Anal. Theory Appl.
Vol. 28, No. 3 (2012), 224–231
DOI : 10.3969/j.issn.1672-4070.2012.03.002

BMO ESTIMATES FOR MULTILINEAR FRACTIONAL INTEGRALS*

Xiangxing Tao and Yunpin Wu
(*Zhejiang University of Science and Technology, China)*

Received July 30, 2010

Abstract. In this paper, the authors prove that the multilinear fractional integral operator $T^{A_1,A_2}_{\Omega,\alpha}$ and the relevant maximal operator $M^{A_1,A_2}_{\Omega,\alpha}$ with rough kernel are both bounded from $L^p(1<p<\infty)$ to L^q and from L^p to $L^{n/(n-\alpha),\infty}$ with power weight, respectively, where

$$T^{A_1,A_2}_{\Omega,\alpha}(f)(x)=\int_{\mathbf{R}^n}\frac{R_{m_1}(A_1;x,y)R_{m_2}(A_2;x,y)}{|x-y|^{n-\alpha+m_1+m_2-2}}\Omega(x-y)f_(y)\mathrm{d}y$$

and

$$M^{A_1,A_2}_{\Omega,\alpha}(f)(x)=\sup_{r>0}\frac{1}{r^{n-\alpha+m_1+m_2-2}}\int_{|x-y|<r}\prod_{i=1}^{2}R_{m_i}(A_i;x,y)\Omega(x-y)f_(y)|\mathrm{d}y,$$

and $0<\alpha<n$, $\Omega\in L^s(S^{n-1})(s\geq 1)$ is a homogeneous function of degree zero in $\mathbf{R}^n$, A_i is a function defined on $\mathbf{R}^n$ and $R_{m_i}(A_i;x,y)$ denotes the m_i-th remainder of Taylor series of A_i at x about y. More precisely, $R_{m_i}(A_i;x,y)=A_i(x)-\sum_{|\gamma|<m_i}\frac{1}{\gamma!}D^\gamma A_i(y)(x-y)^r$, where $D^\gamma(A_i)\in\mathrm{BMO}(\mathbf{R}^n)$ for $|\gamma|=m_i-1(m_i>1)$, $i=1,2$.

Key words: *multilinear operator, fractional integral, rough kernel,* BMO

AMS (2010) subject classification: 42B20, 42B25

1 Introduction

As two of the most important operators in harmonic analysis, the fractional integral operator $T_{\Omega,\alpha}$ and the corresponding maximal operator $M_{\Omega,\alpha}$ are defined by

$$T_{\Omega,\alpha}f(x):=\int_{\mathbf{R}^n}\frac{\Omega(x-y)}{|x-y|^{n-\alpha}}f(y)\mathrm{d}y, \tag{1.1}$$

* Supported in part by the NNSF of China under Grant #10771110 and # 11171306 and by the ZNSF of China under Grant LY12A01024.

$$M_{\Omega,\alpha}f(x) := \sup_{h>0}\int_{|x-y|<h}|\Omega(x-y)f(y)|\mathrm{d}y, \tag{1.2}$$

where $0<\alpha<n$, $1/q=1/p-\alpha/n$ and $\Omega\in L^s(S^{n-1})(s\geq n/(n-\alpha))$ is homogeneous of degree zero in $\mathbf{R}^n$. In 1993 and 1998, Chanillo [1] and Ding [7] proved that $T_{\Omega,\alpha}$ and $M_{\Omega,\alpha}$ are bounded from $L^p(1<p<\infty)$ to L^q respectively. In 1997, Ding [2] gave that if $-1<\beta<0$ and $f\in L^1(|x|^{\beta(n-\alpha)/n})$, then $T_{\Omega,\alpha}$ and $M_{\Omega,\alpha}$ are both bounded from $L^1(|x|^{\beta(n-\alpha)/n})$ to $L^{n/(n-\alpha),\infty}$.

