Anal. Theory Appl.
Vol. 28, No. 3 (2012), 232–241
DOI : 10.3969/j.issn.1672-4070.2012.03.003

STANCU POLYNOMIALS BASED ON THE Q-INTEGERS

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Received Mar. 18, 2011

Abstract. A new generalization of Stancu polynomials based on the q-integers and a nonnegative integer s is firstly introduced in this paper. Moreover, the shape-preserving and convergence properties of these polynomials are also investigated.

Key words: *Stancu polynomial, q-integer, q-derivative, shape-preserving property, convergence rate, modulus of continuity*
AMS (2010) subject classification: 41A10

1 Introduction

In 1981 Stancu proposed a kind of generalized Bernstein polynomials, namely Stancu polynomials, which was defined as:

Definition 1[1]. Let s be an integer and $0 \le s < \frac{n}{2}$, for $f \in C[0,1]$,

$$L_{n,s}(f,x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) b_{n,k,s}(x), \tag{1.1}$$

where

$$b_{n,k,s}(x) = \begin{cases} (1-x)p_{n-s,k}(x), & 0 \le k < s, \\ (1-x)p_{n-s,k}(x) + xp_{n-s,k-s}(x), & s \le k \le n-s, \\ xp_{n-s,k-s}(x), & n-s < k \le n, \end{cases}$$

and $p_{j,k}(x)$ are the base functions of Bernstein polynomials.

It is not difficult to see that for $s = 0,\ 1$ the Stancu polynomials are just the classical Bernstein polynomials. For $s \ge 2$, these polynomials possess many remarkable properties, which have made them an area of intensive research (see [2, 3, 4, 5]).

Throughout this paper we employ the following notations of q-Calculus. Let $q > 0$. For each nonnegative integer k, the q-integer $[k]$ and the q-factorial $[k]!$ are defined by

$$[k] = \begin{cases} \dfrac{1-q^k}{1-q}, & q \ne 1 \\ k, & q = 1, \end{cases}$$

$$[k]! = \begin{cases} [k][k-1]\cdots[1], & k \geq 1 \\ 1, & k = 0. \end{cases}$$

For $n,\ k,\ n \geq k \geq 0$, q-binomial coefficients are defined naturally as

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!}.$$

Now let's introduce a new generalization of Stancu polynomials as below.

Definition 2. Let s be an integer and $0 \leq s < \frac{n}{2}$, $q > 0,\ n > 0$, for $f \in C[0,1]$,

$$L_{n,s}(f,q;x) = \sum_{k=0}^{n} f\left(\frac{[k]}{[n]}\right) b_{n,k,s}(q;x), \tag{1.2}$$

where

$$b_{n,k,s}(q;x) = \begin{cases} (1-q^{n-k-s}x)p_{n-s,k}(q;x), & 0 \leq k < s, \\ (1-q^{n-k-s}x)p_{n-s,k}(q;x) + q^{n-k}xp_{n-s,k-s}(q;x), & s \leq k \leq n-s, \\ q^{n-k}xp_{n-s,k-s}(q;x), & n-s < k \leq n, \end{cases}$$

and

$$p_{n-s,k}(q;x) = \begin{bmatrix} n-s \\ k \end{bmatrix} x^k \prod_{l=0}^{n-s-k-1}(1-q^l x), \qquad k = 0,\ 1,\ \cdots,\ n-s.$$

(agree on $\prod_{l=0}^{0} = 1$).

It is worth mentioning that the q-Stancu polynomials defined as (1.2) differ essentially from the q-Stancu polynomials in [6]. To get their q-Stancu polynomials in [6] the authors just generalized the control points of the Stancu polynomials based on the q-integers leaving alone the basis functions. While in our q-Stancu polynomials both the control points and the basis functions are the q-analogue of those in Stancu polynomials. As a result, it is not a strange thing that these two q-Stancu polynomials behave quite differently properties, especially in the approximation problem.

It can be easily verified that in case $q = 1$, $L_{n,s}(f,q;x)$ reduce to the Stancu polynomials and in case $s = 0,\ 1$, $L_{n,s}(f,q;x)$ coincide with the q-Bernstein polynomials which are defined by Phillips in [7] and have been intensively investigated during these years (see [8-12]).

