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COMPLETE HYPERSURFACES WITH CONSTANT SCALAR CURVATURE IN A SPECIAL KIND OF LOCALLY SYMMETRIC MANIFOLD

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Abstract. In this paper, we investigate n-dimensional complete and orientable hypersurfaces M^n $(n \geq 3)$ with constant normalized scalar curvature in a locally symmetric manifold. Two rigidity theorems are obtained for these hypersurfaces.

Key words: *hypersurfaces, scalar curvture, locally symmetric manifold*

AMS (2010) subject classification: 53C42, 53A10

1 Introduction

When the ambient manifolds possess very nice symmetry, for example the unit sphere, there are many rigidity results for hypersurfaces with constant mean curvature or with constant scalar curvature in these ambient manifolds, such as [2, 3, 4, 6, 7, 11, 12] and the references therein. Recently, many researchers studied the minimal hypersurfaces or hypersurfaces with constant mean curvature in more general Riemanniam manifolds such as the locally symmetric manifolds and the δ-pinched manifolds, and obtained many rigidity results these hypersurfaces, such as [5, 10, 13, 14] and the references therein. It is natural and very important to study n-dimensional complete and orientable hypersurfaces with constant scalar curvature in a locally symmetric manifold. In the paper, we will discuss complete hypersurfaces in this direction.

In order to represent our theorems, we need some notation. Let N^{n+1} be a locally symmetric manifold and M^n be an n-dimensional complete and oriented hypersurface in N^{n+1}. We choose a

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local orthonormal frame $e_1,\cdots,e_n,e_{n+1}$ in N^{n+1} such that $e_1,\cdots,e_n$ are tangent to M^n and e_{n+1} is normal to M^n. We assume that N^{n+1} satisfies the following conditions:

$$K_{n+1in+1i}=c_0, \tag{1}$$

$$\frac{1}{2}<\delta\leq K_N\leq 1, \tag{2}$$

where c_0, δ are constants and K_N denotes the sectional curvature of N^{n+1}. When N^{n+1} satisfies the above conditions (1), (2), it is said simply for N^{n+1} to satisfy the condition $(*)$.

Remark 1.1. If N^{n+1} is a unit sphere $S^{n+1}(1)$, then it satisfies the condition $(*)$, where $c_0=\delta=1$.

It is easy to know that the scalar curvature $\bar{R}$ of locally symmetric manifold is constant. On the other hand, if we denote $\bar{R}_{CD}$ as the components of the Ricci curvature tensor of N^{n+1} satisfying the condition $(*)$, then the scalar curvature $\bar{R}$ of N^{n+1} is

$$\bar{R}=2\sum_k K_{n+1kn+1k}+\sum_{ij}K_{ijij}=2nc_0+\sum_{ij}K_{ijij}, \tag{3}$$

hence, $\sum_{ij}K_{ijij}$ is constant. This fact together with the formula (12) suggests us to define a constant P by

$$n(n-1)P=n(n-1)R-\sum_{ij}K_{ijij}=n^2H^2-S. \tag{4}$$

Using (4), we finally establish our main results:

Theorem 1.2. *Let M^n $(n\geq 3)$ be an n-dimensional complete and orientable hypersurface with constant normalized scalar curvature R in a locally symmetric manifold N^{n+1} satisfying the condition $(*)$. If $P\geq 0$, in the case where $P=0$, assume further that the mean curvature function H does not change sign, then*

(i) *either* $\sup|\Phi|^2=0$ *and M is a totally umbilical hypersurface.*

(ii) *or*

$$\sup|\Phi|^2\geq D(n,P)=\frac{n(n-1)(P+c)^2}{(n-2)(nP+2c)}>0. \tag{5}$$

Moreover, if $P>0$ the equality $\sup|\Phi|^2=D(n,P)$ *holds and this suppremum is attained at some point of M, then M^n has two distinct constant principal curvatures, one of them being simple, where $c=2\delta-c_0>0$ and P determined by* (4).