It is well known that the study of multilinear fractional integral operators are received increasing attentions. Let $\gamma=(\gamma_1,\gamma_2,\cdots,\gamma_n)$, and $\gamma_i(i=1,2,\cdots,n)$ be nonnegtive integers. Denote $|\gamma|=\sum\limits_{i=1}^{n}\gamma_i$, $\gamma!=\gamma_1!\gamma_2!\cdots\gamma_n!$, $x^\gamma=x_1^{\gamma_1}x_2^{\gamma_2}\cdots x_n^{\gamma_n}$

$$D^\gamma=\frac{\partial^{|\gamma|}}{\partial^{\gamma_1}x_1\partial^{\gamma_2}x_2\cdots\partial^{\gamma_n}x_n}.$$

Suppose that A is a function defined on $\mathbf{R}^n$. Denote by $R_m(A;x,y)$ the m-th order remainder of the Taylor series of A at x about y, that is, $R_m(A;x,y)=A(x)-\sum\limits_{|\gamma|<m}\frac{1}{\gamma!}D^\gamma A(y)(x-y)^r$, $m\geq 1$. Then the multilinear fractional integral operator $T^A_{\Omega,\alpha}$ is defined by

$$T^A_{\Omega,\alpha}f(x) := \int_{\mathbf{R}^n}\frac{\Omega(x-y)R_m(A;x,y)}{|x-y|^{n-\alpha+m-1}}f(y)\mathrm{d}y \tag{1.3}$$

and the relevant maximal operator $M^A_{\Omega,\alpha}$ is given by

$$M^A_{\Omega,\alpha}f(x) := \sup_{r>0}\frac{1}{r^{n-\alpha+m-1}}\int_{|x-y|<r}|\Omega(x-y)R_m(A;x,y)f(y)|\mathrm{d}y. \tag{1.4}$$

In 2001, Ding[3] proved that if $D^\gamma A\in L^r(\mathbf{R}^n)(1<r\leq\infty,|\gamma|=m-1)$, then $T^A_{\Omega,\alpha}$, $M^A_{\Omega,\alpha}$ are both weighted bounded operators from $L^p(w^p)$ to $L^q(w^q)$ with the weight $w\in A(p,q)$ and from $L^p(1\leq p<n/\alpha)$ to $L^{n/(n-\alpha),\infty}$ with the power weight. Obviously, when $m=1$, $T^A_{\Omega,\alpha}$ reduces to the commutator generated by the fractional integral $T_{\Omega,\alpha}$ and the function A. In 2002, Yang and Wu [9] proved that if $D^\gamma A\in \mathrm{BMO}(\mathbf{R}^n)$, then $T^A_{\Omega,\alpha}$ and $M^A_{\Omega,\alpha}$ are bounded from $L^p(1<p<\infty)$ to L^q. In 2003, Lu and Zhang[5] proved that if $D^\gamma A\in\dot{\wedge}_\beta$, $s>\dfrac{n}{n-(\alpha+2\beta)}$, $0<\beta<1$, $1/q=1/p-(\alpha+\beta)/n$, then $T^A_{\Omega,\alpha}$ and $M^A_{\Omega,\alpha}$ are bounded from $L^p(1<p<\dfrac{n}{\alpha+\beta})$ to L^q and from L^1 to $L^{n/n-\alpha-\beta,\infty}$. In 2001, Lu and Ding[4] showed that if $D^\gamma A_j\in\mathrm{BMO}(\mathbf{R}^n)$, than the operator

$$T^{A_1,A_2,\cdots,A_k}_{\Omega,\alpha}f(x) := \int_{\mathbf{R}^n}\frac{\Omega(x-y)}{|x-y|^{n-\alpha+N}}\prod_{j=1}^{k}R_{m_j}(A_j;x,y)f(y)\mathrm{d}y \tag{1.5}$$

with $N=\sum\limits_{j=1}^{k}(m_j-1)(m_j\geq 2)$ and the relevant maximal operator

$$M^{A_1,A_2,\cdots,A_k}_{\Omega,\alpha}f(x) := \sup_{r>0}\frac{1}{r^{n-\alpha+N}}\int_{|x-y|<r}|\Omega(x-y)\prod_{j=1}^{k}R_{m_j}(A_j;x,y)f(y)|\mathrm{d}y \tag{1.6}$$

are both weighted bounded operators from $L^p(w^p)$ to $L^q(w^q)$ with $w \in A(p,q)$.