By some direct calculations, one can get the following two representations: for $f \in C[0,1]$, an integer s and $0 \leq s < \frac{n}{2}$,

$$L_{n,s}(f,q;x) = \sum_{k=0}^{n-s}\left\{(1-q^{n-k-s}x)f\left(\frac{[k]}{[n]}\right) + q^{n-k-s}xf\left(\frac{[k+s]}{[n]}\right)\right\} p_{n-s,k}(q;x); \tag{1.3}$$

$$L_{n,s}(f,q;x) = \sum_{k=0}^{n-s+1} \left\{ \frac{[n-s+1-k]}{[n-s+1]} f\left(\frac{[k]}{[n]}\right) + \frac{q^{n-s+1-k}[k]}{[n-s+1]} f\left(\frac{[s-1+k]}{[n]}\right) \right\} p_{n-s+1,k}(q;x). \tag{1.4}$$

Except the above two representations, Stancu polynomials based on q-integers possess the following essential properties.

Proposition 1. *For $0 < q < 1$, $L_{n,s}(\cdot, q)$ is a positive linear operator, while for $q > 1$ it is not true, as the positiveness fails.*

Proposition 2. *Let $q > 0$. For $e_i = x^i$, $i = 0,\ 1,\ 2$, hold $L_{n,s}(e_0, q; x) \equiv 1$, $L_{n,s}(e_1, q; x) = e_1$,*

$$L_{n,s}(e_2, q; x) = e_2 + \left(\frac{[1]}{[n]} + \frac{q^{n-s}[s]^2 - q^{n-s}[s]}{[n]^2}\right) x(1-x).$$

Proposition 3. *For any function $f(x)$ and parameter $q > 0$, hold $L_{n,s}(f, q; 0) = f(0)$, $L_n(f, q; 1) = f(1)$.*

Proposition 4. *Let $0 < q < 1$. For a concave function $f(x)$ on [0,1], holds $L_{n,s}(f, q; x) > B_{n-s+1}(f, q; x)$.*

The following are our main results on shape-preserving properties.

2 Shape-Preserving Properties

To begin with, we should recall the conception of q-derivative. Let $q > 0$ and $q \neq 1$. For a function $f(x)$, its q-derivative denoted by $D_q(f)(x)$, is defined as

$$D_q(f)(x) = \begin{cases} \dfrac{f(qx) - f(x)}{(q-1)x}, & x \neq 0, \\ \lim\limits_{t \to 0} D_q(f)(t), & x = 0; \end{cases}$$

and the higher q-derivatives are defined recursively by

$$D_q^n f = D_q(D_q^{n-1} f), \qquad n = 1,\ 2,\ \cdots,\ D_q^0 f = f.$$

Under the above definition, one can see for $x \neq 0$ the existence of $D_q^n(f)(x)$ is sure and if $f(x)$ is continuous the continuity of $D_q^n(f)(x)$ can also be guaranteed. The usual derivative $f'(x)$ is just equal to the limit of $D_q(f)(x)$ as q trends to 1. Moreover, the following lemma holds.

Lemma 1. *Let $f(x)$ be a continuous function on $[0,1]$ satisfying $f(0) = f(1)$. Then there exists $\xi \in (0,1)$ such that*

$$D_q(f)(\xi) = 0$$

holds for all $q \in (0,1) \bigcup (1, +\infty)$.

This lemma improves the q-Rolle theorem (see [13, Th.2.1]) *with respect to the range of q.*

Proof. As $f(x)$ is continuous on $[0,1]$ and $f(0)=f(1)$, there exist either the maximum or the minimum points in the inner of $[0,1]$. In the following we discuss the sign of $D_q(f)(1)$ under the condition $q\in(0,1)$.