In particular, let $N^{n+1}=S^{n+1}(1)$ in Theorem 1.2, then $c=c_0=\delta=1$, so $P=R-1$ from (4). If $P>0$, i.e., $R>1$ the equality $\sup|\Phi|^2=D(n,P)$ holds and this suppremum is attained at some point of M, following from Theorem 1.2, we know that M^n has two distinct principal curvatures; in fact M^n is the $H(r)$-torus $S^1(\sqrt{1-r^2})\times S^{n-1}(r)\subset S^{n+1}(1)$, with $0<r<\sqrt{(n-2)/nR}$. In this case, Theorem 1.2 generalizes the result in [2],[3] to more general situations.

Theorem 1.3. *Let M^n $(n\geq 3)$ be an n-dimensional complete and orientable hypersurface with constant normalized scalar curvature R in a locally symmetric manifold N^{n+1} satisfying the condition $(*)$. Assume $P>0$. If $S\leq 2\sqrt{n-1}c$, then M^n is totally umbilical hypersurface, or* $\sup S=2\sqrt{n-1}c$. *Moreover, the equality* $\sup S=2\sqrt{n-1}c$ *holds and this suppremum is attained at some point of M, then M^n has two distinct constant principal curvatures, one of them being simple, where $c=2\delta-c_0>0$.*

2 Preliminaries

Let N^{n+1} be a locally symmetric manifold and M^n be an n-dimensional complete and oriented hypersurface in N^{n+1}. We choose a local orthonormal frame $e_1,\cdots,e_n,e_{n+1}$ in N^{n+1} such that $e_1,\cdots,e_n$ are tangent to M^n and e_{n+1} is normal to M^n. Let $\omega_1,\cdots,\omega_{n+1}$ be the dual coframe. We use the following convention on the range of indices:

$$1\leq A,B,\cdots\leq n+1;\quad 1\leq i,j,\cdots\leq n.$$

The structure equations of N^{n+1} are given by

$$d\omega_A=-\sum_B\omega_{AB}\wedge\omega_B,\quad \omega_{AB}+\omega_{BA}=0, \tag{6}$$

$$d\omega_{AB}=-\sum_C\omega_{AC}\wedge\omega_{CB}+\frac{1}{2}\sum_{ABCD}K_{ABCD}\omega_C\wedge\omega_D, \tag{7}$$

where K_{ABCD} are the components of the curvature tensor of N^{n+1}.

Restricting to M^n such that

$$\omega_{n+1}=0,\quad \omega_{n+1i}=\sum_j h_{ij}\omega_j,\quad h_{ij}=h_{ji}, \tag{8}$$

The structure equations of M^n are

$$d\omega_i=-\sum_j\omega_{ij}\wedge\omega_j,\quad \omega_{ij}+\omega_{ji}=0, \tag{9}$$

$$d\omega_{ij} = -\sum_k \omega_{ik}\wedge\omega_{kj} + \frac{1}{2}\sum_{k,l} R_{ijkl}\omega_k\wedge\omega_l, \tag{10}$$

$$R_{ijkl} = K_{ijkl} + (h_{ik}h_{jl} - h_{il}h_{jk}), \tag{11}$$

$$n(n-1)R = \sum_{ij} K_{ijij} + n^2H^2 - S, \tag{12}$$

$$\sum_k h_{ijk}\omega_k = dh_{ij} - \sum_k h_{kj}\omega_{ki} - \sum_k h_{ik}\omega_{kj}, \tag{13}$$

$$\sum_l h_{ijkl}\omega_l = dh_{ijk} - \sum_l h_{ljk}\omega_{li} - \sum_l h_{ilk}\omega_{lj} - \sum_l h_{ijl}\omega_{lk}, \tag{14}$$

where $n(n-1)R$ is the scalar curvature, H is the mean curvature and S is the squared of the second fundamental form of M^n.