In 2006, Lan[8] proved that if $D^\gamma A \in \dot{\Lambda}_\beta$, then the operator

$$T^{A_1,A_2}_{\Omega,\alpha} f(x) := \int_{\mathbf{R}^n} \frac{\Omega(x-y)}{|x-y|^{n-\alpha+N}} \prod_{j=1}^{2} R_{m_j}(A_j;x,y) f(y) \mathrm{d}y \tag{1.7}$$

with $N = \sum_{j=1}^{2}(m_j-1)(m_j \geq 2)$ and the relevant maximal operator

$$M^{A_1,A_2}_{\Omega,\alpha} f(x) := \sup_{r>0} \frac{1}{r^{n-\alpha+N}} \int_{|x-y|<r} |\Omega(x-y) \prod_{j=1}^{2} R_{m_j}(A_j;x,y) f(y)| \mathrm{d}y \tag{1.8}$$

are bounded from $L^1(\mathbf{R}^n)$ to $L^{\frac{n}{n-(\alpha+2\beta)},\infty}(\mathbf{R}^n)$ and from $L^1(|x|^{\frac{t(n-(\alpha+2\beta))}{n}})$ to $L^{\frac{n}{n-(\alpha+2\beta)},\infty}(|x|^t)$.

Our aim in the paper is to establish the boundedness for the multilinear fractional integral operators $T^{A_1,A_2}_{\Omega,\alpha}$ and $M^{A_1,A_2}_{\Omega,\alpha}$, and obtain the following theorems:

Theorem 1.1. *Let $0 < \alpha < n$, A_i is a function defined on $\mathbf{R}^n$, $D^\gamma(A_i) \in \mathrm{BMO}(\mathbf{R}^n)(|\gamma| = m_i - 1)$, $i = 1,2$ and Ω is homogeneous of degree zero on $\mathbf{R}^n$ with zero mean value on S^{n-1}, $\Omega \in L^s(S^{n-1})$ with $s > \frac{n}{n-\alpha}$. Then if $1 < p,q < \infty$, and $1/q = 1/p - \alpha/n$, there exists a constant C, independent of A and f, such that*

$$\| T^{A_1,A_2}_{\Omega,\alpha} f \|_{L^q} \leq C \prod_{i=1}^{2} \sum_{|\gamma|=m_i-1} \| D^\gamma A_i \|_{\mathrm{BMO}} \| f \|_{L^p}, \tag{1.9}$$

$$\| M^{A_1,A_2}_{\Omega,\alpha} f \|_{L^q} \leq C \prod_{i=1}^{2} \sum_{|\gamma|=m_i-1} \| D^\gamma A_i \|_{\mathrm{BMO}} \| f \|_{L^p} . \tag{1.10}$$

Theorem 1.2. *Let $0 < \alpha < n$, A_i a function defined on $\mathbf{R}^n$, $D^\gamma(A_i) \in \mathrm{BMO}(\mathbf{R}^n)(|\gamma| = m_i - 1)$, $i = 1,2$ and $\Omega \in L^s(S^{n-1})$, $s \geq 1$. Then if $-1 < \beta < 0$, there exists a constant C, independent of A and f, such that for any $\lambda > 0$ and any $f \in L^1(|x|^{\beta(n-\alpha)/n})$, the following conclusions hold:*