Case 1 $D_q(f)(1)<0$. In this case, we have $f(q)>f(1)$ as $q\in(0,1)$. Then without loss of generality, we can assume that there exists $x_0\in(0,1)$ such that $f(x_0)=\max\limits_{0\leq x\leq 1}f(x)$. Evidently, $D_q(f)(x_0)>0$. From the continuity of $D_q(f)(x),\ x\in(0,1]$, we can conclude that there exists $\xi\in(x_0,1)\subsetneqq(0,1)$ such that $D_q(f)(\xi)=0$.

Case 2 $D_q(f)(1)>0$. Using the similar method of Case 1, we get that there exists $\xi\in(0,1)$ such that $D_q(f)(\xi)=0$.

Case 3 $D_q(f)(1)=0$. In this case, we have $f(q)=f(1)=f(0)$. Repeat the above discussion for $D_q(f)(q)$, then we get: for $D_q(f)(q)\neq 0$, there exists $\xi\in(0,q)$ such that $D_q(f)(\xi)=0$; otherwise the result of the lemma holds naturally as $\xi=q$.

As a conclusion, the result holds for all $1>q>0$.

For $q\in(1,+\infty)$, discussing $D_q(f)(\frac{1}{q})$ instead of $D_q(f)(1)$, we can prove the result of the lemma by the similar way.

Furthermore, based on Lemma 1, we get a more explicit result of Theorem 2.3 in [13].

Lemma 2. *Let x and $x_0,\ x_1,\ \cdots,\ x_n$ be any distinct points in the interval $[0,1]$. Let $f(x)$ be a continuous function on $[0,1]$. Then there exists $\xi_x\in(0,1)$ such that for all $q\in(0,1)\bigcup(1,+\infty)$ holds*

$$f[x,\ x_0,\ x_1,\ \cdots,\ x_n]=\frac{D_q^{n+1}(f)(\xi_x)}{[n+1]!},$$

where $f[x,\ x_0,\ x_1,\ \cdots,\ x_n]$ denotes the divided difference of $f(x)$ at points $\{x,\ x_0,\ x_1,\ \cdots,\ x_n\}$.

Proof. Because of the continuity of $f(x)$ and the definition of $D_q^k(f),\ k=0,\ 1,\ \cdots,$ $D_q^{n+1}(f)(x)$ exists in $(0,1)$. Using Lemma 1 to replace the q-Rolle theorem in the proof of Theorem 2.3 in [13], we can get the result of Lemma 2.

In this section, we use $\Delta_q f$ to denote the q-differences of function $f(x)$. Especially, $\Delta_q^0 f_i=f_i$ for $i=0,\ 1,\ \cdots,\ n$ and

$$\Delta_q^{k+1}f_i=\Delta_q^k f_{i+1}-q^k\Delta_q^k f_i,$$

for $k=0,\ 1,\ \cdots,\ n-i-1$, where f_i denotes $f\left(\frac{[i]}{[n]}\right)$.

Theorem 1. *Let $0<q<1$, abd s an integer satisfying $0\leq s<\frac{n}{2}$, and $f(x)$ be a continuous, increasing function on $[0,1]$, then $L_{n,s}(f,q;x)$ is increasing on $[0,1]$.*

Proof. As for $s=0,1$, q-Stancu polynomials coincide with the q-Berntein polynomials, which possess the shape preserving properties[8], we just focus on the case $2\leq s<\frac{n}{2}$. By directly computing, we get

$$\begin{aligned} D_q(L_{n,s}(f,q))(x) &= \sum_{k=0}^{n-s} \Big\{ [n-s-k]\Delta_q^1 f_k + q^{n-s-k}[k+1]\Delta_q^1 f_{s-1+k} \\ &\quad + q^{n-s-k}[1]\left[f\left(\frac{[s-1+k]}{[n]}\right) - f\left(\frac{[k]}{[n]}\right)\right]\Big\} \frac{p_{n-s,k}(q;qx)}{q^k}. \end{aligned}$$

As $f(x)$ is an increasing function, for $k=0,\ 1,\ \cdots,\ n-s$, hold $\Delta_q^1 f_k > 0$ and $\Delta_q^1 f_{s-1+k} > 0$, $f\left(\frac{[s-1+k]}{[n]}\right) - f\left(\frac{[k]}{[n]}\right) > 0$. Then $D_q(L_{n,s}(f,q;x)) > 0$ in $(0,1]$.