The Laplacian $\triangle h_{ij}$ of the second fundamental form of M^n is defined by $\triangle h_{ij} = \sum_k h_{ijkk}$. By a simple and direct calculation, we have

$$\begin{aligned}\triangle h_{ij} &= (nH)_{ij} + nHK_{n+1in+1j} - \sum_k K_{n+1kn+1k}h_{ij} + nH\sum_k h_{ik}h_{kj} \\ &\quad -Sh_{ij} + \sum_k [K_{mkjk}h_{mi} + K_{mkik}h_{mj} + 2K_{mijk}h_{km}].\end{aligned} \tag{15}$$

Choose a local frame of orthonormal vectors fields $\{e_i\}$ such that at arbitrary point x of M^n

$$h_{ij} = \lambda_i\delta_{ij}, \tag{16}$$

then at point x we have

$$\begin{aligned}\frac{1}{2}\triangle S &= \sum_{ijk} h_{ijk}^2 + \sum_{ij} h_{ij}\triangle h_{ij} \\ &= \sum_{ijk} h_{ijk}^2 + \sum_i \lambda_i(nH)_{ii} + nH\sum_i \lambda_i K_{n+1in+1i} - S\sum_i k_{n+1in+1i} \\ &\quad + \sum_{ij}(\lambda_i-\lambda_j)^2 K_{ijij} - S^2 + nH\sum_i \lambda_i^3,\end{aligned} \tag{17}$$

where we use the fact that the Riemannian curvature of locally symmetric manifold is covariant constant.

Set $\Phi = h_{ji} - nH\delta_{ij}$, it is easy to check that Φ is traceless and $|\Phi|^2 = S - nH^2 \geq 0$, with equality if and only if M^n is totally umbilical. For this reason, Φ is also called the total umbilicity tensor of M^n.

According to Cheng-Yau[14], we introduce the following operator $\square$ acting on any C^2-function f by

$$\square(f)=\sum_{ij}(nH\delta_{ij}-h_{ij})f_{ij}. \tag{18}$$

We also need the following algebraic Lemmas.

Lemma 2.1.[1,8] *Let $\mu_1,\cdots,\mu_n$ be real numbers such that*

$$\sum_i \mu_i=0 \quad \text{and} \quad \sum_i \mu_i^2=\beta^2,$$

where $\beta\geq 0$ is constant. Then

$$|\sum_i \mu_i^3|\leq \frac{n-2}{\sqrt{n(n-1)}}\beta^3, \tag{19}$$

and equality holds if and only if at least $n-1$ of μ_i's are equal.

Lemma 2.2[9]. *Let M^n be an n-dimensional complete Riemannian manifold whose sectional curvature is bounded from below and $F: M\to R$ be a smooth function which is bounded above on M^n. Then there exists a sequence of points $x_k\in M^n$ such that*

$$\lim_{k\to\infty} F(x_k)=\sup F,$$
$$\lim_{k\to\infty}|\nabla F(x_k)|=0,$$
$$\lim_{k\to\infty}\sup\max\{(\nabla^2(F(x_k)))(X,X):|X|=1\}\leq 0.$$

3 Proof of Theorems

First, we give the following Lemma.

Lemma 3.1. *With the same assumptions as Theorem* 1.2.

(1) *we have the following inequality,*

$$\square(nH)\geq \frac{1}{n-1}|\Phi|^2 Q_P(|\Phi|), \tag{20}$$

where

$$Q_P(x)=-(n-2)x^2-(n-2)x\sqrt{x^2+n(n-1)P}+n(n-1)(P+c),$$

and

$$c=2\delta-c_0>0.$$

(2) *If the mean curvature H is bounded, then there is a sequence of points $\{x_k\}$ in M such that*

$$\lim_{k\to\infty} nH(x_k)=\sup(nH),\quad \lim_{k\to\infty}|\nabla(nH)(x_k)|=0,\quad \lim_{k\to\infty}\sup(\square(nH)(x_k))\leq 0. \tag{21}$$