$$\int_{\{x:|T^{A_1,A_2}_{\Omega,\alpha} f(x)|>\lambda\}} |x|^\beta \mathrm{d}x \leq C \left(\frac{1}{\lambda} \prod_{i=1}^{2} \sum_{|\gamma|=m_i-1} \| D^\gamma A_i \|_{\mathrm{BMO}} \| f \|_{L^1(|x|^{\beta(n-\alpha)/n})} \right)^{n/(n-\alpha)}, \tag{1.11}$$

$$\int_{\{x:|M^{A_1,A_2}_{\Omega,\alpha} f(x)|>\lambda\}} |x|^\beta \mathrm{d}x \leq C \left(\frac{1}{\lambda} \prod_{i=1}^{2} \sum_{|\gamma|=m_i-1} \| D^\gamma A_i \|_{\mathrm{BMO}} \| f \|_{L^1(|x|^{\beta(n-\alpha)/n})} \right)^{n/(n-\alpha)}. \tag{1.12}$$

2 Proof of Theorems

Lemma 2.1[6]**.** *Let A be a function on $\mathbf{R}^n$ and $D^\gamma A \in L^l_{loc}(\mathbf{R}^n)$ for $|\gamma| = m$ and some $l > n$. Then*

$$|R_m(A;x,y)| \leq C|x-y|^m \sum_{|\gamma|=m} \left(\frac{1}{|\widetilde{Q}(x,y)|} \int_{\widetilde{Q}(x,y)} |D^\gamma A(z)|^l \mathrm{d}z \right)^{\frac{1}{l}},$$

where $\widetilde{Q}(x,y)$ *is the cube centered at* x *with edges parallel to the axes and having diameter* $16\sqrt{n}|x-y|$.

Lemma 2.2[7]**.** *Let* $0<\alpha<n$, Ω *be homogeneous of degree zero on* $\mathbf{R}^n$, $\Omega\in L^s(S^{n-1})$, $s\geq\dfrac{n}{n-\alpha}$, $1<p,q<\infty$ *and* $1/q=1/p-\alpha/n$, *then*

$$\| M_{\Omega,\alpha}f \|_{L^q}\leq C\| f \|_{L^p}\ . \tag{2.1}$$

Lemma 2.3[2]**.** *Suppose that* $0<\alpha<n$, $-1<\beta<0$, $\Omega\in L^s(S^{n-1}), s\geq\frac{n}{n-\alpha}$. *Then for any* $\lambda>0$ *and any* $f\in L^1(|x|^{\beta(n-\alpha)/n})$, *there exists a constant* C, *independent of* f, *such that*

$$\int_{\{x:|T_{\Omega,\alpha}f(x)|>\lambda\}}|x|^\beta \mathrm{d}x\leq C\left(\frac{1}{\lambda}\| f \|_{L^1(|x|^{\beta(n-\alpha)/n})}\right)^{n/n-\alpha}, \tag{2.2}$$

$$\int_{\{x:|M_{\Omega,\alpha}f(x)|>\lambda\}}|x|^\beta \mathrm{d}x\leq C\left(\frac{1}{\lambda}\| f \|_{L^1(|x|^{\beta(n-\alpha)/n})}\right)^{n/n-\alpha}. \tag{2.3}$$

Lemma 2.4[4]**.** *Suppose that* $0<\alpha<n$, Ω *is homogeneous of degree zero on* $\mathbf{R}^n$, $\Omega\in L^s(S^{n-1})(s\geq1)$, *and define the operator* $\widetilde{M}^{A_1,A_2}_{\Omega,\alpha}$ *by*

$$\widetilde{M}^{A_1,A_2}_{\Omega,\alpha}f(x):=\sup_{r>0}\frac{1}{r^{n-\alpha+N}}\int_{r/2\leq|x-y|<r}|\Omega(x-y)\prod_{i=1}^2 R_{m_i}(A_i;x,y)f(y)|\mathrm{d}y.$$

Let $|\gamma|=m_i-1$ *for* $i=1,2$, *and* $D^\gamma(A_i)\in \mathrm{BMO}(\mathbf{R}^n)$, *then for any* $1<t<\infty$, *we have*

$$\widetilde{M}^{A_1,A_2}_{\Omega,\alpha}f(x)\leq C\prod_{i=1}^2\sum_{|\gamma|=m_i-1}\| D^\gamma A_i \|_{BMO}\left[M_{\Omega,\alpha}f(x)+(M_{\Omega^t,\alpha t}(|f|^t)(x))^{1/t}\right], \tag{2.4}$$

where C *is a constant independent of* f *and* t.