By Lemma 2, we have: for any $x_1,\ x_2 \in [0,1]$, there exists $\xi \in (0,1)$ such that

$$L_{n,s}(f,q)[x_1,\ x_2] = D_q(L_{n,s}(f,q))(\xi).$$

Thus, for any $x_1 \le x_2 \in [0,1]$, hold $L_{n,s}(f,q)[x_1,\ x_2] > 0$. Up to now, the monotonic increasing property of $L_{n,s}(f,q;x)$ can be got directly.

For the convex function $f(x)$ which is the linear spline joining up the points $(0,0)$, $(0.2,0.6)$, $(0.6,0.8)$, $(0.9,0.7)$ and $(1,0)$, it is illustrated by **Figure 1** that $L_{n,s}(f,q;x)$ is also convex on $[0,1]$ with $q=0.7,\ 0.5$ and $s=3,\ 5$. In fact, we will show that it possesses more than this.

Theorem 2. *Let $0<q<1$, and s an integer satisfying $0 \le s < \frac{n}{2}$, and $f(x)$ be a continuous convex function on $[0,1]$, then $L_{n,s}(f,q;x)$ is also convex on $[0,1]$ and $L_{n,s}(f,q;x) \le f(x)$. Moreover, for any $x \in [0,1]$, $L_{n,s}(f,q;x)$ is monotonic decreasing in the parameter n.*

Proof. Firstly, we have

$$\begin{aligned} D_q^2(L_{n,s}(f,q))(x) &= [n-s]\sum_{k=0}^{n-s-1} \Big\{ [n-s-k-1]\Delta_q^2 f_k + q^{n-s-k-1}[k+2]\Delta_q^2 f_{s-1+k} \\ &\quad + \frac{q^{n-s}[2][s-1]}{[n]}\left[f[\frac{[s+k]}{[n]}, \frac{[k+1]}{[n]}] - f[\frac{[s-1+k]}{[n]}, \frac{[k]}{[n]}]\right]\Big\} \frac{p_{n-s-1,k}(q;q^2x)}{q^{2k}}. \end{aligned}$$

As $f(x)$ is convex on $[0,1]$, for any $k=0,\ 1,\ \cdots,\ n-s-1$, holds

$$\Delta_q^2 f_k = f\left(\frac{[k+2]}{[n]}\right) - (1+q)f\left(\frac{[k+1]}{[n]}\right) + qf\left(\frac{[k]}{[n]}\right) > 0.$$

In the same way, we get for $k=0,\ 1,\ \cdots,\ n-s-1$, $\Delta_q^2 f_{s-1+k} > 0$. And for $k=0,\ 1,\ \cdots,\ n-s-1$, the differences $f[\frac{[s+k]}{[n]}, \frac{[k+1]}{[n]}] - f[\frac{[s-1+k]}{[n]}, \frac{[k]}{[n]}] > 0$ are also guaranteed by the increasing property of the convex function in the slope of chord. Therefore,

$$D_q^2(L_{n,s}(f,q))(x) > 0, \qquad x \in (0,1]. \tag{2.1}$$

Combining (2.1) with Lemma 2, we obtain that $L_{n,s}(f,q;x)$ is convex on $[0,1]$.

Secondly, using the Jessen inequality for the convex function and the proposition 2, we get

$$\begin{aligned}
L_{n,s}(f,q;x) &= \sum_{k=0}^{n-s}\left\{(1-q^{n-k-s}x)f\left(\frac{[k]}{[n]}\right)+q^{n-k-s}xf\left(\frac{[k+s]}{[n]}\right)\right\}p_{n-s,k}(q;x)\\
&\geq \sum_{k=0}^{n-s} f\left((1-q^{n-k-s}x)\cdot\frac{[k]}{[n]}+q^{n-k-s}x\cdot\frac{[k+s]}{[n]}\right)p_{n-s,k}(q;x)\\
&\geq f\left(\sum_{k=0}^{n-s}\left\{(1-q^{n-k-s}x)\frac{[k]}{[n]}+q^{n-k-s}x\frac{[k+s]}{[n]}\right\}p_{n-s,k}(q;x)\right)\\
&= f(x).
\end{aligned}$$