Proof. (1) Putting $\mu_i = \lambda_i - H$ and $|\Phi|^2 = \sum_i \mu_i^2 = S - nH^2$. From (12),(17), we have

$$\begin{aligned} \square(nH) &= \sum_{ij}(nH\delta_{ij} - h_{ij})(nH)_{ij} = nH\triangle(nH) - \sum_{ij} h_{ij}(nH)_{ij} \\ &= \frac{1}{2}\triangle[(nH)^2] - n^2|\nabla H|^2 - \sum_{ij} h_{ij}(nH)_{ij} \\ &= \frac{1}{2}\triangle S - n^2|\nabla H|^2 - \sum_{ij} h_{ij}(nH)_{ij} \\ &= \underbrace{\sum_{ijk} h_{ijk}^2 - n^2|\nabla H|^2}_{I} \underbrace{- S^2 + nH\sum_i \lambda_i^3}_{II} \\ & \quad \underbrace{+ nH\sum_i \lambda_i K_{n+1in+1i} - S\sum_i k_{n+1in+1i} + \sum_{ij}(\lambda_i - \lambda_j)^2 K_{ijij}}_{III}. \end{aligned} \tag{22}$$

Firstly, we estimate (I):

Taking the covariant derivative of the equation (12), we have

$$2n^2 HH_k = 2\sum_{ij} h_{ij}h_{ijk}. \tag{23}$$

Therefore

$$n^4H^2|\nabla H|^2 = \sum_k(\sum_{ij} h_{ij}h_{ijk})^2 \leq S(\sum_{ijk} h_{ijk}^2). \tag{24}$$

Since $P \geq 0$, we have $n^2H^2 \geq S$, so from (24), we obtain

$$\sum_{ijk} h_{ijk}^2 - n^2|\nabla H|^2 \geq 0. \tag{25}$$

Secondly, we estimate (II):

It is easy to know that $\sum_i \lambda_i^3 = nH^3 + 3H\sum_i \mu_i^2 + \sum_i \mu_i^3$. By applying Lemma 2.1 to real numbers $\mu_1, \cdots, \mu_n$, we obtain

$$\begin{aligned} -S^2 + nH\sum_i \lambda_i^3 &= -(|\Phi|^2 + nH^2)^2 + n^2H^4 + 3nH^2|\Phi|^2 + nH\sum_i \mu_i^3 \\ &\geq -|\Phi|^4 - \frac{n(n-2)}{\sqrt{n(n-1)}}|H||\Phi|^3 + nH^2|\Phi|^2. \end{aligned} \tag{26}$$

Finally, we estimate (III):

Using curvature condition $(*)$, we get

$$nH\sum_i \lambda_i K_{n+1in+1i} - S\sum_i k_{n+1in+1i} = nc_0(nH^2 - S) = -nc_0|\Phi|^2 \tag{27}$$

and

$$\sum_{ij}(\lambda_i-\lambda_j)^2K_{ijij}=\sum_{ij}(\mu_i-\mu_j)^2K_{ijij}\geq\delta\sum_{ij}(\mu_i-\mu_j)^2=2n\delta|\Phi|^2. \tag{28}$$

From (27) and (28), we have

$$III\geq n(2\delta-c_0)|\Phi|^2. \tag{29}$$

From (22),(25), (26),(29) and set $c=2\delta-c_0$, we have

$$\square(nH)\geq-|\Phi|^2[|\Phi|^2+\frac{n(n-2)}{\sqrt{n(n-1)}}|H||\Phi|-n(c+H^2)]. \tag{30}$$

From Gauss equation, we have

$$H^2=\frac{1}{n(n-1)}[|\Phi|^2+n(n-1)P]. \tag{31}$$

From (30) and (31), we have

$$\square(nH)\geq\frac{1}{n-1}|\Phi|^2Q_P(|\Phi|), \tag{32}$$

where $Q_P(x)=-(n-2)x^2-(n-2)x\sqrt{x^2+n(n-1)P}+n(n-1)(P+c)$.