Proof of Theorem 1.1. By Lemma 2.4 and noting that $t/q=t/p-\alpha t/n$, we obtain

$$\begin{aligned}\| \widetilde{M}^{A_1,A_2}_{\Omega,\alpha}f \|_{L^q}&\leq C\prod_{i=1}^2\sum_{|\gamma|=m_i-1}\| D^\gamma A_i \|_{\mathrm{BMO}}\left(\| M_{\Omega,\alpha}f \|_{L^q}+\| M_{\Omega^t,\alpha t}(|f|^t) \|^{1/t}_{q/t}\right)\\ &\leq C\prod_{i=1}^2\sum_{|\gamma|=m_i-1}\| D^\gamma A_i \|_{\mathrm{BMO}}\| f \|_{L^p}\ .\end{aligned}$$

Since $M^{A_1,A_2}_{\Omega,\alpha}f(x)\leq C\widetilde{M}^{A_1,A_2}_{\Omega,\alpha}f(x)$ for all $x\in\mathbf{R}^n$, we have

$$\| M^{A_1,A_2}_{\Omega,\alpha}f \|_{L^q}\leq\| \widetilde{M}^{A_1,A_2}_{\Omega,\alpha}f \|_{L^q}\leq C\prod_{i=1}^2\sum_{|\gamma|=m_i-1}\| D^\gamma A_i \|_{\mathrm{BMO}}\| f \|_{L^p}\ .$$

This finishes the proof of (1.10).

Before showing (1.9), we give a proposition.

Proposition 2.5. *For any $\varepsilon > 0$ with $0 < \alpha - \varepsilon < \alpha + \varepsilon < n$, we have*

$$|T_{\Omega,\alpha}^{A_1,A_2} f(x)| \le C_\varepsilon [M_{\Omega,\alpha+\varepsilon}^{A_1,A_2} f(x) M_{\Omega,\alpha-\varepsilon}^{A_1,A_2} f(x)]^{1/2}.$$

Proof. The basic idea of the proof is taken from [7]. Given $x \in \mathbf{R}^n$ and $\varepsilon > 0$ with $0 < \alpha - \varepsilon < \alpha + \varepsilon < n$, we choose $\delta > 0$ such that $\delta^{2\varepsilon} = M_{\Omega,\alpha+\varepsilon}^{A_1,A_2} f(x) / M_{\Omega,\alpha-\varepsilon}^{A_1,A_2} f(x)$. Now we put

$$\begin{aligned} T_{\Omega,\alpha}^{A_1,A_2} f(x) &= \int_{|x-y|<\delta} \frac{R_{m_1}(A_1;x,y) R_{m_2}(A_2;x,y)}{|x-y|^{n-\alpha+m_1+m_2-2}} \Omega(x-y) f_(y) \mathrm{d}y \\ &+ \int_{|x-y|\ge\delta} \frac{R_{m_1}(A_1;x,y) R_{m_2}(A_2;x,y)}{|x-y|^{n-\alpha+m_1+m_2-2}} \Omega(x-y) f_(y) \mathrm{d}y \\ &= I_1 + I_2. \end{aligned}$$