Thirdly, before the proof of the monotonic property of $L_{n,s}(f,q;x)$ in the parameter n, it is necessary to recommend some notations. We denote

$$\varphi_{n,k}(x)=\begin{bmatrix} n-s+2\\ k\end{bmatrix}x^k\prod_{l=n-s+2-k}^{n-s+1}(1-q^lx)^{-1},\qquad x_{n,k}=\frac{[k]}{[n]},\ k=0,\ 1,\ \cdots,\ n.$$

It follows from the convex inequality of $f(x)$ that for $s\geq 1$, $0<q<1$ and $x\in[0,1]$,

$$\begin{aligned}
&\{L_{n+1,s}(f,q;x)-L_{n,s}(f,q;x)\}\prod_{l=0}^{n-s+1}(1-q^lx)^{-1}\\
&=\sum_{k=1}^{n-s+1}\left\{\frac{[n-s+2-k]}{[n-s+2]}f(x_{n+1,k})+\frac{q^{n-s+2-k}[k]}{[n-s+2]}f(x_{n+1,s-1+k})\right.\\
&\quad-\frac{[n-s+2-k]}{[n-s+2]}\left(\frac{[n-s+1-k]}{[n-s+1]}f(x_{n,k})+\frac{q^{n-s+1-k}[k]}{[n-s+1]}f(x_{n,s-1+k})\right)\\
&\quad\left.-\frac{q^{n-s+2-k}[k]}{[n-s+2]}\left(\frac{[n-s+2-k]}{[n-s+1]}f(x_{n,k-1})+\frac{q^{n-s+2-k}[k-1]}{[n-s+1]}f(x_{n,s-2+k})\right)\right\}\varphi_{n,k}(x)\\
&\leq\sum_{k=1}^{n-s+1}\left\{\frac{[n-s+2-k]}{[n-s+2]}\left(f(x_{n+1,k})-f(\eta_1)\right)+\frac{q^{n-s+2-k}[k]}{[n-s+2]}\left(f(x_{n+1,s-1+k})-f(\eta_2)\right)\right.\\
&\quad\left.+\frac{q^{n-s+1-k}[k][n-s+2-k](1-q)}{[n-s+1][n-s+2]}\left(f(x_{n,k})-f(x_{n,s-2+k})\right)\right\}\varphi_{n,k}(x),
\end{aligned}$$

where

$$\eta_1=\frac{q^{n-s+2-k}[k]}{[n-s+1]}\cdot\frac{[k-1]}{[n]}+(1-\frac{q^{n-s+2-k}[k]}{[n-s+1]})\cdot\frac{[k]}{[n]},$$

$$\eta_2=\frac{[n-s+2-k]}{[n-s+1]q}\cdot\frac{[s-1+k]}{[n]}+(1-\frac{[n-s+2-k]}{q[n-s+1]})\cdot\frac{[s-2+k]}{[n]}.$$

For the sake of convenience, we denote

$$\lambda_1=\frac{q^{n-s+1}[k][n-s+2-k]\{q^n+[s-1]\}}{[n-s+1][n-s+2][n][n+1]},$$

$$\lambda_2 = \frac{q^{n-s+1}[k][n-s+2-k]\left\{q^{s-2}+q^n[s-1]\right\}}{[n-s+1][n-s+2][n][n+1]},$$

then we have

$$\begin{aligned}&\{L_{n+1,s}(f,q;x)-L_{n,s}(f,q;x)\}\prod_{l=0}^{n-s+1}(1-q^l x)^{-1}\\ \le\ &\sum_{k=1}^{n-s+1}\{\lambda_1\left(f[\eta_1,x_{n+1,k}]-f[x_{n,k},x_{n,s-2+k}]\right)+\lambda_2\left(f[x_{n,k},x_{n,s-2+k}]-f[x_{n+1,s-1+k},\eta_2]\right)\}\varphi_{n,k}(x).\end{aligned}$$

