1/\overline{z})}$. Therefore, we have for every α, $\beta\in\mathbf{C}$ with $|\alpha|\le 1$, $|\beta|\le 1$ and $R>r\geqslant 1$,

$$\left|p(Rz)-\beta p(rz)+\alpha\left\{\left(\frac{R+1}{r+1}\right)^n-|\beta|\right\}p(rz)\right|=\left|q(Rz)-\beta q(rz)+\alpha\left\{\left(\frac{R+1}{r+1}\right)^n-|\beta|\right\}q(rz)\right|.$$

Hence we can write

$$|F(\theta)|=|G(\theta)|,\quad 0\le\theta<2\pi, \tag{54}$$

where

$$\begin{aligned}
F(\theta)&=p(Re^{i\theta})-\beta p(re^{i\theta})+\alpha\left\{\left(\frac{R+1}{r+1}\right)^n-|\beta|\right\}p(re^{i\theta}),\\
G(\theta)&=R^n p(e^{i\theta}/R)-\overline{\beta}r^n p(e^{i\theta}/r)+\overline{\alpha}r^n\left\{\left(\frac{R+1}{r+1}\right)^n-|\beta|\right\}p(e^{i\theta}/r).
\end{aligned}$$

By applying Lemma 5, we have

$$\begin{aligned}
\int_0^{2\pi}\left|F(\theta)+e^{i\gamma}G(\theta)\right|^s d\theta\le\Bigg|&R^n-\beta r^n+\alpha r^n\left\{\left(\frac{R+1}{r+1}\right)^n-|\beta|\right\}\\
&+e^{i\gamma}\left[1-\overline{\beta}+\overline{\alpha}\left\{\left(\frac{R+1}{r+1}\right)^n-|\beta|\right\}\right]\Bigg|^s\int_0^{2\pi}|p(e^{i\theta})|^s d\theta.
\end{aligned} \tag{55}$$

By using the similar argument as in the proof of Theorem 2, we conclude the desired result. And this completes the proof of Theorem 3.

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Department of Mathematics
Semnan University
Semnan
Iran

M. Bidkham
E-mail: mdbidkham@gmail.com

H. A. Soleiman Mezerji
E-mail: soleiman50@gmail.com

A. Mir
Department of Mathematics
University of Kashmir
Srinagar, 19006
India

E-mail: mabdullahmir@yahoo.com