(2) Since M^n is orientable, $P\geq0$ and in the case where $P=0$, the mean curvature function H does not change sign, we can assume that $H\geq0$ (by changing the orientation of M^n if necessary).

If $H\equiv0$ the result is obvious. Let suppose that H is not identically zero, we may assume that $\sup H>0$. From

$$(\lambda_i)^2\leq S\leq n^2H^2,\quad \text{i.e.}\quad |\lambda_i|\leq n|H|. \tag{33}$$

Since H is bounded and (33), we know that S is also bounded. From (11), we have

$$R_{ijij}\geq\delta-\lambda_i\lambda_j\geq\delta-S. \tag{34}$$

This shows that the sectional curvatures of M^n are bounded from below because S is bounded. Therefore we may apply Lemma 2.2 to the function nH and obtain a sequence of points $\{x_k\}\in M^n$ such that

$$\lim_{k\to\infty}nH(x_k)=n\sup H,\quad \lim_{k\to\infty}|\nabla H(x_k)|=0,\quad \lim_{k\to\infty}\sup(nH_{ii}(x_k))\leq0. \tag{35}$$

From (33), we have

$$0\leq nH-|\lambda_i|\leq nH-\lambda_i, \tag{36}$$

By applying $\square(nH)$ at x_k, we have

$$\lim_{k\to\infty}\sup(\square(nH)(x_k)) = \lim_{k\to\infty}\sum_i \sup[nH(x_k)-\lambda_i(x_k)]nH_{ii}(x_k) \le 0. \tag{37}$$

Proof of Theorem 1.2. From the assumptions of Theorem 1.2, we can assume that $H \ge 0$ on M^n. If $\sup|\Phi|^2 = +\infty$, then (ii) of Theorem 1.2 is trivially satisfied and there is nothing to prove. If $\sup|\Phi|^2 = 0$, then (i) of Theorem 1.2 holds and there is nothing to prove. Then, let us assume that $0 < \sup|\Phi|^2 < +\infty$. From (31), we know that H is bounded. According to (2) of Lemma 3.1, there exists a sequence of points $\{x_k\}$ in M^n such that

$$\lim_{k\to\infty} nH(x_k) = \sup(nH), \quad \lim_{k\to\infty}|\nabla(nH)(x_k)| = 0, \quad \lim_{k\to\infty}\sup(\square(nH)(x_k)) \le 0. \tag{38}$$

From (31), we have $\lim_{\to\infty}|\Phi|^2(x_k) = \sup|\Phi|^2$. Evaluating (20) at the point x_k of the sequence, taking the limit and using (38), we obtain that

$$0 \ge \lim_{k\to\infty}\square(nH)(x_k) \ge \frac{1}{n-1}\sup|\Phi|^2 Q_P(\sup|\Phi|). \tag{39}$$

Since $P \ge 0$ and $c = 2\delta - c_0 > 0$, $Q_P(0) = P + c > 0$ and $Q_P(x)$ is strictly decreasing for $x \ge 0$, with $Q_P(x_0) = 0$ at

$$x_0 = \sqrt{\frac{n(n-1)}{(n-2)(nP+2c)}}(P+c) > 0. \tag{40}$$

Therefore (39) implies

$$\sup|\Phi|^2 \ge \frac{n(n-1)(P+c)^2}{(n-2)(nP+2c)} = D(n,P). \tag{41}$$

This proves the inequality in (ii) of Theorem 1.2.