As

$$x_{n,k-1}<\eta_1<x_{n+1,k}<x_{n,k}<x_{n,s-2+k}<x_{n+1,s-1+k}<\eta_2<x_{n,s-1+k},$$

$\lambda_i \ge 0, i=1,2$, and $f(x)$ is convex on $[0,1]$, we have for n sufficiently large that

$$L_{n+1,s}(f,q;x)-L_{n,s}(f,q;x)\le 0, \tag{2.2}$$

holds for all $x\in[0,1]$. For $s=0$, (2.2) is clear. The proof of Theorem 2 is complete.

3 Approximation Theorem

For $0<q<1,\ f\in C[0,1]$, it is not difficult to get for $x\in[0,1]$,

$$|L_{n,s}(f,q;x)-f(x)|\le 2\,\omega\left(f,\sqrt{\left(\frac{[1]}{[n]}+\frac{q^{n-s}[s]^2-q^{n-s}[s]}{[n]^2}\right)x(1-x)}\right), \tag{3.1}$$

where $\omega(f,t)$ is the usual modulus of continuity of the function $f(x)$.

As for a fixed q satisfying $0<q<1$, $\lim\limits_{n\to\infty}[n]^{-1}=0$ does not hold, we can conclude that the generalization of Stancu operator $L_{n,s}(f,q)$ does not converge to the mother function $f(x)$ any more, whatever the parameter s is. While for $q=q(n)\in(0,1]$ and $\lim\limits_{n\to\infty}q_n=1$, $L_{n,s}(f,q_n;x)$ converges to the continuous function $f(x)$ uniformly for $x\in[0,1]$. However, the approximation rate can not be better than the Stancu polynomials. Actually, under some necessary condition of integer s, for $f\in C[0,1]$, $L_{n,s}(f,q;x)$ converges to a limit operator which is defined as:

Definition 3[7]. For any nonnegative integer n, $f(x)\in C[0,1]$,

$$B_\infty(f,q;x)=\begin{cases}\displaystyle\sum_{k=0}^{\infty} f(1-q^k)p_{\infty,k}(q;x), & 0\le x<1,\\ f(1), \quad x=1,\end{cases} \tag{3.2}$$

where $p_{\infty,k}(q;x)=\dfrac{x^k}{(1-q)^k[k]!}\displaystyle\prod_{s=0}^{\infty}(1-q^s x)$.

In detail, we have the following theorem.

Theorem 3. *Let $f(x)\in C[0,1]$, abd s an integer with $0\le s<\frac{n}{2}$, and $0<q<1$, then holds*

$$\|L_{n,s}(f,q;x)-B_\infty(f,q;x)\|_C\le\Big(4-\frac{4\ln(1-q)}{q(1-q)}\Big)\omega(f,q^{n-s+1}). \tag{3.3}$$

It can be seen from this theorem that for fixed integer s or $s=s(n), n-s(n)\to\infty$,

$$\lim_{n\to\infty}\|L_{n,s}(f,q;x)-B_\infty(f,q;x)\|_C=0$$

holds for $0<q<1$. This result has some slightly difference with the corresponding result of Stancu operator in [2]. To Stancu operator, when $s=s(n)$ it should satisfy $s=o(n)$ as $n\to\infty$ to make sure the convergence of the relevant Stancu polynomial. While to q-Stancu operator it only needs $n-s(n)\to\infty$. Hereby for $s=s(n)=\frac{n-1}{2},\ \frac{n}{3},\ \frac{n}{4},\ \cdots$, we still have $\lim\limits_{n\to\infty}\|L_{n,s}(f,q;x)-B_\infty(f,q;x)\|_C=0$, but for Stancu operator it doesn't hold any longer.

Proof of Theorem 3. Based on the proposition 2 and the linear preserving properties of the limit operator $B_\infty(\cdot,q)$ [7], we can assume $f(0)=f(1)=0$ without loss of generality.