Thus

$$\begin{aligned} |I_1| &\le \sum_{j=1}^{\infty} \int_{2^{-j}\delta \le |x-y| < 2^{-j+1}\delta} \frac{|R_{m_1}(A_1;x,y) R_{m_2}(A_2;x,y)|}{|x-y|^{n-\alpha+m_1+m_2-2}} |\Omega(x-y)||f_(y)| \mathrm{d}y \\ &\le \sum_{j=1}^{\infty} (2^{-j}\delta)^{-(n-\alpha+m_1+m_2-2)} \int_{|x-y|<2^{-j+1}\delta} |R_{m_1}(A_1;x,y) R_{m_2}(A_2;x,y)||\Omega(x-y)||f_(y)| \mathrm{d}y \\ &\le C_\varepsilon \sum_{j=1}^{\infty} (2^{-j+1}\delta)^{-(n-\alpha+\varepsilon+m_1+m_2-2)} \int_{|x-y|<2^{-j+1}\delta} |R_{m_1}(A_1;x,y) R_{m_2}(A_2;x,y)||\Omega(x-y)||f_(y)| \mathrm{d}y \\ &\le C_\varepsilon \delta^\varepsilon M_{\Omega,\alpha-\varepsilon}^{A_1,A_2} f(x). \end{aligned}$$

Similarly,

$$|I_2| \le C_\varepsilon \delta^{-\varepsilon} M_{\Omega,\alpha+\varepsilon}^{A_1,A_2} f(x).$$

Therefore, we get

$$\begin{aligned} |T_{\Omega,\alpha}^{A_1,A_2} f(x)| &\le C_\varepsilon [\delta^\varepsilon M_{\Omega,\alpha-\varepsilon}^{A_1,A_2} f(x) + \delta^{-\varepsilon} M_{\Omega,\alpha+\varepsilon}^{A_1,A_2} f(x)] \\ &\le C_\varepsilon [M_{\Omega,\alpha+\varepsilon}^{A_1,A_2} f(x)]^{1/2} [M_{\Omega,\alpha-\varepsilon}^{A_1,A_2} f(x)]^{1/2}. \end{aligned}$$

The proposition is proved.

Take $0 < \varepsilon < \alpha$ satisfying $\alpha + \varepsilon/n < 1/p < 1$, $1 < q_1, q_2 < \infty$ satisfying $1/q_1 = 1/p - (\alpha+\varepsilon)/n$ and $1/q_2 = 1/p - (\alpha-\varepsilon)/n$. Noting that $1/q = 1/2q_1 + 1/2q_2$, by Proposition 2.5. Holder's inequality and (1.10), we obtain

$$\begin{aligned} \| T_{\Omega,\alpha}^{A_1,A_2} f \|_{L^q} &\le C \| [M_{\Omega,\alpha+\varepsilon}^{A_1,A_2} f]^{1/2} \|_{2q_1} \| [M_{\Omega,\alpha-\varepsilon}^{A_1,A_2} f]^{1/2} \|_{2q_2} \\ &\le C \prod_{i=1}^{2} \sum_{|\gamma|=m_i-1} \| D^\gamma A_i \|_{\mathrm{BMO}} \| f \|_{L^p} . \end{aligned}$$

This finishes the proof of (1.9).

Proof of Theorem 1.2. At first we show (1.12). By Lemma 2.4, we get

$$\widetilde{M}^{A_1,A_2}_{\Omega,\alpha} f(x) \le C \prod_{i=1}^{2} \sum_{|\gamma|=m_i-1} \| D^\gamma A_i \|_{\rm BMO} \left[M_{\Omega,\alpha} f(x) + (M_{\Omega^t,\alpha t}(|f|^t)(x))^{1/t} \right].$$

For any $\lambda > 0$, if $C \prod_{i=1}^{2} \sum_{|\gamma|=m_i-1} \| D^\gamma A_i \|_{\rm BMO} \doteq C_0$, we get

$$\int_{\{x:|M^{A_1,A_2}_{\Omega,\alpha} f(x)|>\lambda\}} |x|^\beta {\rm d}x \le \int_{\{x:C_0 M_{\Omega,\alpha} f(x)>\lambda/2\}} |x|^\beta {\rm d}x + \int_{\{x:C_0 M_{\Omega^t,\alpha t}(|f|^t)(x))^{1/t}>\lambda/2\}} |x|^\beta {\rm d}x := J_1 + J_2.$$