If $P > 0$, from Gauss equation, we know $n^2H^2 > S \ge \lambda_i^2$, so $nH - \lambda_i \ge nH - |\lambda_i| > 0$, i.e.

$$nH - \lambda_i > 0. \tag{42}$$

From (42), we know the operator $\square$ is positive definite, that is, the operator $\square$ is elliptic. From (31), we have

$$\begin{aligned}
\square(|\Phi|^2) &= \frac{n-1}{n}\square(n^2H^2) \\
&= 2\frac{n-1}{n}nH\square(nH) + 2\frac{n-1}{n}(nH-\lambda_i)(nH_i)^2 \\
&\ge 2\frac{n-1}{n}nH\square(nH) \\
&\ge 2\frac{n-1}{n}nH|\Phi|^2 Q_P(|\Phi|) = 2(n-1)H|\Phi|^2 Q_P(|\Phi|).
\end{aligned} \tag{43}$$

If $\sup_M |\Phi| = x_0$, then $0 \le |\Phi| \le \sup_M |\Phi| = x_0$, so we have

$$Q_P(|\Phi|) \ge 0. \tag{44}$$

From (43) and (44), we have

$$\square(|\Phi|^2) \ge 0. \tag{45}$$

If $\sup|\Phi|^2 = D(n,P)$ and this supremum is attained at some point of M^n, then by the maximum principle $|\Phi|$ must be constant, $|\Phi| = x_0$. From (31), we know that H is constant. Thus, (20) becomes trivially an equality

$$\square(nH) = 0 = \frac{1}{n-1}|\Phi|^2 Q_P(|\Phi|). \tag{46}$$

Therefore, all the inequality in the proof of (1) of Lemma 3.1 must be equalities. So (27) becomes an equality, i.e. $\sum_{ijk} h_{ijk}^2 = n^2|\nabla H|^2$, since H is constant, we know that $\sum_{ijk} h_{ijk}^2 = 0$, i.e. $h_{ijk} = 0$, for $i,j,k \in \{1,\cdots,n\}$. From (13), we have $0 = d\lambda_i - 2\sum_k h_{ik}\omega_{ki} = d\lambda_i$, hence λ_i is constant.

From (26) and Lemma 2.1, we know that M^n has two distinct principal curvatures, one of them being simple, after reenumeration if necessary, we can assume that $\mu_1 = \cdots = \mu_{n-1} \ge 0$, $\mu_n \ne \mu_1$, where $\mu_i = \lambda_i - H$, $i = 1,\cdots,n$. Thus $\lambda_i \ge H \ge 0$ for $i = 1,\cdots,n-1$, we set $\lambda = \lambda_1 = \cdots = \lambda_{n-1} \ge 0$, $\mu = \lambda_n$.

From the equality of (28), we have $\sum_{ij}(\lambda_i - \lambda_j)^2(K_{ijij} - \delta) = 0$. Since $K_N \ge \delta$, so if $i \ne j$, then $(\lambda_i - \lambda_j)^2(K_{ijij} - \delta) = 0$, so $K_{ijij} = \delta$ or $\lambda_i = \lambda_j$ for $i \ne j$. If $\lambda_i \ne \lambda_j$, from (13), we have $(\lambda_i - \lambda_j)\omega_{ij} = 0$, so $\omega_{ij} = 0$. From (10) and $\omega_{ij} = 0$, we have

$$R_{ijij} = 0(\lambda_i \ne \lambda_j). \tag{47}$$

From (11), we have

$$\lambda\mu + \delta = 0. \tag{48}$$

On other hand

$$(n-1)\lambda + \mu = nH = \text{constant}. \tag{49}$$

From (48) and (49), we have

$$\lambda = \frac{1}{2(n-1)}[nH + \sqrt{n^2H^2 + 4(n-1)\delta}], \quad \mu = \frac{1}{2}[nH - \sqrt{n^2H^2 + 4(n-1)\delta}].$$

So M^n has two distinct constant principal curvatures, one of them being simple. This proves the Theorem 1.2.