Then we have

$$\begin{aligned}
&|L_{n,s}(f,q;x)-B_\infty(f,q;x)|\\
=\ &\Big|\sum_{k=0}^{n-s+1}\left\{\frac{[n-s+1-k]}{[n-s+1]}f\left(\frac{[k]}{[n]}\right)+\frac{q^{n-s+1-k}[k]}{[n-s+1]}f\left(\frac{[s-1+k]}{[n]}\right)\right\}p_{n-s+1,k}(q;x)\\
&-\sum_{k=0}^{\infty}f(1-q^k)p_\infty(q;x)\Big|\\
\le\ &\Big|\sum_{k=0}^{n-s+1}\left\{\frac{[n-s+1-k]}{[n-s+1]}\Big(f\left(\frac{[k]}{[n]}\right)-f(1-q^k)\Big)+\frac{q^{n-s+1-k}[k]}{[n-s+1]}\Big(f\left(\frac{[s-1+k]}{[n]}\right)\right.\\
&\left.-f(1-q^k)\Big)\right\}p_{n-s+1,k}(q;x)\Big|+\Big|\sum_{k=0}^{n-s+1}(f(1-q^k)-f(1))(p_{n-s+1,k}(q;x)-p_{\infty,k}(q;x))\Big|\\
&+\Big|\sum_{k=n-s+2}^{\infty}(f(1-q^k)-f(1))p_{\infty,k}(q;x)\Big|:=I_1+I_2+I_3.
\end{aligned}$$

From the proof of Theorem 1 in [11], we know

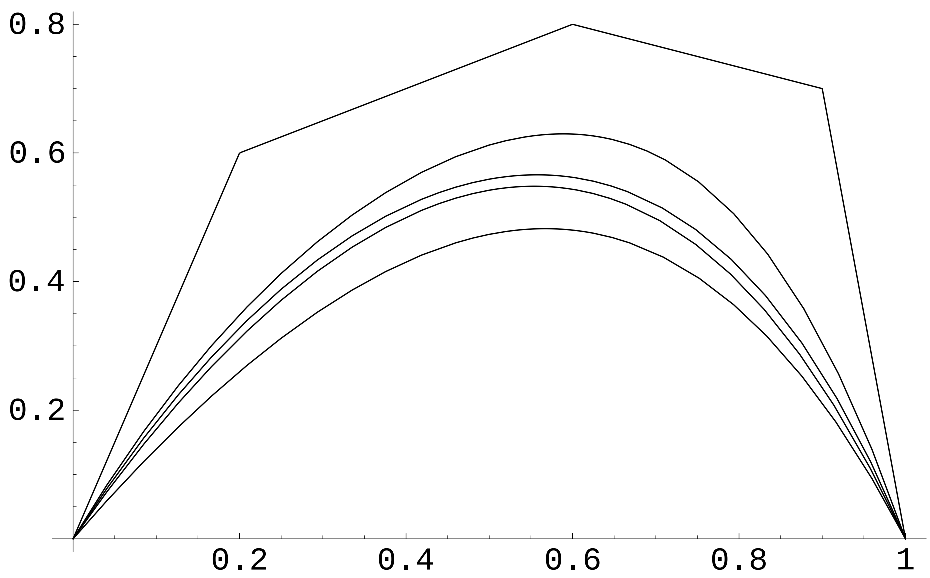
$$I_2\le\frac{-4\ln(1-q)}{q(1-q)}\omega(f,q^{n-s+1}),\qquad\qquad I_3\le\omega(f,q^{n-s+1}).$$