Using Lemma 2.3, we have

$$\begin{aligned} J_1 &\le \left(C_0 \frac{1}{\lambda} \| f \|_{L^1(|x|^{\beta(n-\alpha)/n})} \right)^{n/(n-\alpha)} \\ &\le C \left(\frac{1}{\lambda} \prod_{i=1}^{2} \sum_{|\gamma|=m_i-1} \| D^\gamma A_i \|_{\rm BMO} \| f \|_{L^1(|x|^{\beta(n-\alpha)/n})} \right)^{n/(n-\alpha)}. \end{aligned}$$

Now let us give the estimation of J_2. Note that

$$\begin{aligned} J_2 &\le (\frac{2C_0}{\lambda})^{n/(n-\alpha)} \int_{\mathbf{R}^n} (M_{\Omega^t,\alpha t}(|f|^t)(x))^{1/t \cdot n/(n-\alpha)} |x|^\beta {\rm d}x \\ &\le (\frac{2C_0}{\lambda})^{n/(n-\alpha)} \left(\int_{\mathbf{R}^n} \left[(M_{\Omega^t,\alpha t}(|f|^t)(x))^{1/t} |x|^{\beta(n-\alpha)/n} \right]^{n/(n-\alpha)} {\rm d}x \right)^{\frac{n-\alpha}{n}\frac{n}{n-\alpha}} \\ &\le \left(\frac{C_0}{\lambda} \| (M_{\Omega^t,\alpha t}|f|^t)^{1/t} |x|^{\beta(n-\alpha)/n} \|_{L^{\frac{n}{n-\alpha}}} \right)^{n/(n-\alpha)}. \end{aligned}$$

By $-1<\beta<0$ and $0<\alpha<n$, $\dfrac{1}{n/(n-\alpha)} = 1 - \frac{\alpha}{n}$, applying $(L^1, L^{\frac{n}{n-\alpha}})$ boundedness of $M_{\Omega,\alpha}$ with power weights [11], and that $\dfrac{t}{n/(n-\alpha)} = t - \dfrac{\alpha t}{n}$, we have

$$\begin{aligned} \| (M_{\Omega^t,\alpha t}|f|^t)^{1/t} |x|^{\beta(n-\alpha)/n} \|_{L^{\frac{n}{n-\alpha}}} &\le C \| (M_{\Omega^t,\alpha t}|f|^t) |x|^{\beta(n-\alpha)/n} \|^{1/t}_{L^{\frac{n}{(n-\alpha)t}}} \\ &\le C \| f |x|^{\beta(n-\alpha)/n} \|_{L^1} \\ &= C \| f \|_{L^1(|x|^{\beta(n-\alpha)/n})}. \end{aligned}$$

So

$$\begin{aligned} J_2 &\le \left(C_0 \frac{1}{\lambda} \| f \|_{L^1(|x|^{\beta(n-\alpha)/n})} \right)^{n/(n-\alpha)} \\ &\le C \left(\frac{1}{\lambda} \prod_{i=1}^{2} \sum_{|\gamma|=m_i-1} \| D^\gamma A_i \|_{\rm BMO} \| f \|_{L^1(|x|^{\beta(n-\alpha)/n})} \right)^{n/(n-\alpha)}. \end{aligned}$$

Combining the estimates for J_1 and J_2 we get

$$\int_{\{x:|M_{\Omega,\alpha}^{A_1,A_2}f(x)|>\lambda\}}|x|^{\beta}\mathrm{d}x\le C\left(\frac{1}{\lambda}\prod_{i=1}^{2}\sum_{|\gamma|=m_i-1}\|D^{\gamma}A_i\|_{\mathrm{BMO}}\|f\|_{L^1(|x|^{\beta(n-\alpha)/n})}\right)^{n/(n-\alpha)}.$$

We finish the proof of (1.12).