Proof of Theorem 1.3. From the assumptions of Theorem 1.3, we can assume that $H>0$ on M^n. From (30), we have

$$\Box(nH) \geq -|\Phi|^2[|\Phi|^2+\frac{n(n-2)}{\sqrt{n(n-1)}}H|\Phi|-n(c+H^2)]. \tag{50}$$

Consider the quadratic form $P(x,y)=-x^2-\frac{n-2}{\sqrt{n-1}}xy+y^2$. By the orthogonal transformation

$$\begin{aligned} u &= \frac{1}{\sqrt{2n}}((1+\sqrt{n-1})y+(1-\sqrt{n-1})x) \\ v &= \frac{1}{\sqrt{2n}}((-1+\sqrt{n-1})y+(1+\sqrt{n-1})x) \end{aligned}$$

$P(x,y)=\frac{n}{2\sqrt{n-1}}(u^2-v^2)$. Take $x=|\Phi|$ and $y=\sqrt{n}H$; we obtain $u^2+v^2=x^2+y^2$, and by (50), we have

$$\begin{aligned} \Box(nH) &\geq |\Phi|^2(nc+\frac{n}{2\sqrt{n-1}}(u^2-v^2)) \\ &\geq |\Phi|^2(nc-\frac{n}{2\sqrt{n-1}}(u^2+v^2)+\frac{n}{2\sqrt{n-1}}2u^2) \\ &\geq |\Phi|^2(nc-\frac{n}{2\sqrt{n-1}}(u^2+v^2)) \\ &\geq |\Phi|^2(nc-\frac{n}{2\sqrt{n-1}}S). \end{aligned} \tag{51}$$

From (12) and $S\leq 2\sqrt{n-1}c$, we know that H is bounded. According to (2) of Lemma 3.1, there exists a sequence of points $\{x_k\}$ in M^n such that

$$\lim_{k\to\infty} nH(x_k)=\sup(nH), \quad \lim_{k\to\infty}|\nabla(nH)(x_k)|=0, \quad \lim_{k\to\infty}\sup(\Box(nH)(x_k))\leq 0. \tag{52}$$

From (31) we have

$$\lim_{k\to\infty}|\Phi|^2(x_k)=\sup|\Phi|^2. \tag{53}$$

and

$$\lim_{k\to\infty} S(x_k)=\lim_{k\to\infty}|\Phi|^2(x_k)+\lim_{k\to\infty}(nH)(x_k)=\sup S. \tag{54}$$

Evaluating (51) at the points x_k of the sequence, taking the limit and using (52), we obtain that

$$\begin{aligned} 0 &\geq \lim_{k\to\infty}\sup(\Box(nH)(x_k)) \\ &\geq \sup|\Phi|^2(nc-\frac{n}{2\sqrt{n-1}}\sup S)\geq 0. \end{aligned} \tag{55}$$

we have $\sup|\Phi|^2 = 0$, that is $\Phi = 0$ or $\sup S = 2\sqrt{n-1}c$. If $\sup\Phi = 0$, then $S = nH^2$ and M^n is totally umbilical.

Since $P > 0$, we know that $\Box$ is an elliptic operator. From (12) and $S \le 2\sqrt{n-1}c$, we have

$$\begin{aligned}\Box(S) &= \Box(n^2H^2) = 2nH\Box(nH) + 2(nH-\lambda_i)(nH_i)^2 \\ &\ge 2nH\Box(nH) \ge 2nH|\Phi|^2(nc - \frac{n}{2\sqrt{n-1}}S) \ge 0. \end{aligned} \tag{56}$$

If $\sup S = 2\sqrt{n-1}c$ and this supremum is attained at some point of M^n, then by the maximum principle S must be constant, $S = 2\sqrt{n-1}c$. From (12), we know that H is constant. Thus, (51) becomes trivially an equality

$$\Box(nH) = 0 = |\Phi|^2(nc - \frac{n}{2\sqrt{n-1}}S). \tag{57}$$

Therefore, all inequalities in proof of (51) must be equalities. From $u = 0$, we have

$$|\Phi| = \frac{\sqrt{n-1}+1}{\sqrt{n-1}-1}\sqrt{n}H > 0. \tag{58}$$

By using Lemma 2.1 and (58), we know that M^n has two distinct principal curvature, one of them being simple. Since S and H are constants, it is easy to know that M^n has two distinct constant principal curvatures, one of them being simple. This proves Theorem 1.3.

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