Since for $0 < \delta \le \eta \le 1$, holds $\dfrac{\omega(f,\eta)}{\eta} \le 2\dfrac{\omega(f,\delta)}{\delta}$, then we have

$$\begin{aligned}
I_1 &\le \sum_{k=0}^{n-s+1} \left\{ \frac{[n-s+1-k]}{[n-s+1]}\omega(f,\frac{[k]}{[n]}q^n) + \frac{q^{n-s+1-k}[k]}{[n-s+1]}\omega(f,\frac{[s-1]}{[n]}q^k + \frac{[k]}{[n]}q^n) \right\} p_{n-s+1,k}(q;x) \\
&\le \sum_{k=0}^{n-s+1} \omega(f,\frac{[k]}{[n]}q^n) p_{n-s+1,k}(q;x) + \sum_{k=0}^{n-s+1} \frac{q^{n-s+1}[k]}{[n-s+1]}\frac{[s-1]}{[n]}\frac{\omega(f,\frac{[s-1]}{[n]}q^k)}{\frac{[s-1]}{[n]}q^k} p_{n-s+1,k}(q;x) \\
&\le \omega(f,q^n) + \sum_{k=0}^{n-s+1} \frac{q^{n-s+1}[k]}{[n-s+1]}\frac{[s-1]}{[n]}\frac{2\omega(f,\frac{[s-1]}{[n]}q^{n-s+1})}{\frac{[s-1]}{[n]}q^{n-s+1}} p_{n-s+1,k}(q;x) \\
&\le \omega(f,q^n) + 2\omega(f,\frac{[s-1]}{[n]}q^{n-s+1}) \sum_{k=0}^{n-s+1} \frac{[k]}{[n-s+1]} p_{n-s+1,k}(q;x) \\
&\le \omega(f,q^n) + 2x\omega(f,\frac{[s-1]}{[n]}q^{n-s+1}).
\end{aligned}$$

Combining the results of I_1, I_2, I_3 we complete the proof of Theorem 3.

Figure 1 The function $f(x)$ is the segment by segment linear function combining $(0,0)$, $(0.2,0.6)$, $(0.6,0.8)$, $(0.9,0.7)$ and $(1,0)$. The others are $L_{15,3}(f,0.7;x)$, $L_{11,5}(f,0.7;x)$, $L_{7,3}(f,0.7;x)$ and $L_{20,3}(f,0.5;x)$ from up to down.



References

[1] Stancu, D. D., Oberwolfach (ed by H*ä*mmerlin G.), Pro. Conf. Math. Res. Inst., 1981, 241-251.

[2] Cao, F. L., The Approximation Theorems for Stancu Polynomials, Journal of Qufu Normal University 24:3(1998), 25-30.

[3] Cao, F. L. and Xu, Z. B., Stancu Polynomials Defined on a Simplex and Best Polynomial Approximation, Acta Mathematic Sinica, 46:1(2003), 189-196.

[4] Cao, F. L. and Yang, R. Y., Optimal Approximation Order and its Characterization for Multivariate Stancu Polynomials, Acta Mathematicae Applicatae Sinica, 27:2(2004), 218-229.

[5] Cao, F. L., Multivariate Stancu Polynomials and Modulus of Continuity, Acta Mathematic Sinica, 48:1(2005), 51-62.

[6] Li, F. J., Xu, Z. B. and Zhen, K. J., Optimal Approximation Order for q-Stancu Operators Defined on a Simplex, Acta Mathematica Sinica, 51(2008), 135-144.

[7] Phillips, G. M., Bernstein Polynomials Based on the q-integers, Ann. Numer Math., 4(1997), 511-518.

[8] Goodman, T. N. T., Oruc, H. and Phillips, G. M., Convexity and Generalized Bernstein Polynomials, Pro. Edinburgh Math. Soc., 42:1(1999), 179-190.

[9] Videnskii, V. S., On Some Classes of q-parametric Positive Linear Operators, Oper. Theory Adv. Appl., 158(2005), 213-222.

[10] Il'inskii, A. and Ostrovska, S., Convergence of Generalized Bernstein Polynomials, J. Approx. Theory , 116(2002), 100-112.

[11] Wang, H. P., The Rate of Convergence of q-Bernstein Polynomials for $0 < q < 1$, J. Approx. Theory, 136(2005), 151-158.

[12] Wang, H. P., Saturation of Cconvergence for q-Bernstein Polynomials in the Case $q \geq 1$, J. Math. Anal. Appl., 337(2008), 744-750.

[13] Aral, A. and Gupta, V., The q-derivative and Application to q-Sz*á*sz Mirakyan Operators, CALCOLO, 43(2006), 151-170.

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