From the proof of Proposition 2.5, we have

$$|T_{\Omega,\alpha}^{A_1,A_2}f(x)|\le C_{\varepsilon}[\delta^{\varepsilon}M_{\Omega,\alpha-\varepsilon}^{A_1,A_2}f(x)+\delta^{-\varepsilon}M_{\Omega,\alpha+\varepsilon}^{A_1,A_2}f(x)].$$

For any $\lambda>0$, by (1.12) we get

$$\begin{aligned}\int_{\{x:|T_{\Omega,\alpha}^{A_1,A_2}f(x)|>\lambda\}}|x|^{\beta}\mathrm{d}x&\le\int_{\{x:C_{\varepsilon}\delta^{-\varepsilon}M_{\Omega,\alpha+\varepsilon}f(x)>\lambda/2\}}|x|^{\beta}\mathrm{d}x\\&\quad+\int_{\{x:C_{\varepsilon}\delta^{\varepsilon}M_{\Omega,\alpha-\varepsilon}^{A_1,A_2}f(x)>\lambda/2\}}|x|^{\beta}\mathrm{d}x\\&\le C\left(\frac{1}{\lambda}\prod_{i=1}^{2}\sum_{|\gamma|=m_i-1}\|D^{\gamma}A_i\|_{\mathrm{BMO}}\|f\|_{L^1(|x|^{\beta(n-\alpha)/n})}\right)^{n/(n-\alpha)}.\end{aligned}$$

We finish the proof of (1.11). The proof of the theorems are complete.

Remark 1. Theorem 1.1 also holds for the operators $T_{\Omega,\alpha}^{A_1,A_2,\cdots,A_k}$ and $M_{\Omega,\alpha}^{A_1,A_2,\cdots,A_k}$, $k\in\mathbf{N}$.

Acknowledgment The authors would like to thank the referees for their useful comments.

References

[1] Chanillo, S., Watson, D. and Wheeden, R. L., Some Integral and Maximal Operators Related to Star-like, Studia. Math., 107(1993), 233-255.

[2] Ding Y., Weak Type Bounds for a Class of Rough Operators with Power Weights, Proc. Amer. Math. Soc., 125(1997), 2939-2942.

[3] Ding Y., A Note on Multiliner Fractional Integrals with Rough Kernel, Adv. Math. J., 30(2001), 238-246.

[4] Ding, Y. and Lu, S. Z., Weighted Boundedness for a Class of Rough Multilinear Operators, Acta. Math. Sinica., 17(2001), 517-526.

[5] Lu, S. Z. and Zhang P., Lipschitz Estimates for Generalized Commutators of Fractional Integrals with Rough Kernel, Nachr. Math., 252(2003), 70-85.

[6] Cohen, J. J. and Gosselin, A BMO Estimate for Multilinear Singular Integral Operators, Illionois J. Math., 30(1986), 445-465.

[7] Ding, Y. and Lu, S. Z., Weighted Norm Inequalities for Fractional Integral Operators with Rough Kernel, Canad. J. Math., 58(1998), 29-39.

[8] Lan, J. C., Hardy-Littlewood-Sobolev, Theorem of Multilinear Fractional Integral Operators, Act. Math. Scientia. ser. A., 26(2006), 449-457.

[9] Wu, Q. and Yang, D. C., On Fractional Multilinear Singular Integrals, Nachr. Math. J., 239/240(2002), 215-235.

[10] Ding, Y. and Lu, S. Z., Multilinear Singular and Fractional Integrals, Acta. Math. Sinica, 22(2006):347-356.

[11] Muckenhoupt, B. and Wheeden, R., Weighted Norm Inequalities for Singular and Fractional Integrals, Tran. Amer. Math. Soc., 161(1971):249-258.

Department of Mathematics
Zhejiang University of Science and Technology
Hangzhou, 310023
P. R. China

X. X. Tao
E-mail: xxtau@163.com

Y. P. Wu
E-mail: wyp19860917@163.com